Nanophotonics

Mie theory for describing particle scattering

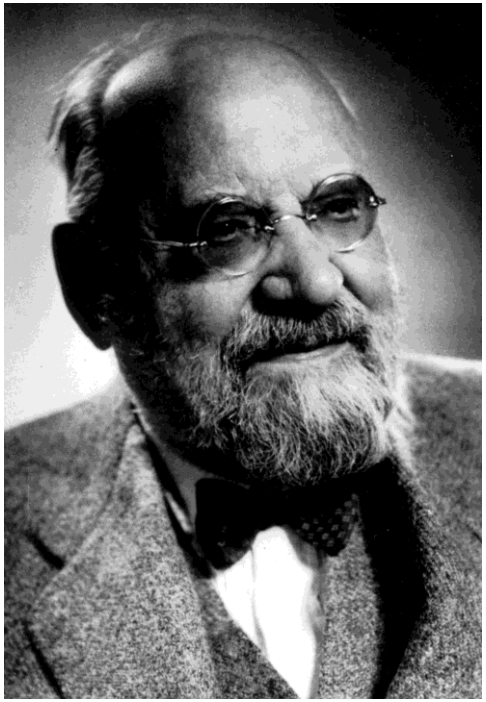
Brian Stout

C.L.A.R.T.E / *Institut Fresnel, U.M.R. 6133*

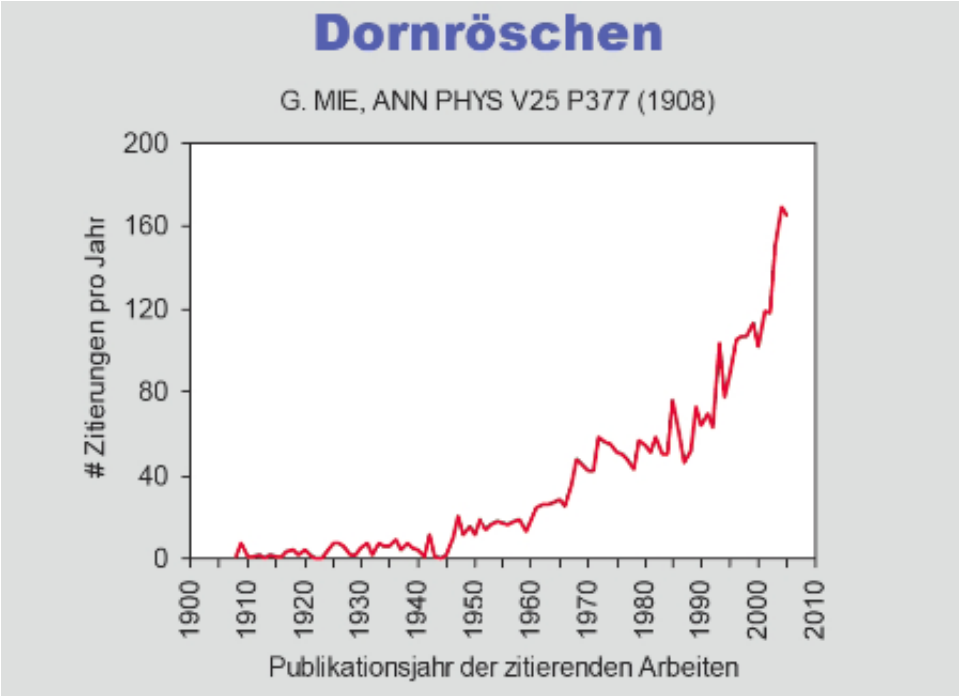
http://www.fresnel.fr/perso/stout/

Institut Fresnel, CNRS Aix Marseille Université,
Domaine Universitaire de Saint Jérôme,
13397 Marseille, France

Lorenz(1890)-Mie(1908)-Debye(1909) theory ?



Gustav Mie
(1868-1957)



Ludvig Lorenz (1829–91)

“Light scattering and reflection by a transparent sphere (surface)”

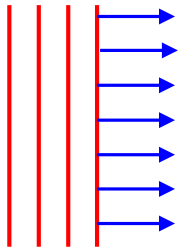
in Oeuvres scientifiques de **L. Lorenz**. revues et annotées par H. Valentiner.

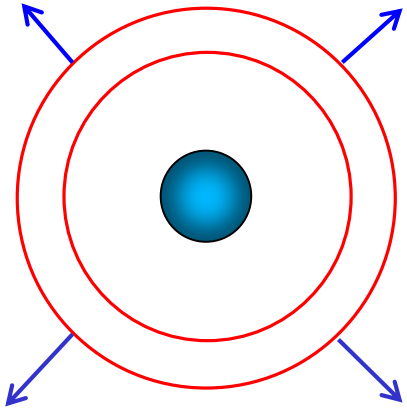
Tome Premier, Libraire Lehmann & Stage, Copenhague, 1898, p 403-529.

Scattering from a homogeneous sphere

Incident plane waves :

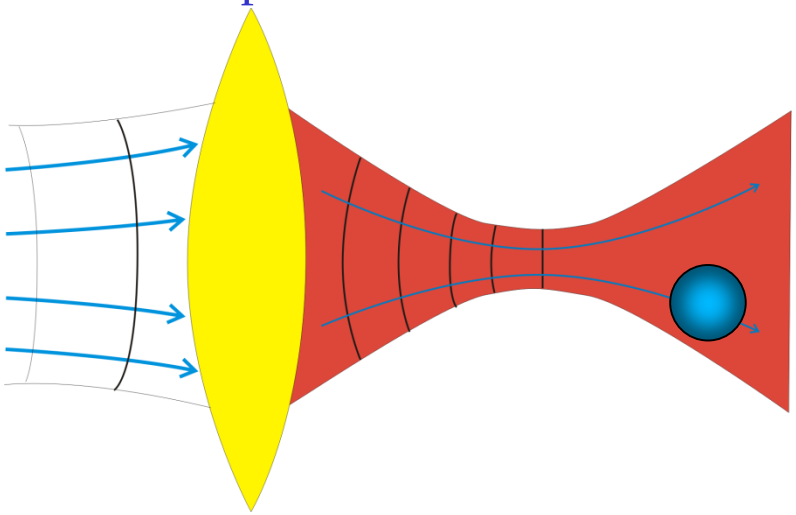
<u>Lorenz</u>-Mie-Debye theory:





Particles in shaped beams :

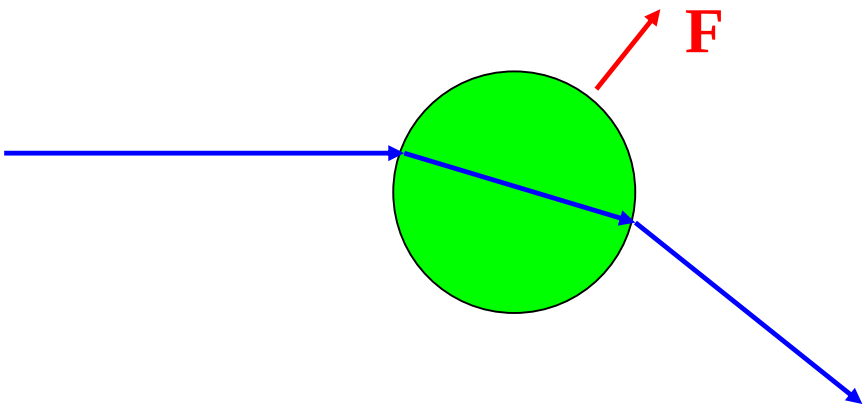
"Generalized" Mie theory :



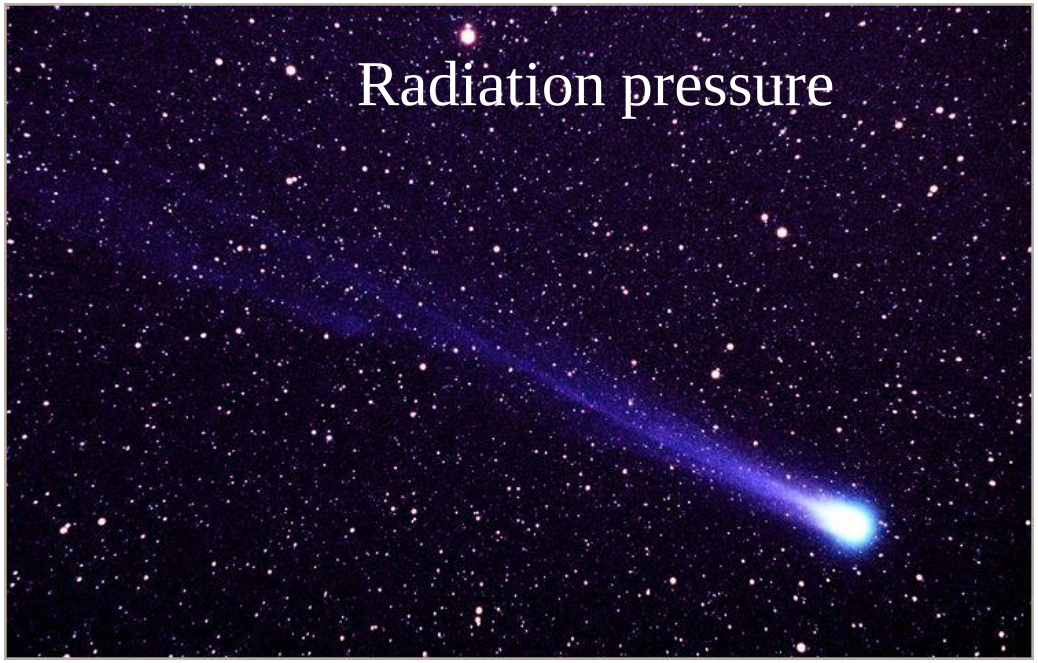
Peter Debye (1909) – radiation pressure

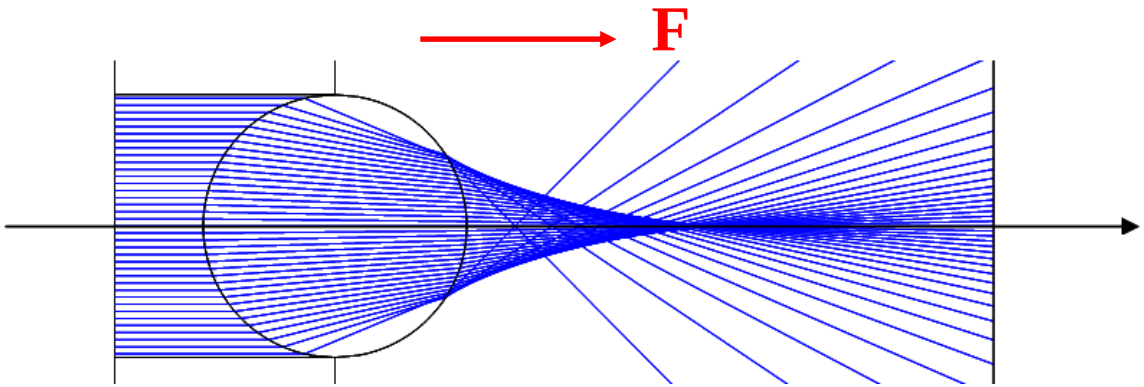
A photon transports
both energy and momentum

$E = cp$



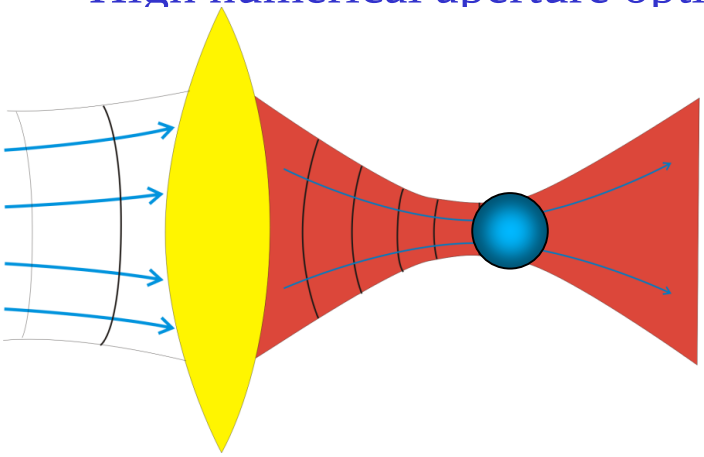
Radiation forces

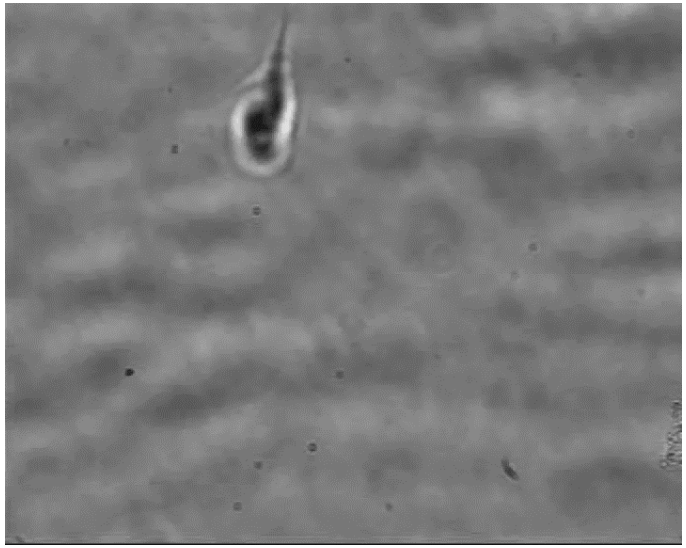




Optical Tweezers

High numerical aperture optics :





Mie theory from nano to milli-metric particles

Rayleigh scattering

Radiation pressure

Glory – backscattering

Rainbows, clouds

Scattering and absorption by nano-particles (plasmonics)

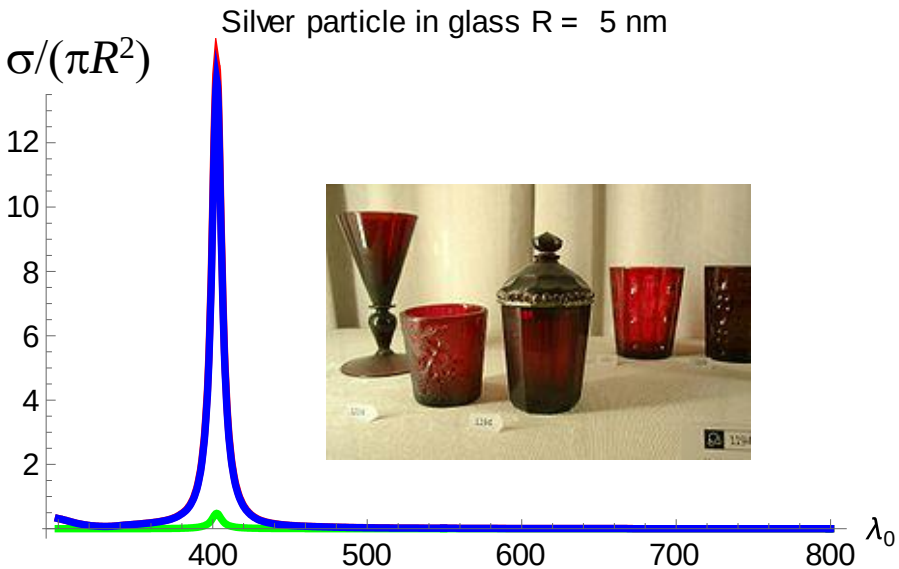
Mie(1908) - Contributions to the optics of turbid media, particularly of colloidal metal solutions

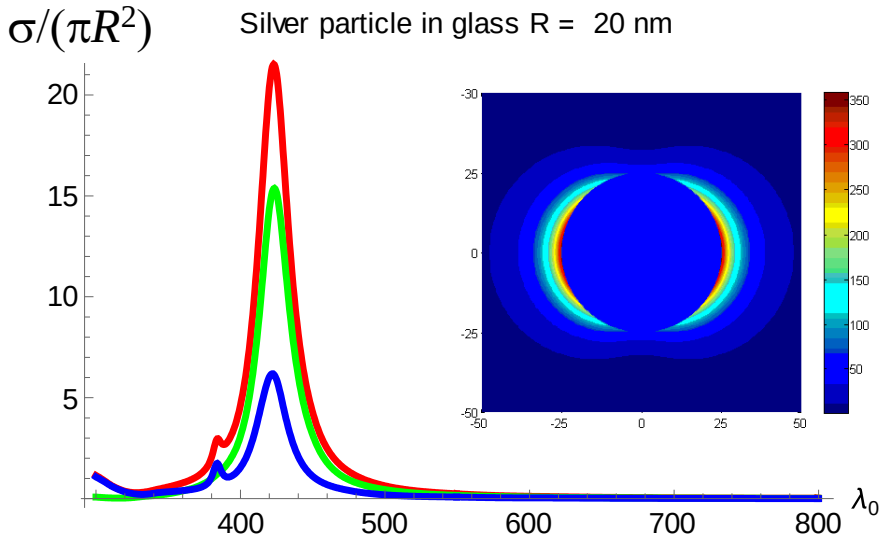
Stain glass ~ 17th century

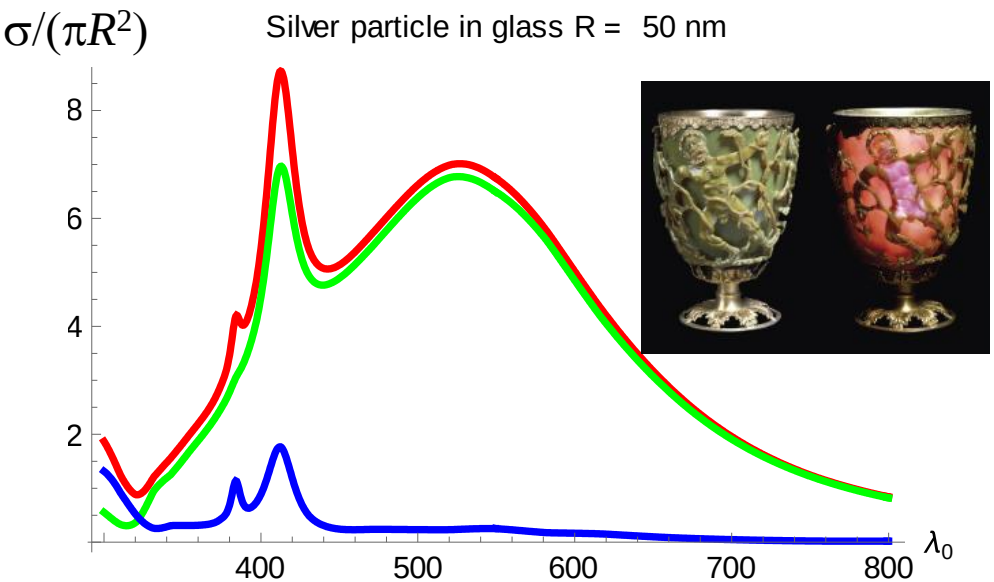
Gold “Ruby” glass

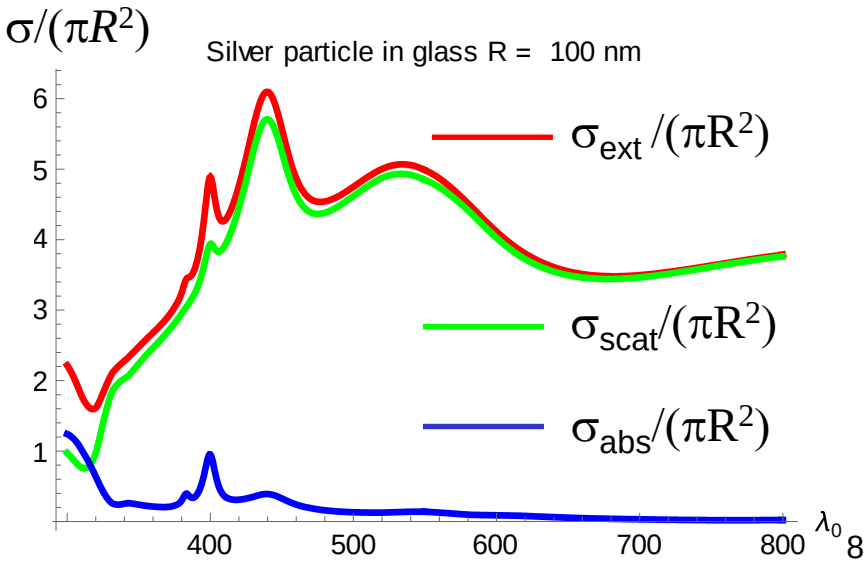
Lycurgus cup ~ 4th century

Silver spheres in glass
(Localized surface plasmon resonances)

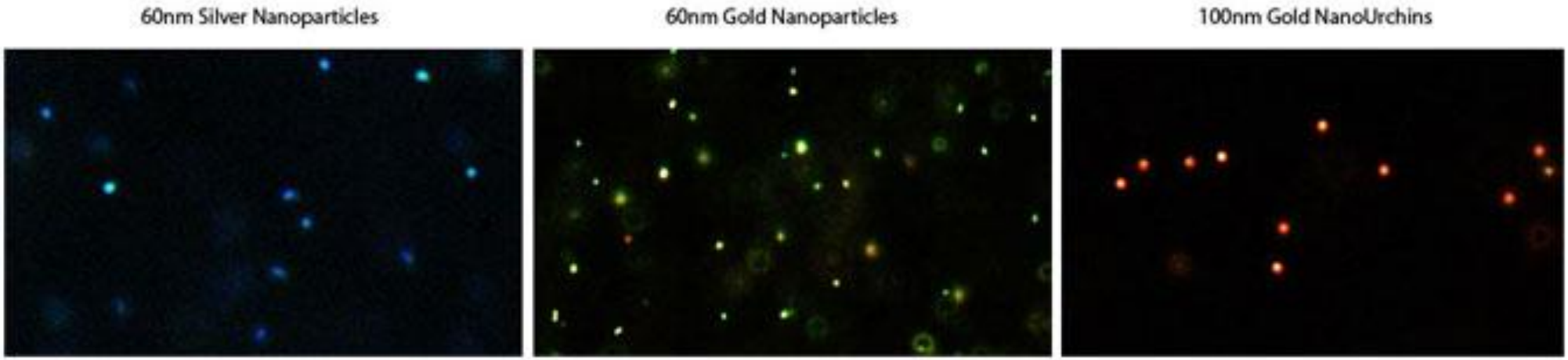








Dark field imaging of plasmonic particles (imaging "below the diffraction limit")



Wave equations
(spherically symmetric potentials)

Acoustic waves:

$$\nabla \cdot \frac{1}{\rho(r)} \nabla \psi - \frac{1}{\kappa(r)} \left(\frac{\omega}{c}\right)^2 \psi = 0$$

Electromagnetic waves:

$$\nabla \times \frac{1}{\mu(r)} \nabla \times \vec{\mathbf{E}} - \varepsilon(r) \left(\frac{\omega}{c}\right)^2 \vec{\mathbf{E}} = \vec{\mathbf{0}}$$

Schrödinger's equation:

$$\Delta\psi + \frac{2m}{\hbar^2} E\psi - \frac{2m}{\hbar^2} V(r)\psi = 0$$

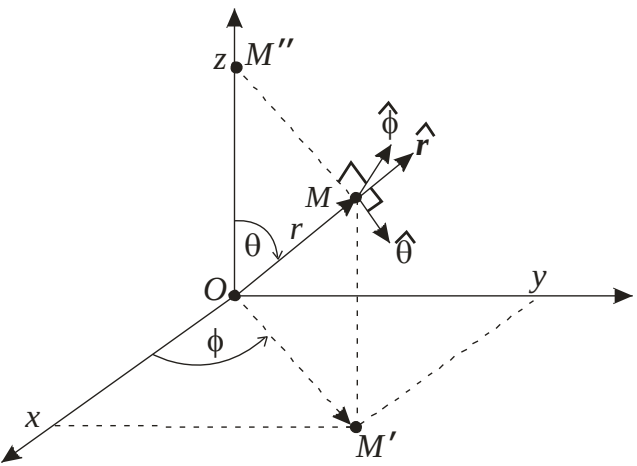
Scalar "light" ($\rho = 1 \quad \varepsilon = 1/\kappa$):

$$\Delta\psi + \varepsilon(r)\left(\frac{\omega}{c}\right)^2 \psi = 0$$

Optical fields ($\mu = 1$) :

$$\nabla \times \nabla \times \vec{\mathbf{E}} - \varepsilon(r) \left(\frac{\omega}{c}\right)^2 \vec{\mathbf{E}} = \vec{\mathbf{0}}$$

Scalar Helmholtz equation in a spherically symmetric potential:



$$\Delta\psi + \left(\frac{\omega}{c}\right)^2 \varepsilon(r)\psi = 0$$

$$\frac{1}{r}\frac{\partial^2(r\psi)}{\partial r^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2} + \left(\frac{\omega}{c}\right)^2 \varepsilon(r)\psi = 0$$

Separation of variables : $\psi(r,\theta,\phi) = \psi_r(r)\psi_\theta(\theta)\psi_\phi(\phi)$

$$\frac{d^2\psi_\phi}{d\varphi^2} + a\psi_\phi = 0 \quad \Rightarrow \psi_\phi(\phi) = \exp(im\phi) \qquad m = -n,\ldots,n$$

$$\frac{1}{\sin^2\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\psi_\theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta}\psi_\theta = -n(n+1)\psi_\theta \quad \Rightarrow \quad \psi_\theta(\theta) = P_n^m(\cos\theta) \qquad n = 0,1,2\ldots\infty$$

Legendre Polynomials : $P_n(x)$
Associated Legendre functions : $P_n^m(x)$

$$\left[\frac{1}{\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi_\theta e^{im\phi}=-n(n+1)\psi_\theta e^{im\phi}$$

$$-\mathcal{L}^2\psi_\theta e^{im\phi}=-n(n+1)\psi_\theta e^{im\phi}$$

$$\left[\frac{1}{\sin^2\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\psi_\theta}{d\theta}\right)e^{im\phi}-\frac{m^2}{\sin^2\theta}\psi_\theta\right]e^{im\phi}=-n(n+1)\psi_\theta e^{im\phi}\quad\Rightarrow\quad \psi_\theta(\theta)=P_n^m(\cos\theta)\qquad n=0,1,2\ldots\infty$$

Legendre Polynomials: $P_n(x)=P_n^0(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}\left[\left(x^2-1\right)^n\right]$ (Rodrigues' Formula)

$$P_0(x)=1$$

$$P_1(x)=x$$

$$P_2(x)=\frac{1}{2}\left(3x^2-1\right)$$

$$P_3(x)=\frac{1}{2}\left(5x^2-3x\right)$$

$$\vdots$$

Associated Legendre functions:

$$P_n^m(x)=(-1)^m\left(1-x^2\right)^{m/2}\frac{d^m}{dx^m}P_n(x)$$

$$=(-1)^m(\sin\theta)^{m/2}\frac{d^m}{dx^m}P_n(x)$$

Scalar Spherical harmonics

$$Y_{n,m}(\theta,\phi) = \left[\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}\right]^{1/2} e^{im\phi} P_n^m(\cos\theta) \qquad n = 0,\dots,\infty \qquad m = -n,\dots,n$$

$$\equiv e^{im\phi}\bar{P}_n^m(\cos\theta)$$

Angular momentum operator

$$\left[\frac{1}{\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}\right]Y_{nm}(\theta,\phi)$$

$$=-\bar{\bar{L}}^2 Y_{nm}(\theta,\phi) = -n(n+1)Y_{nm}(\theta,\phi)$$

Orthogonal basis:

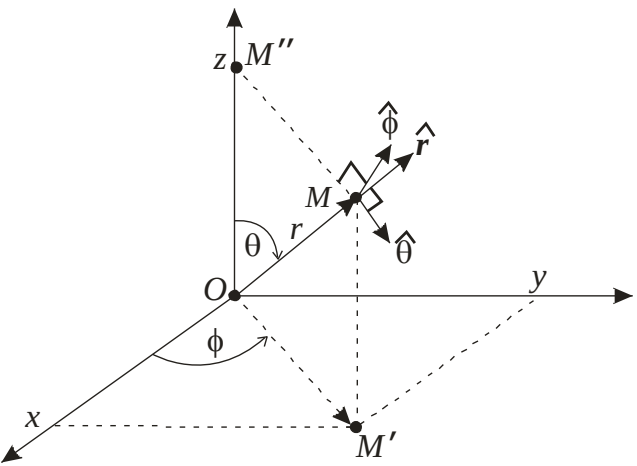
$$\boxed{\int_0^{4\pi} d\Omega Y_{\nu,\mu}^*(\theta,\phi)Y_{n,m}(\theta,\phi) \equiv \delta_{n,\nu}\delta_{m,\mu}}$$

$$n=0 \quad Y_{00}(x) = \frac{1}{\sqrt{4\pi}}$$

$$n=1 \quad \begin{cases} Y_{11} = -\sqrt{\dfrac{3}{8\pi}}\sin\theta e^{i\phi} \\ Y_{10} = \sqrt{\dfrac{3}{4\pi}}\cos\theta \end{cases}$$

$$n=2 \quad \begin{cases} Y_{22} = \dfrac{1}{4}\sqrt{\dfrac{15}{2\pi}}\sin^2\theta e^{i2\phi} \\ Y_{21} = -\sqrt{\dfrac{15}{8\pi}}\sin\theta\cos\theta e^{i\phi} \\ Y_{20} = \sqrt{\dfrac{5}{4\pi}}\left(\dfrac{3}{2}\cos^2\theta - \dfrac{1}{2}\right) \end{cases}$$

Scalar Helmholtz equation in a spherically symmetric potential:



$$\Delta\psi + \left(\frac{\omega}{c}\right)^2 \varepsilon(r)\psi = 0$$

$$\frac{1}{r}\frac{\partial^2(r\psi)}{\partial r^2} - \frac{\bar{\bar{L}}^2}{r^2}\psi + \left(\frac{\omega}{c}\right)^2 \varepsilon(r)\psi = 0$$

$$\psi(r,\theta,\phi) = \psi_r(r) Y_{n,m}(\theta,\phi) \qquad \bar{\bar{L}}^2 Y_{n,m}(\theta,\phi) = n(n+1) Y_{n,m}(\theta,\phi)$$

$$Y_{n,m}(\theta,\phi) = \bar{P}_n^m(\cos\theta)\exp(im\phi) \qquad \begin{cases} n = 0,1,2\ldots\infty \\ m = -n,\ldots,n \end{cases}$$

$$\boxed{\frac{d^2u}{dr^2} - \frac{n(n+1)}{r^2}u(r) + \left(\frac{\omega}{c}\right)^2 \varepsilon(r)u(r) = 0}$$

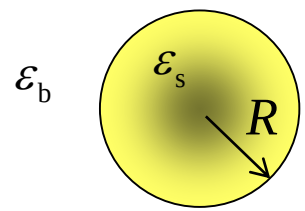
$$u(r) \equiv r\psi_r(r)$$

Scattering from a spherically symmetric potential
(Quantum mechanical analogy)

Scalar light : $k^2 = \varepsilon_b \left(\frac{\omega}{c}\right)^2$

$$\frac{d^2}{dr^2}u_n(r) + k^2 u_n(r) - V_{\text{eff}}^{(\text{opt})}(r)\, u_n(r) = 0$$

$$V_{\text{eff}}^{(\text{opt})}(r) = \frac{n(n+1)}{r^2} - (\varepsilon_{\text{s}} - \varepsilon_{\text{b}})\left(\frac{\omega}{c}\right)^2 \theta(R-r)$$



Schrödinger equation :

$$\frac{d^2}{dr^2}u_n(r) + \frac{2m}{\hbar^2}E u_n(r) - V_{\text{eff}}^{(\text{Sch})}(r)\, u_n(r) = 0$$

$$V_{\text{eff}}^{(\text{Sch})}(r) = \frac{n(n+1)}{r^2} - \frac{2m}{\hbar^2}\frac{e^2}{4\pi\varepsilon_0 r}$$

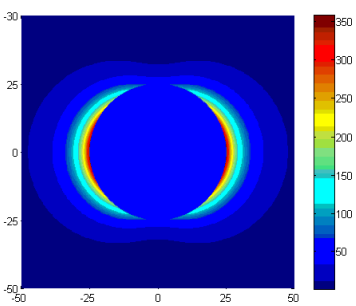
Effective potential – Photonic “atom”

Scalar light : $k^2 = \varepsilon_b \left(\frac{\omega}{c}\right)^2$

$$V_{\text{eff}}^{(\text{opt})}(r) = \frac{n(n+1)}{r^2} - (\varepsilon_s - \varepsilon_b)\left(\frac{\omega}{c}\right)^2 \theta(R-r)$$

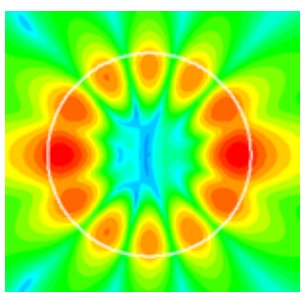
Plasmonic resonances
“repulsive” potential :

$\varepsilon_s < \varepsilon_b$

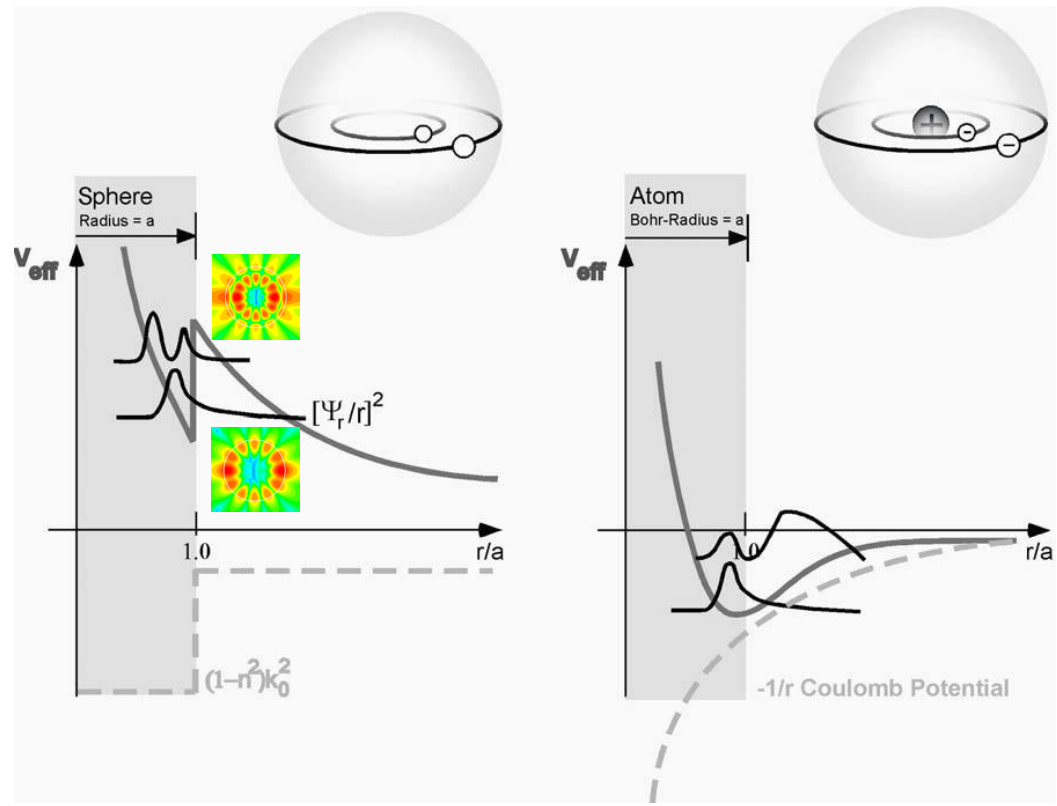


“Morphologic” resonances
“attractive” potential :

$\varepsilon_s > \varepsilon_b$

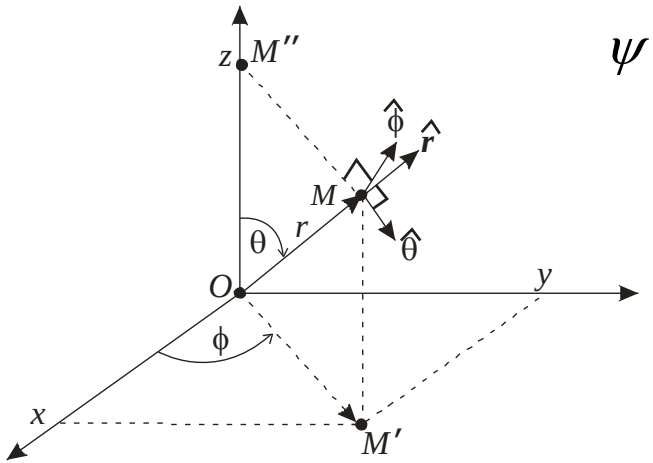


Whispering Gallery modes $\varepsilon_s > \varepsilon_b$
“attractive” potential with radiative loss :



$$V_{\text{eff}}^{(\text{Sch})}(r) = \frac{n(n+1)}{r^2} - \frac{2m}{\hbar^2}\frac{e^2}{4\pi\varepsilon_0 r}$$

Homogeneous Helmholtz equation in spherical coordinates



Separation of variables:

$$\psi(r,\theta,\phi)=\psi_r(r)Y_{n,m}(\theta,\phi) \quad n=0,1,2...\infty \quad m=-n,...,n$$

$$\Delta\psi+k^2\psi=0 \quad \rightarrow \quad \left[\frac{1}{r}\frac{\partial^2(r\psi_r)}{\partial r^2}-\frac{\psi_r L^2}{r^2}+k^2\psi_r\right]Y_{n,m}=0$$

$$\left[r\frac{d^2(r\psi_r)}{dr^2}-n(n+1)\psi_r+\left(k^2r^2\right)\psi_r\right]Y_{n,m}=0$$

$$k^2=\varepsilon\left(\frac{\omega}{c}\right)^2$$

Change of variables: $\psi_r(r)=\dfrac{Z(r)}{(kr)^{1/2}}$

Spherical Bessel function equation:

$$\boxed{r^2\frac{d^2Z}{dr^2}+r\frac{dZ}{dr}+\left[k^2r^2-\left(n+\frac{1}{2}\right)^2\right]Z=0}$$

Homogeneous Helmholtz equation in spherical coordinates

$$r^2\frac{d^2Z}{dr^2}+r\frac{dZ}{dr}+\left[k^2r^2-\left(n+\frac{1}{2}\right)^2\right]Z=0$$

$$\psi_r(r)=\frac{Z(r)}{(kr)^{1/2}}$$

Spherical Bessel, Neumann and Hankel functions:

$$\psi(x)=\begin{cases} j_n(x)\equiv\sqrt{\frac{\pi}{2x}}J_{n+1/2}(x) \\ y_n(x)\equiv\sqrt{\frac{\pi}{2x}}Y_{n+1/2}(x) \\ h_n^+(x)\equiv\sqrt{\frac{\pi}{2x}}H_{n+1/2}(x)=j_n(x)+iy_n(x) \\ h_n^-(x)\equiv\sqrt{\frac{\pi}{2x}}H_{n-1/2}(x)=j_n(x)-iy_n(x) \end{cases}$$

Bessel (1):

$$j_0(x)=\sum_{s=0}^{\infty}\frac{(-1)^s}{(2s+1)!}x^{2s}=\frac{\sin x}{x}$$

$$j_1(x)=\frac{\sin x}{x^2}-\frac{\cos x}{x}$$

$$\vdots$$

Neumann (2):

$$y_0(x)=-\frac{\cos x}{x}$$

$$y_1(x)=\frac{\cos x}{x^2}-\frac{\sin x}{x}$$

$$\vdots$$

Hankel + (3):

$$h_0(x)=-\frac{i}{x}e^{ix}$$

$$h_1(x)=-e^{ix}\left(\frac{1}{x}+\frac{i}{x^2}\right)$$

$$\vdots$$

Linearly independent solutions

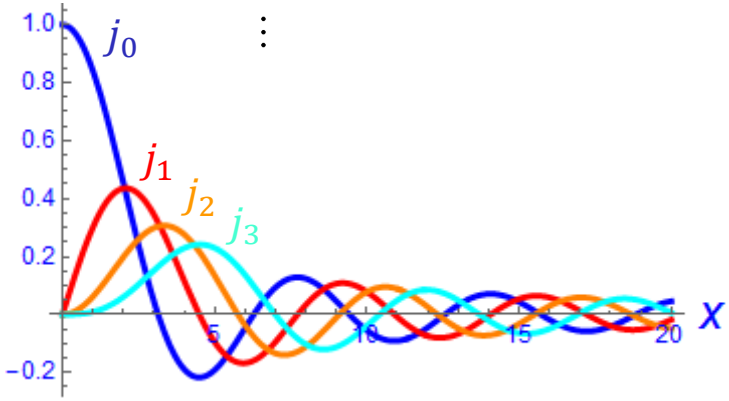
$\nabla \cdot \vec{\mathbf{M}}=0 \qquad \nabla \cdot \vec{\mathbf{N}}=0$

Spherical Bessel functions (1)

$$j_0(x)=\sum_{s=0}^{\infty}\frac{(-1)^s}{(2s+1)!}x^{2s}=\frac{\sin x}{x}$$

$$j_1(x)=\frac{\sin x}{x^2}-\frac{\cos x}{x}$$

$$\vdots$$

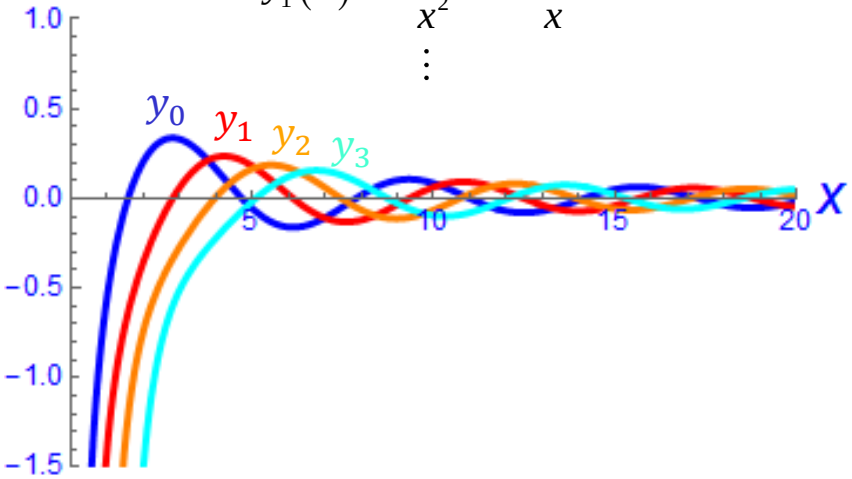


Spherical Neumann functions (2)

$$y_0(x)=-\frac{\cos x}{x}$$

$$y_1(x)=\frac{\cos x}{x^2}-\frac{\sin x}{x}$$

$$\vdots$$



Outgoing spherical Hankel functions (+)

$$h_0^{(+)}(x)=-\frac{i}{x}e^{ix}$$

$$h_1^{(+)}(x)=-e^{ix}\left(\frac{1}{x}+\frac{i}{x^2}\right)$$

$$\vdots$$

Incoming spherical Hankel functions (-)

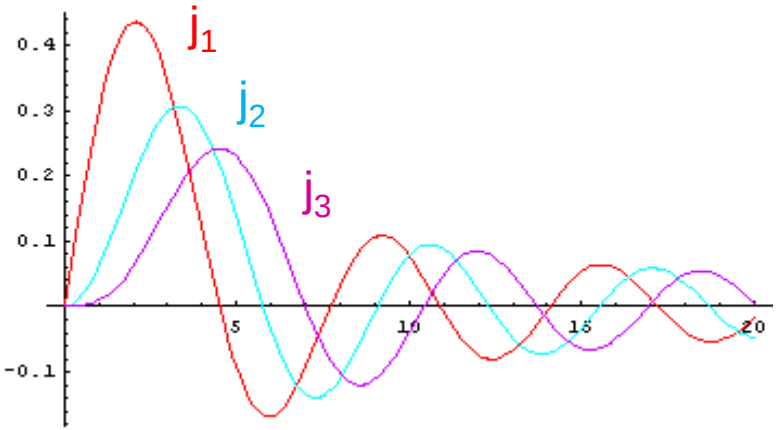
$$h_0^{(-)}(x)=\frac{i}{x}e^{-ix}$$

$$h_1^{(-)}(x)=-e^{-ix}\left(\frac{1}{x}-\frac{i}{x^2}\right)$$

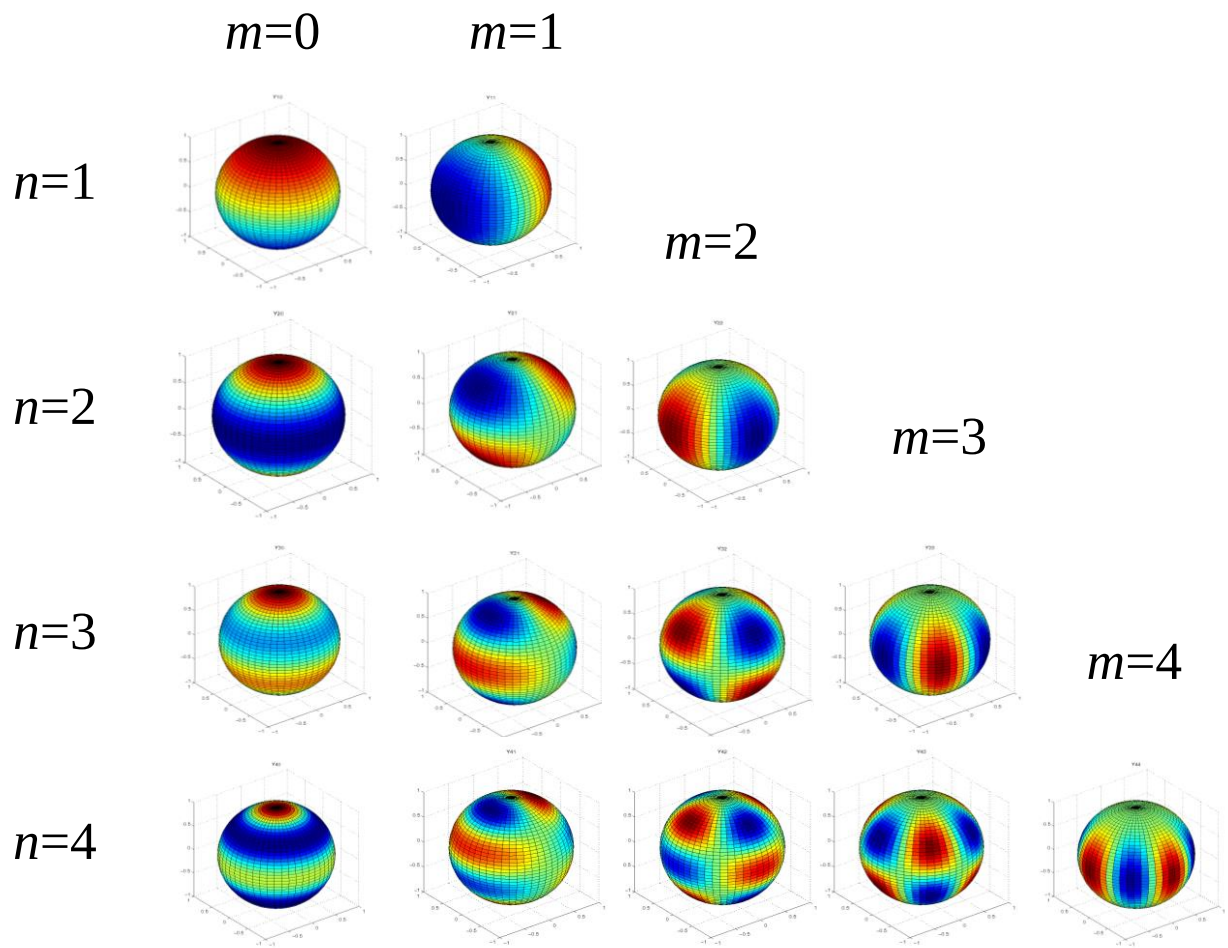
$$\vdots$$

Regular partial wave basis : $\varphi_{n,m}(k\mathbf{r}) = j_n(kr)Y_{n,m}(\theta,\phi)$

Bessel functions : $j_n(r)$



Spherical Harmonics : $Y_{n,m}(\theta,\phi)$



Partial wave basis

Plane wave expansion :

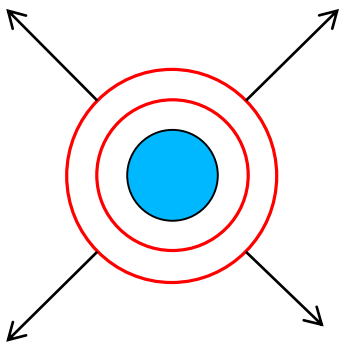
$$e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} a_{nm}\varphi_{nm}\left(k\mathbf{r}\right)$$

Plane wave Coefficients :

$$p_{n,m} = 4\pi i^n Y_{n,m}^*\left(\hat{\mathbf{k}}\right)$$

Regular partial wave :

$$\varphi_{n,m}(k\boldsymbol{r}) = j_n(kr)Y_{n,m}(\theta,\phi)$$



Outgoing partial wave :

$$\varphi_{n,m}^{(+)}(k\boldsymbol{r}) = h_n^{(+)}(kr)Y_{n,m}(\theta,\phi)$$

Helmholtz equation in spherical coordinates

$$\Delta\varphi + k^2\varphi = 0$$

Scalar "partial" waves

$$\boxed{\varphi_{n,m}(k\mathbf{r}) = j_n(kr)Y_{n,m}(\theta,\phi)}$$

$$n = 0,1,2\ldots\infty \qquad m = -n,\ldots,n$$

Bouwkamp-Casimir (Jackson) approach

$$\Delta\left(\vec{\mathbf{r}}\cdot\vec{\mathbf{H}}\right) + k^2\vec{\mathbf{r}}\cdot\vec{\mathbf{H}} = 0$$

$$\Delta\left(\vec{\mathbf{r}}\cdot\vec{\mathbf{E}}\right) + k^2\vec{\mathbf{r}}\cdot\vec{\mathbf{E}} = 0$$

$$\nabla\times\nabla\times\vec{\mathbf{E}} - k^2\vec{\mathbf{E}} = 0$$

Transverse vector "partial" waves : $\nabla\cdot\vec{\mathbf{M}}_{n,m} = \nabla\cdot\vec{\mathbf{N}}_{n,m} = 0$

$$\vec{\mathbf{M}}_{n,m}(k\mathbf{r}) = \frac{\nabla\times\left[\vec{\mathbf{r}}\varphi_{n,m}(k\mathbf{r})\right]}{\sqrt{n(n+1)}}$$ magnetic

$$\vec{\mathbf{N}}_{n,m}(k\mathbf{r}) = \frac{\nabla\times\vec{\mathbf{M}}_{n,m}(k\mathbf{r})}{k}$$ electric

Longitudinal vector "partial" waves

$$\vec{\nabla}\left(\vec{\nabla}\cdot\vec{\mathbf{L}}\right) + k^2\vec{\mathbf{L}} = 0$$

$$\vec{\mathbf{L}}_{n,m}(k\mathbf{r}) = \frac{\nabla\varphi_{n,m}(k\mathbf{r})}{\sqrt{n(n+1)}}$$

$$\Delta\vec{\mathbf{A}} + k^2\vec{\mathbf{A}} = 0 \qquad \vec{\mathbf{A}} = \vec{\mathbf{L}},\vec{\mathbf{M}},\vec{\mathbf{N}}$$

Vector spherical harmonics

Scalar spherical harmonics : $$Y_{nm}(\theta,\phi) \equiv \left[\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}\right] e^{im\phi} P_n^m(\cos\theta)$$

$$\bar{\bar{\mathbf{L}}}_{\text{op}} \equiv \frac{1}{i}(\vec{\mathbf{r}} \times \nabla)$$

3 types of <u>vector</u> spherical harmonics :

$$\vec{\mathbf{W}}_{n,m}^{(1)}(\theta,\phi) \equiv \vec{\mathbf{Y}}_{n,m}(\theta,\phi) \equiv \hat{\mathbf{r}} Y_{n,m}(\theta,\phi) \qquad n = 0,\ldots\infty \qquad m = -n,\ldots n$$

$$\vec{\mathbf{W}}_{n,m}^{(2)}(\theta,\phi) \equiv \vec{\mathbf{X}}_{n,m}(\theta,\phi) \equiv \vec{\mathbf{Z}}_{n,m}(\theta,\phi) \times \hat{\mathbf{r}} = \frac{1}{i}\bar{\bar{\mathbf{L}}}_{\text{op}} Y_{n,m}(\theta,\phi)$$

$$n = 1,\ldots,\infty \qquad m = -n,\ldots,n$$

$$\vec{\mathbf{W}}_{n,m}^{(3)}(\theta,\phi) \equiv \vec{\mathbf{Z}}_{n,m}(\theta,\phi) \equiv \frac{r\nabla Y_{n,m}}{\sqrt{n(n+1)}} = \hat{\mathbf{r}} \times \vec{\mathbf{X}}_{n,m}(\theta,\phi)$$

$$\int_0^{4\pi} d\Omega\, \mathbf{W}_{\nu,\mu}^{(j),*}(\theta,\phi)\, \mathbf{W}_{n,m}^{(k)}(\theta,\phi) \equiv \delta_{j,k}\delta_{n,\nu}\delta_{m,\mu}$$

Vector spherical harmonics

$$\vec{\mathbf{Y}}_{n,m}(\theta,\phi) = \hat{\mathbf{r}} Y_{n,m} \qquad n = 0,\ldots,\infty \qquad m = -n,\ldots,n$$

$$\vec{\mathbf{X}}_{n,m}(\theta,\phi) = e^{im\phi}\left[\hat{\boldsymbol{\theta}} i\bar{u}_n^m - \hat{\boldsymbol{\varphi}}\bar{s}_n^m\right]$$

$$\vec{\mathbf{Z}}_{n,m}(\theta,\phi) = e^{im\phi}\left[\hat{\boldsymbol{\theta}}\bar{s}_n^m + \hat{\boldsymbol{\varphi}} i\bar{u}_n^m\right] \qquad n = 1,\ldots,\infty \qquad m = -n,\ldots,n$$

$$\bar{u}_{n,m}(\theta,\phi) = \frac{1}{\sqrt{n(n+1)}}\frac{m}{\sin\theta}\bar{P}_n^m(\cos\theta)$$

$$\bar{s}_{n,m}(\theta,\phi) = \frac{1}{\sqrt{n(n+1)}}\frac{d}{d\theta}\bar{P}_n^m(\cos\theta)$$

Vector partial waves and Vector spherical Harmonics

$$\nabla \times \nabla \times \vec{\mathbf{E}} - k^2 \vec{\mathbf{E}} = 0$$

$$\vec{\mathbf{M}}_{n,m}^{(1)}(k\mathbf{r}) = j_n(kr)\vec{\mathbf{X}}_{n,m}(\theta,\phi)$$

$$\vec{\mathbf{N}}_{n,m}^{(1)}(k\mathbf{r}) = \frac{1}{kr}\left[j_n(kr)\sqrt{n(n+1)}\vec{\mathbf{Y}}_{n,m}(\theta,\phi) + \psi_n'(x)\vec{\mathbf{Z}}_{n,m}(\theta,\phi)\right]$$

$n=1,2,\ldots,\infty$ $\quad$ $-n < m < n$

Ricatti-Bessel functions

$$\psi_n(x) \equiv x j_n(x)$$

$$\psi_n'(x) = \left[j_n(x) + x j_n'(x)\right]$$

$$\nabla \cdot \vec{\mathbf{M}}_{n,m} = \nabla \cdot \vec{\mathbf{N}}_{n,m} = 0 \qquad \nabla \times \vec{\mathbf{M}}_{n,m} = k\vec{\mathbf{N}}_{n,m} \qquad \nabla \times \vec{\mathbf{N}}_{n,m} = k\vec{\mathbf{M}}_{n,m}$$

Matrix expressions for field developments

Excitation field development

$$\vec{\mathbf{E}}_{\text{exc}}(k\mathbf{r}) = \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left\{e_{n,m}^{(h)}\vec{\mathbf{M}}_{n,m}^{(1)}(k\mathbf{r}) + e_{n,m}^{(e)}\vec{\mathbf{N}}_{n,m}^{(1)}(k\mathbf{r})\right\}$$

$$\vec{\mathbf{E}}_{\text{exc}} = \left[\vec{\mathbf{M}}^{(1)}(k\mathbf{r}), \vec{\mathbf{N}}^{(1)}(k\mathbf{r})\right].e$$

$$\begin{bmatrix} e_{1,-1}^{(h)} \\ e_{1,0}^{(h)} \\ e_{1,1}^{(h)} \\ \vdots \\ e_{1,-1}^{(e)} \\ e_{1,0}^{(e)} \\ e_{1,1}^{(e)} \\ \vdots \end{bmatrix} \equiv e$$

$$p(n,m) = n(n+1) - m$$

p=1,2,…, ∞

Scattering field development

$$\vec{\mathbf{E}}_{\text{s}} = \left[\vec{\mathbf{M}}^{(+)}(k\mathbf{r}), \vec{\mathbf{N}}^{(+)}(k\mathbf{r})\right].f$$

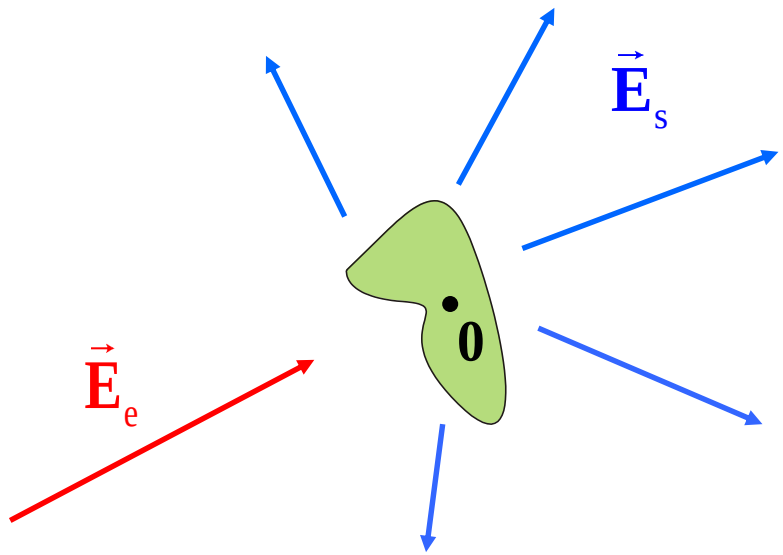
Multipole representation of the T-matrix of a scatterer

Excitation field

$$\vec{\mathbf{E}}_{\mathrm{e}}(k\mathbf{r}) = \left[\vec{\mathbf{M}}^{(1)}(k\vec{\mathbf{r}}), \vec{\mathbf{N}}^{(1)}(k\vec{\mathbf{r}})\right] . e$$

Scattering field

$$\vec{\mathbf{E}}_{\mathrm{s}}(k\mathbf{r}) = \left[\vec{\mathbf{M}}^{(+)}(k\vec{\mathbf{r}}), \vec{\mathbf{N}}^{(+)}(k\vec{\mathbf{r}})\right] . f$$



T-matrix : t

$$f = t . e$$

Mie coefficients : a_n , b_n

Homogeneous sphere T-matrix is diagonal in multipole space

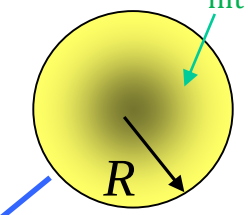
Index contrast :

$$\rho_s = \frac{k_s}{k} = \frac{N_s}{N}$$

$$\vec{E}_e(k\vec{r}) = \left[\vec{M}^{(1)}(k\vec{r}) \ , \quad \vec{N}^{(1)}(k\vec{r})\right].e$$

$$\vec{E}_s(k\vec{r}) = \left[\vec{M}^{(+)}(k\vec{r}) \ , \quad \vec{N}^{(+)}(k\vec{r})\right].f$$

$$\vec{E}_{\text{int}}(k_s\vec{r}) = \left[\vec{M}^{(1)}(k_s\vec{r}) \ , \quad \vec{N}^{(1)}(k_s\vec{r})\right].s$$



Continuity of Tangential **E** and **H** fields at the surface (r=R)

$$\begin{cases} j_n(k_sR)s_p^{(h)} = j_n(kR)e_p^{(h)} + h_n(kR)\,f_p^{(h)} \\ \dfrac{\mu}{\mu_s}\psi_n'(k_sR)s_p^{(h)} = \psi_n'(kR)e_p^{(h)} + \xi_n'(kR)\,f_p^{(h)} \\ \dfrac{\mu}{\mu_s}\rho_s\, j_n(k_sR)s_p^{(e)} = j_n(kR)e_p^{(e)} + h_n(kR)\,f_p^{(e)} \\ \psi_n'(k_sR)s_p^{(e)} = \rho_s\psi_n'(kR)e_p^{(e)} + \rho_s\xi_n'(kR)\,f_p^{(e)} \end{cases}$$

$$\begin{cases} f_p^{(e)} = -a_n\, e_p^{(e)} \\ f_p^{(h)} = -b_n\, e_p^{(h)} \end{cases}$$

$$a_n = \frac{(\mu_s/\mu)\, j_n(kR)\psi_n'(k_sR) - \rho_s^2\, j_n(k_sR)\psi_n'(kR)}{(\mu_s/\mu)\psi_n'(k_sR)h_n(kR) - \rho_s^2\, j_n(k_sR)\xi_n'(kR)}$$

$$b_n = \frac{j_n(kR)\psi_n'(k_sR) - (\mu_s/\mu)\, j_n(k_sR)\psi_n'(kR)}{\psi_n'(k_sR)h_n(kR) - (\mu_s/\mu)\, j_n(k_sR)\xi_n'(kR)}$$

Ricatti-Bessel functions

$$\begin{cases} \psi_n(x) = x j_n(x) \\ \xi_n(x) = x h_n^{+}(x) \end{cases}$$

“Symmetrized” Mie coefficients (Mie : Bohren & Huffman)

$$\begin{cases} f_p^{(e)} = -a_n \, e_p^{(e)} \\ f_p^{(h)} = -b_n \, e_p^{(h)} \end{cases}$$

$$a_n = \frac{\rho_s \psi_n(k_s R)\psi_n'(kR) - (\mu_s/\mu)\psi_n(kR)\psi_n'(k_s R)}{\rho_s \psi_n(k_s R)\xi_n'(kR) - (\mu_s/\mu)\xi_n(kR)\psi_n'(k_s R)}$$

$$b_n = \frac{\rho_s \psi_n(kR)\psi_n'(k_s R) - (\mu_s/\mu)\psi_n(k_s R)\psi_n'(kR)}{\rho_s \xi_n(kR)\psi_n'(k_s R) - (\mu_s/\mu)\psi_n(k_s R)\xi_n'(kR)}$$

T-matrix of a sphere

The T-matrix of a spherically symmetric object is diagonal in multipole space

$$p(n,m) = n(n+1) - m$$

$$\begin{bmatrix} f_p^{(h)} \\ f_p^{(e)} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} t_1^{(h)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & t_n^{(h)} \end{pmatrix} & \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} & \begin{pmatrix} t_1^{(e)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & t_n^{(e)} \end{pmatrix} \end{bmatrix} \begin{bmatrix} e_p^{(h)} \\ e_p^{(e)} \end{bmatrix} = [\![t]\!] \begin{bmatrix} e_p^{(h)} \\ e_p^{(e)} \end{bmatrix}$$

Mie Coefficients : a_n , b_n

$$t_n^{(e)} = -a_n \qquad t_n^{(h)} = -b_n$$

$$a_n = \frac{\rho_s \psi_n(k_s R)\psi_n'(kR) - (\mu_s/\mu)\psi_n(kR)\psi_n'(k_s R)}{\rho_s \psi_n(k_s R)\xi_n'(kR) - (\mu_s/\mu)\xi_n(kR)\psi_n'(k_s R)}$$

$$b_n = \frac{\rho_s \psi_n(kR)\psi_n'(k_s R) - (\mu_s/\mu)\psi_n(k_s R)\psi_n'(kR)}{\rho_s \xi_n(kR)\psi_n'(k_s R) - (\mu_s/\mu)\psi_n(k_s R)\xi_n'(kR)}$$

$$\rho_s = \frac{k_s}{k} = \frac{N_s}{N}$$

Development of a vector plane wave (Vector partial wave basis)

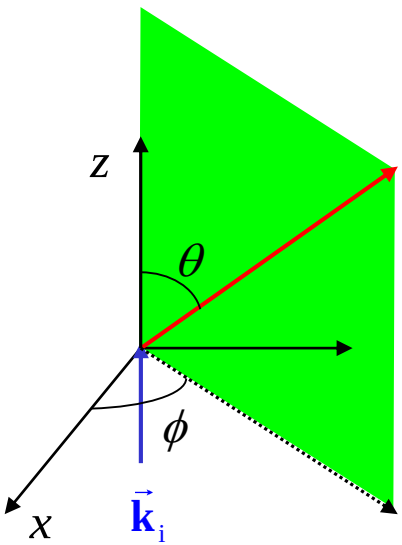
Plane wave expansion :

$$\boxed{e^{i\mathbf{k}\cdot\mathbf{r}}\hat{\mathbf{e}}_{\mathrm{i}} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[p_{n,m}^{(h)}\vec{\mathbf{M}}_{n,m}^{(1)}(k\mathbf{r}) + p_{n,m}^{(e)}\vec{\mathbf{N}}_{n,m}^{(1)}(k\mathbf{r})\right]}$$

Plane wave coefficients :

$$p_{n,m}^{(h)} = i^{n}4\pi\vec{\boldsymbol{X}}_{n,m}^{*}(\hat{\boldsymbol{k}}_{\mathrm{i}})\cdot\hat{\boldsymbol{e}}_{i} \qquad p_{n,m}^{(e)} = i^{n-1}4\pi\vec{\boldsymbol{Z}}_{n,m}^{*}(\hat{\boldsymbol{k}}_{\mathrm{i}})\cdot\hat{\boldsymbol{e}}_{i}$$

"Scattering" matrix



$$\begin{bmatrix} E_{\parallel,s} \\ E_{\perp,s} \end{bmatrix} = \frac{\exp(ikr)}{-ikr} \begin{bmatrix} S_2 & S_3 \\ S_4 & S_1 \end{bmatrix} \begin{bmatrix} E_{\parallel,i} \\ E_{\perp,i} \end{bmatrix}$$

For a spherical scatterer, $S_3 = S_4 = 0$: and

$$S_1(\theta) = \frac{1}{2}\sum_{n=1}^{\infty}(2n+1)[a_n\pi_n(\cos\theta) + b_n\tau_n(\cos\theta)]$$

$$S_2(\theta) = \frac{1}{2}\sum_{n=1}^{\infty}(2n+1)[b_n\pi_n(\cos\theta) + a_n\tau_n(\cos\theta)]$$

$$\pi_n(\cos\theta) \equiv \frac{1}{\sin\theta}\bar{P}_n^1(\cos\theta) \qquad \tau_n(\cos\theta) \equiv \frac{d}{d\theta}\bar{P}_n^1(\cos\theta)$$

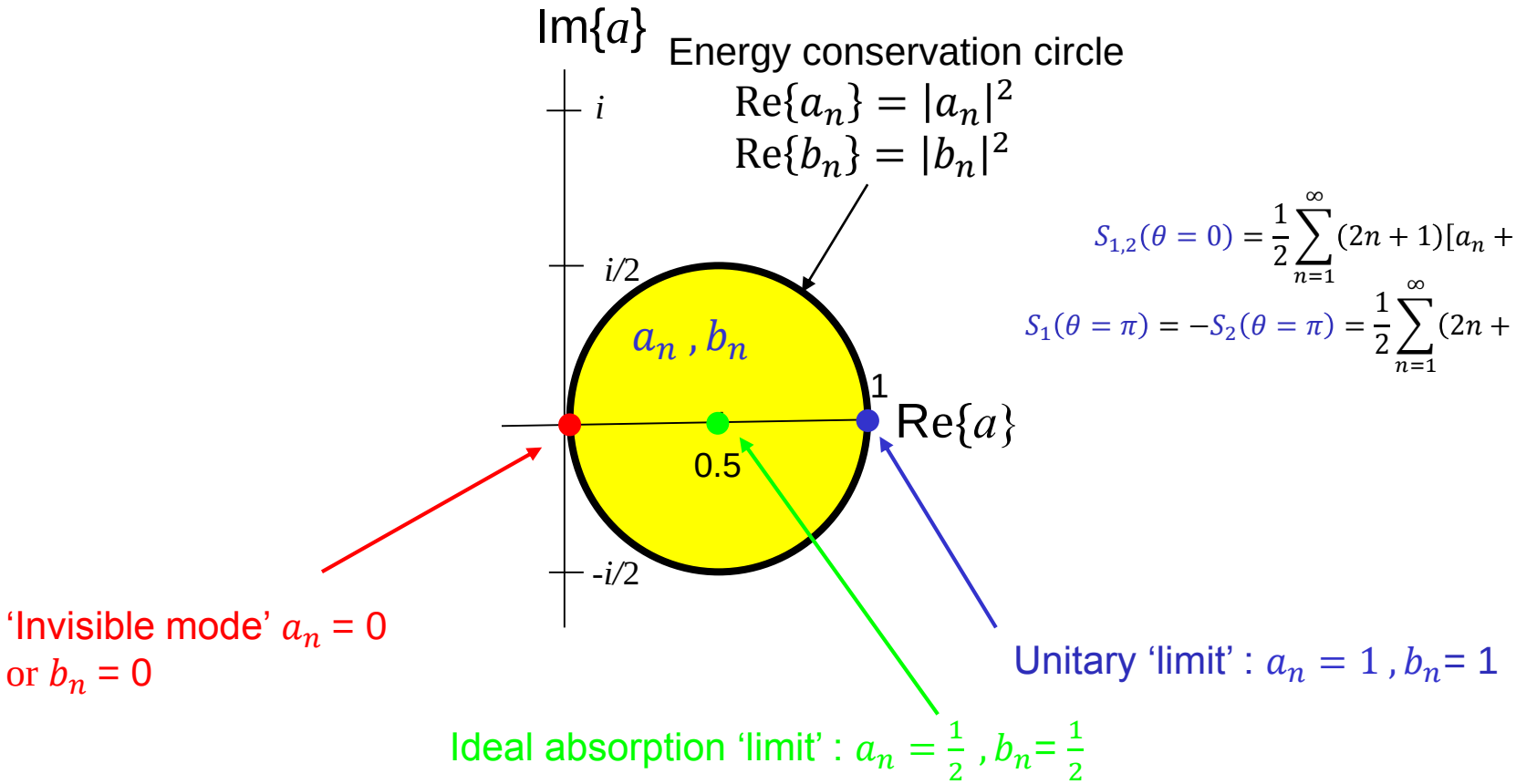
Forward scattering : $S_1(\theta = 0) = S_2(\theta = 0) = \frac{1}{2}\sum_{n=1}^{\infty}(2n+1)[a_n + b_n]$

Back scattering : $S_1(\theta = \pi) = -S_2(\theta = \pi) = \frac{1}{2}\sum_{n=1}^{\infty}(2n+1)[a_n - b_n]$

Mie coefficients are limited by the underlying physics !

Mie Coefficients

$a_n(\omega)$ and $b_n(\omega)$

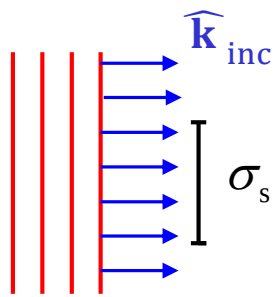


$$S_{1,2}(\theta = 0) = \frac{1}{2}\sum_{n=1}^{\infty}(2n+1)[a_n + b_n]$$

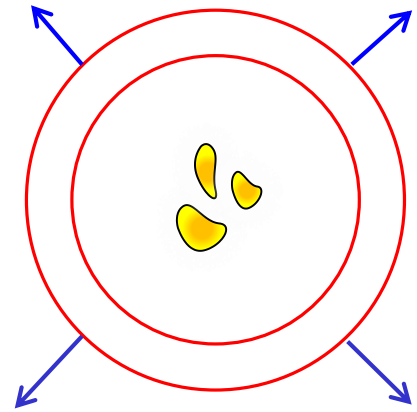
$$S_1(\theta = \pi) = -S_2(\theta = \pi) = \frac{1}{2}\sum_{n=1}^{\infty}(2n+1)[a_n - b_n]$$

Scattering Cross section

$$\vec{\mathbf{S}}_{\text{inc}} = \frac{1}{2}\text{Re}\{\vec{\mathbf{E}}^*{}_{\text{inc}} \times \vec{\mathbf{H}}_{\text{inc}}\} = \frac{1}{2}\sqrt{\frac{\varepsilon_b}{\mu_b}\frac{\epsilon_0}{\mu_0}}\,\|\vec{\mathbf{E}}_{\text{inc}}\|^2\,\widehat{\mathbf{k}}_{\text{inc}}$$

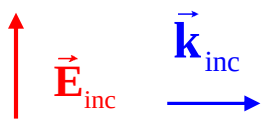


$$\lim_{r\to\infty} \vec{\mathbf{S}}_s = \lim_{r\to\infty}\frac{1}{2}\text{Re}\{\vec{\mathbf{E}}^*{}_{\text{scat}} \times \vec{\mathbf{H}}_{\text{scat}}\} \propto \lim_{r\to\infty}\frac{1}{2}\sqrt{\frac{\varepsilon_b}{\mu_b}\frac{\epsilon_0}{\mu_0}}\,\|\vec{\mathbf{E}}_{\text{scat}}\|^2\,\hat{\mathbf{r}}$$



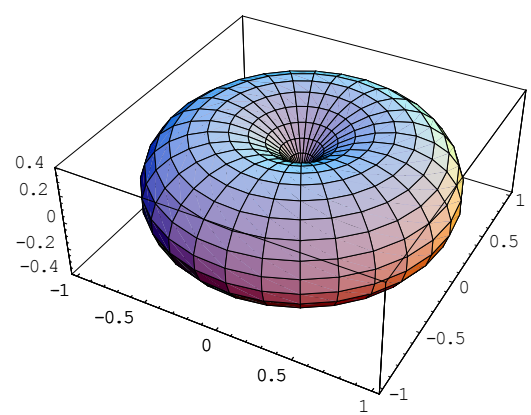
Differential scattering cross section

$$\frac{d\sigma_s}{d\Omega}(\theta,\phi) \equiv \lim_{r\to\infty} r^2 \frac{\vec{\mathbf{S}}_s \cdot \hat{\mathbf{r}}}{\vec{\mathbf{S}}_{\text{inc}} \cdot \widehat{\mathbf{k}}_{\text{inc}}} = \lim_{r\to\infty} r^2 \frac{\|\vec{\mathbf{E}}_{\text{scat}}\|^2}{\|\vec{\mathbf{E}}_{\text{inc}}\|^2}$$

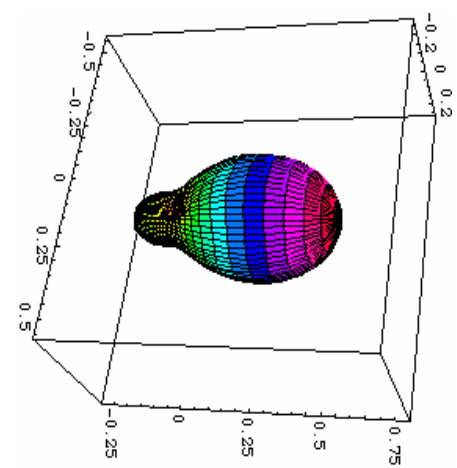


$$\sigma_s \equiv \int_0^{4\pi} d\Omega \frac{d\sigma_s}{d\Omega}(\theta,\phi)$$

Dipole scattering :



Multipole scattering :

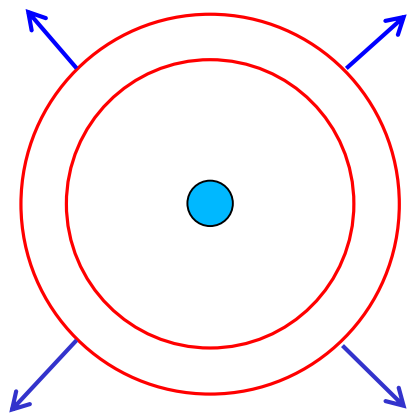


Absorption cross section

Absorption cross section : $\sigma_a = \sigma_e - \sigma_s$

$$\vec{\mathbf{S}}_{\text{tot}} = \frac{1}{2}\text{Re}\left(\vec{\mathbf{E}}^*_{\text{tot}} \times \vec{\mathbf{H}}_{\text{tot}}\right)$$

$$\vec{\mathbf{E}}_{\text{tot}} = \vec{\mathbf{E}}_{\text{inc}} + \vec{\mathbf{E}}_{\text{scat}}$$



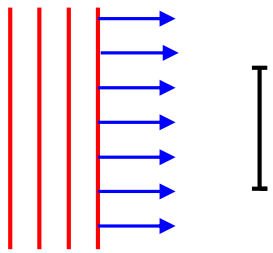
$$\sigma_a \equiv -\frac{r^2 \int \vec{\mathbf{S}}_{\text{tot}} \cdot \hat{\boldsymbol{r}}\, d\Omega}{\vec{\mathbf{S}}_{\text{inc}} \cdot \hat{\boldsymbol{k}}_{\text{inc}}}$$

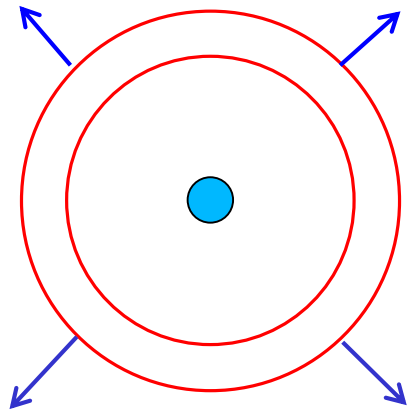
$$\vec{\mathbf{S}}_{\text{tot}} = \frac{1}{2}\text{Re}\left\{\vec{\mathbf{E}}^*_{\text{tot}} \times \vec{\mathbf{H}}_{\text{tot}}\right\} = \underbrace{\frac{1}{2}\text{Re}\left\{\vec{\mathbf{E}}^*_{\text{inc}} \times \vec{\mathbf{H}}_{\text{inc}}\right\}}_{\vec{\mathbf{S}}_{\text{inc}}} + \underbrace{\frac{1}{2}\text{Re}\left\{\vec{\mathbf{E}}^*_{\text{inc}} \times \vec{\mathbf{H}}_{\text{scat}} + \vec{\mathbf{E}}^*_{\text{scat}} \times \vec{\mathbf{H}}_{\text{inc}}\right\}}_{\vec{\mathbf{S}}_{\text{exc}}} + \underbrace{\frac{1}{2}\text{Re}\left\{\vec{\mathbf{E}}^*_{\text{scat}} \times \vec{\mathbf{H}}_{\text{scat}}\right\}}_{\vec{\mathbf{S}}_{\text{scat}}}$$

Extinction cross section

$$\vec{\mathbf{S}}_{\text{ext}} = \frac{1}{2}\text{Re}\{\vec{\mathbf{E}}^*_{\text{inc}} \times \vec{\mathbf{H}}_{\text{scat}} + \vec{\mathbf{E}}^*_{\text{scat}} \times \vec{\mathbf{H}}_{\text{inc}}\} = \vec{\mathbf{S}}_{\text{tot}} - \vec{\mathbf{S}}_{\text{scat}} - \vec{\mathbf{S}}_{\text{inc}}$$

$$\vec{\mathbf{S}}_{\text{inc}} = \frac{1}{2}\text{Re}\left(\vec{\mathbf{E}}^*_{\text{inc}} \times \vec{\mathbf{H}}_{\text{inc}}\right)$$





$$\sigma_{\text{ext}} \equiv -\lim_{r\to\infty} \frac{r^2 \int \vec{\mathbf{S}}_{\text{exc}} \cdot \hat{\boldsymbol{r}} d\Omega}{\vec{\mathbf{S}}_{\text{inc}} \cdot \hat{\boldsymbol{k}}_{\text{inc}}}$$

“Authentic” optical theorem :

$$\sigma_{\text{ext}} = \frac{4\pi}{k^2}\text{Re}\left[S_{1,2}(0°)\right]$$

$$= \frac{2\pi}{k^2}\sum_{n=1}^{\infty}(2n+1)\text{Re}[a_n + b_n]$$

For a sphere :

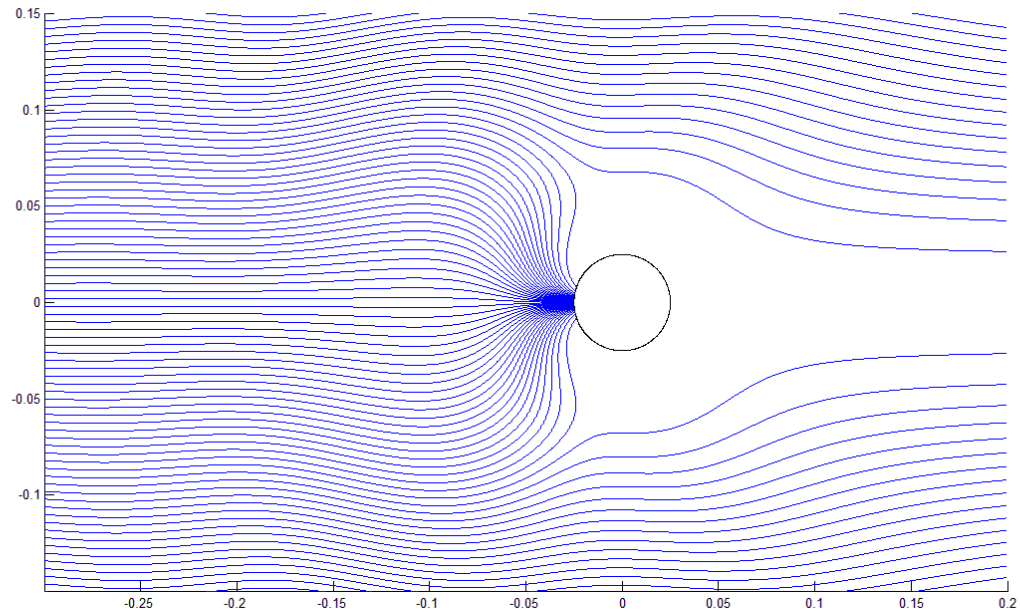
$$S_1(0°) = S_2(0°)$$

Extinction cross section

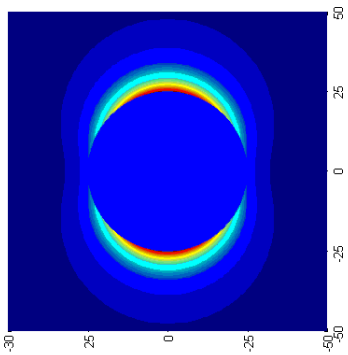
Extinction cross section :

$$\vec{\mathbf{S}}_{\mathrm{ext}} + \vec{\mathbf{S}}_{\mathrm{inc}} = \vec{\mathbf{S}}_{\mathrm{tot}} - \vec{\mathbf{S}}_{\mathrm{scat}}$$

σ_{ext}



$\|\vec{\mathbf{E}}\|^2$



Cross sections in Mie theory

Valid for any scatterer

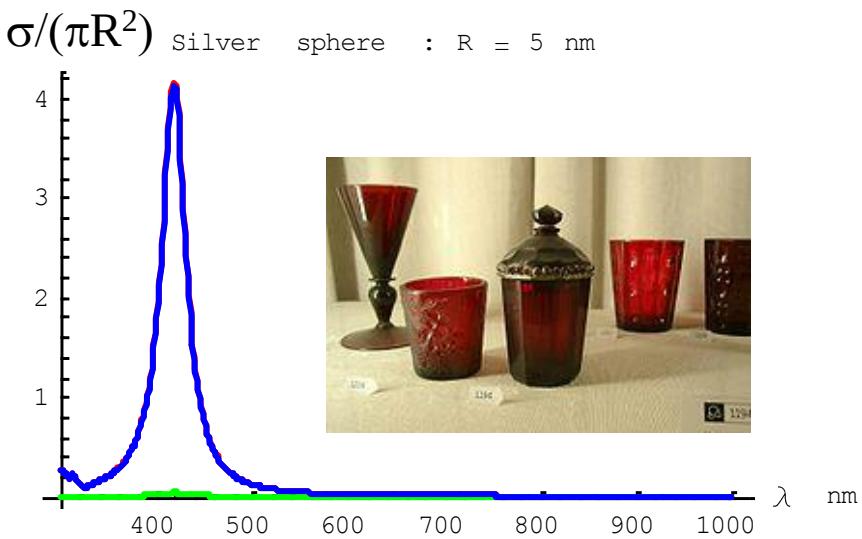
Extinction : $\sigma_e = -\frac{1}{k^2}\mathrm{Re}\left(f^\dagger.p\right) = -\frac{2\pi}{k^2}\sum_{n=1}^{\infty}(2n+1)\mathrm{Re}\left(t_n^{(e)} + t_n^{(h)}\right) = +\frac{2\pi}{k^2}\sum_{n=1}^{\infty}(2n+1)\mathrm{Re}\left(a_n + b_n\right)$

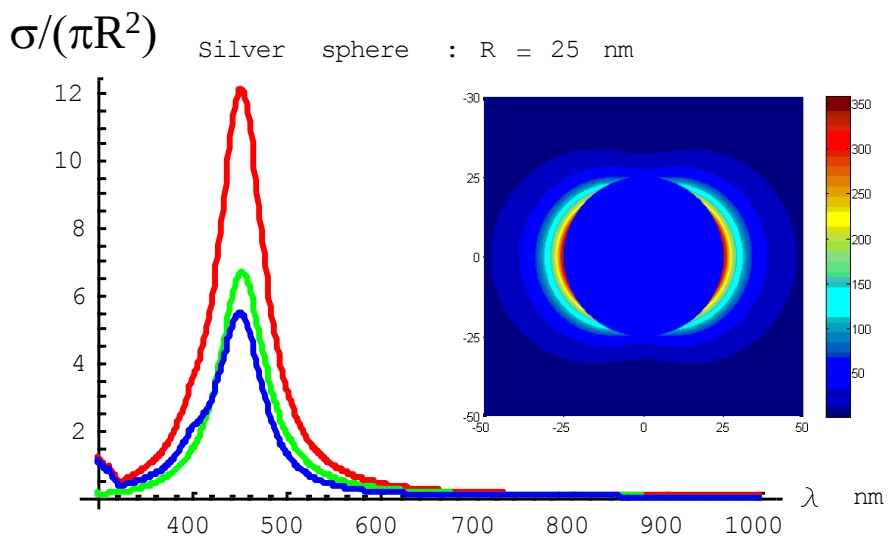
Scattering : $\sigma_s = \frac{1}{k^2} f^\dagger.f = \frac{2\pi}{k^2}\sum_{n=1}^{\infty}(2n+1)\left(\left|t_n^{(e)}\right|^2 + \left|t_n^{(h)}\right|^2\right) = \frac{2\pi}{k^2}\sum_{n=1}^{\infty}(2n+1)\left(\left|a_n\right|^2 + \left|b_n\right|^2\right)$

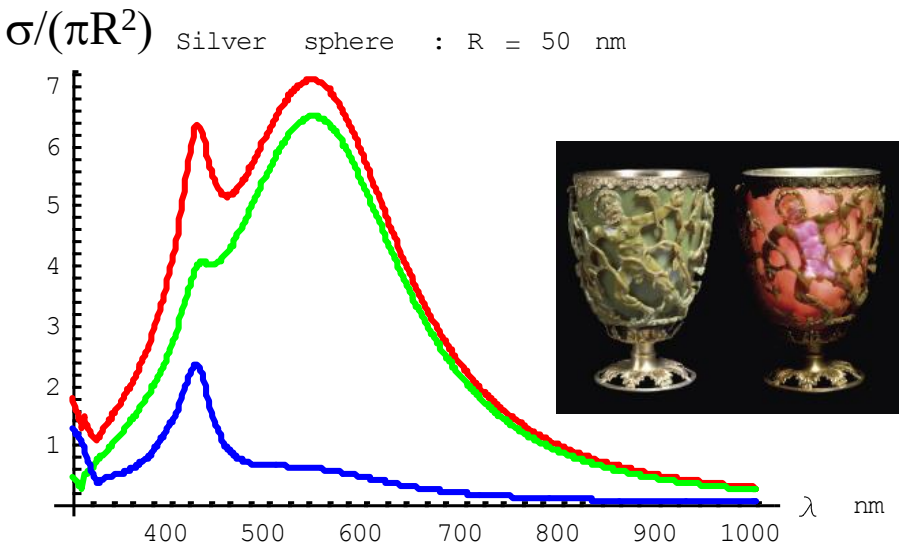
Absorption : $\sigma_a = \sigma_e - \sigma_s$

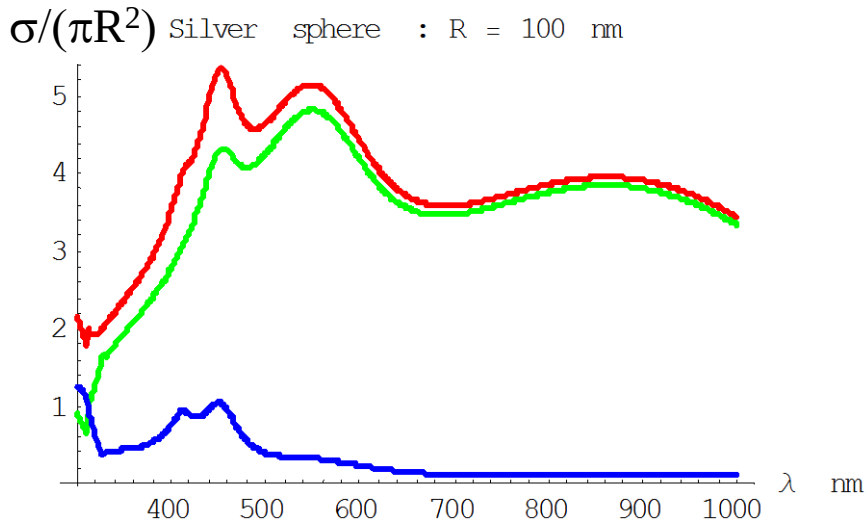
Radiation pressure cross section on a sphere : $\sigma_p = \sigma_e + \frac{4\pi}{k^2}\left[\sum_{n=1}^{\infty}\frac{n(n+2)}{n+1}\mathrm{Re}\left(t_n^{(e)}t_{n+1}^{(e),*} + t_n^{(h)}t_{n+1}^{(h),*}\right) + \frac{2n+1}{n(n+1)}\mathrm{Re}\left(t_n^{(e)}t_n^{(h),*}\right)\right]$

Ready for application !

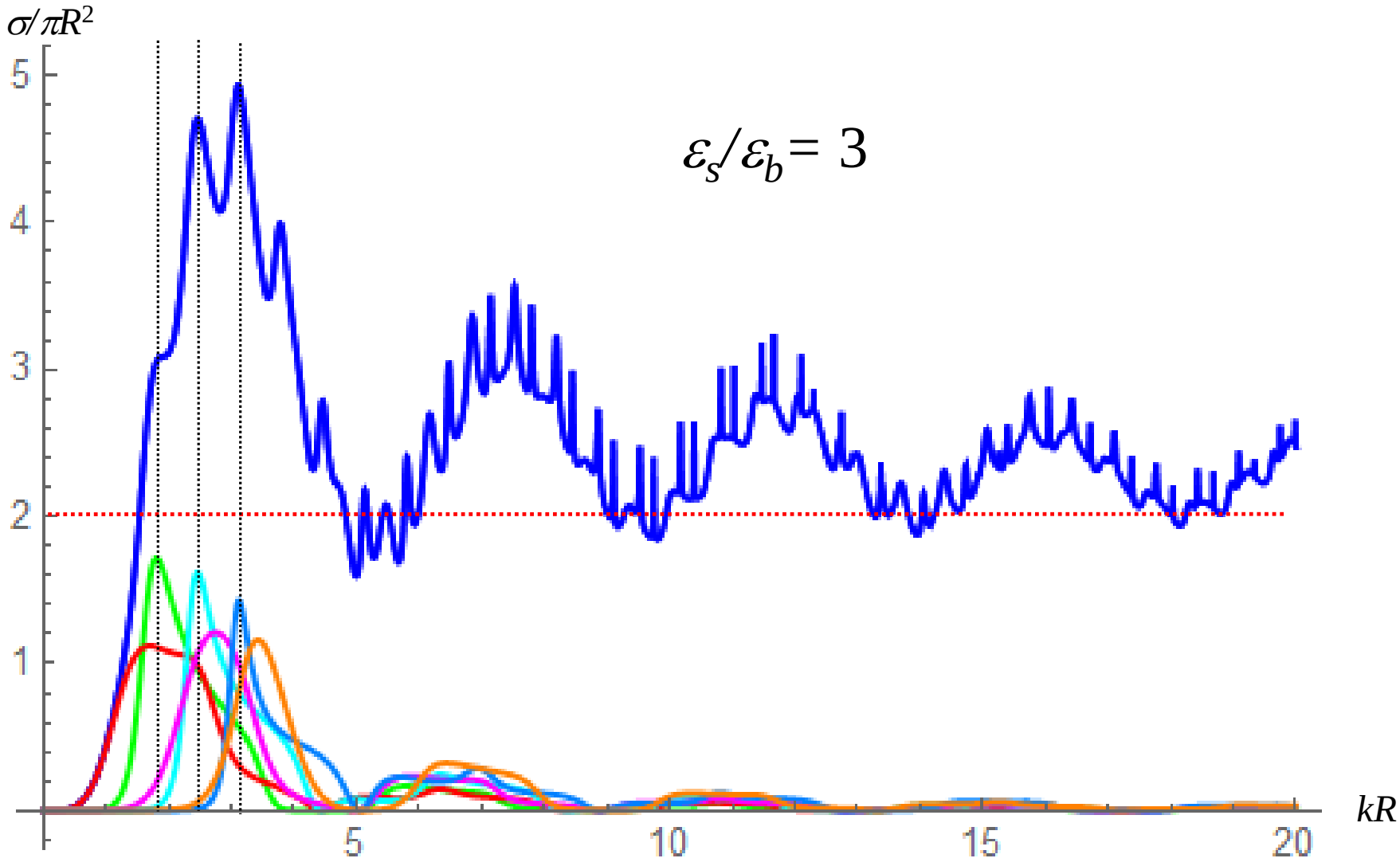




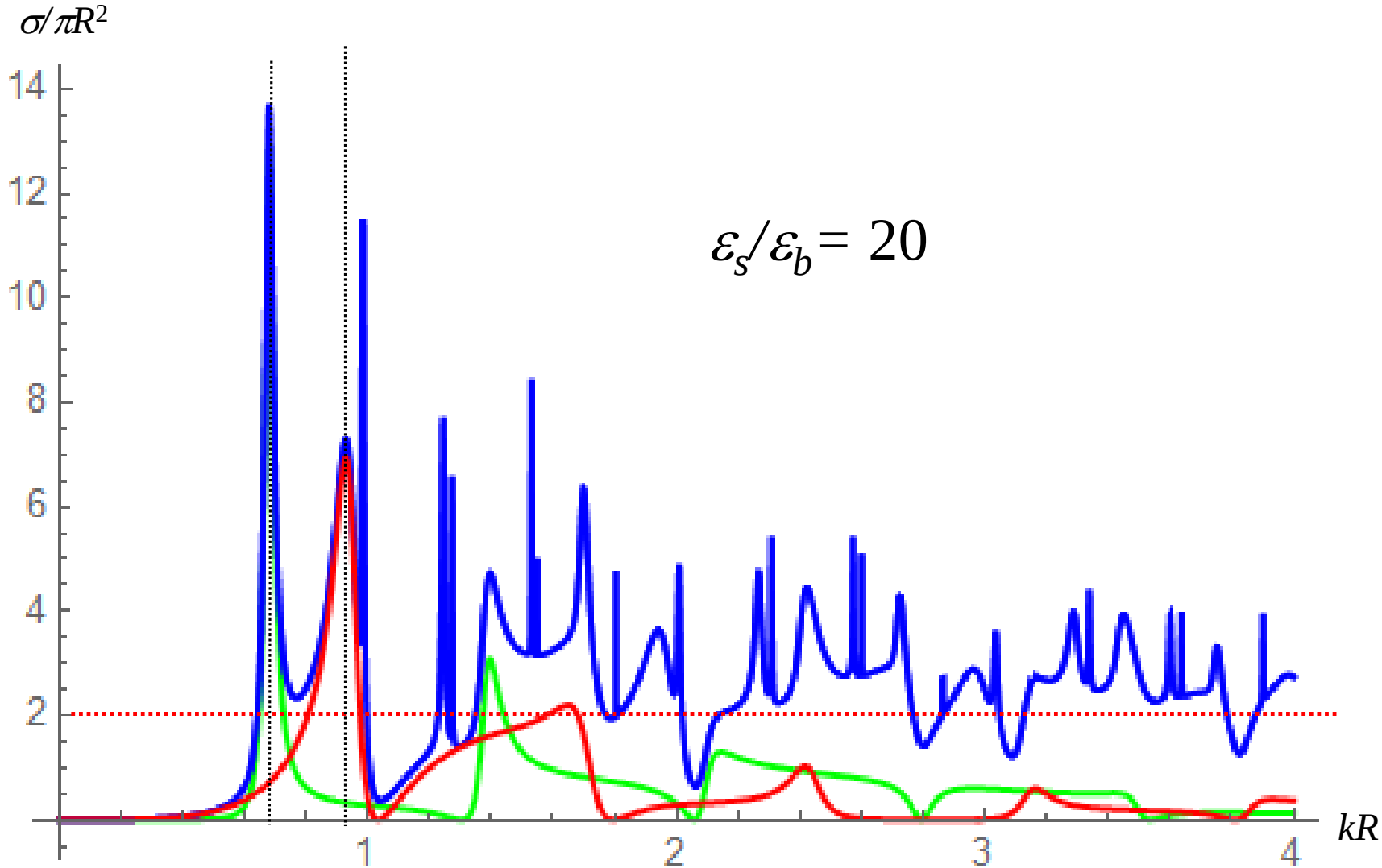




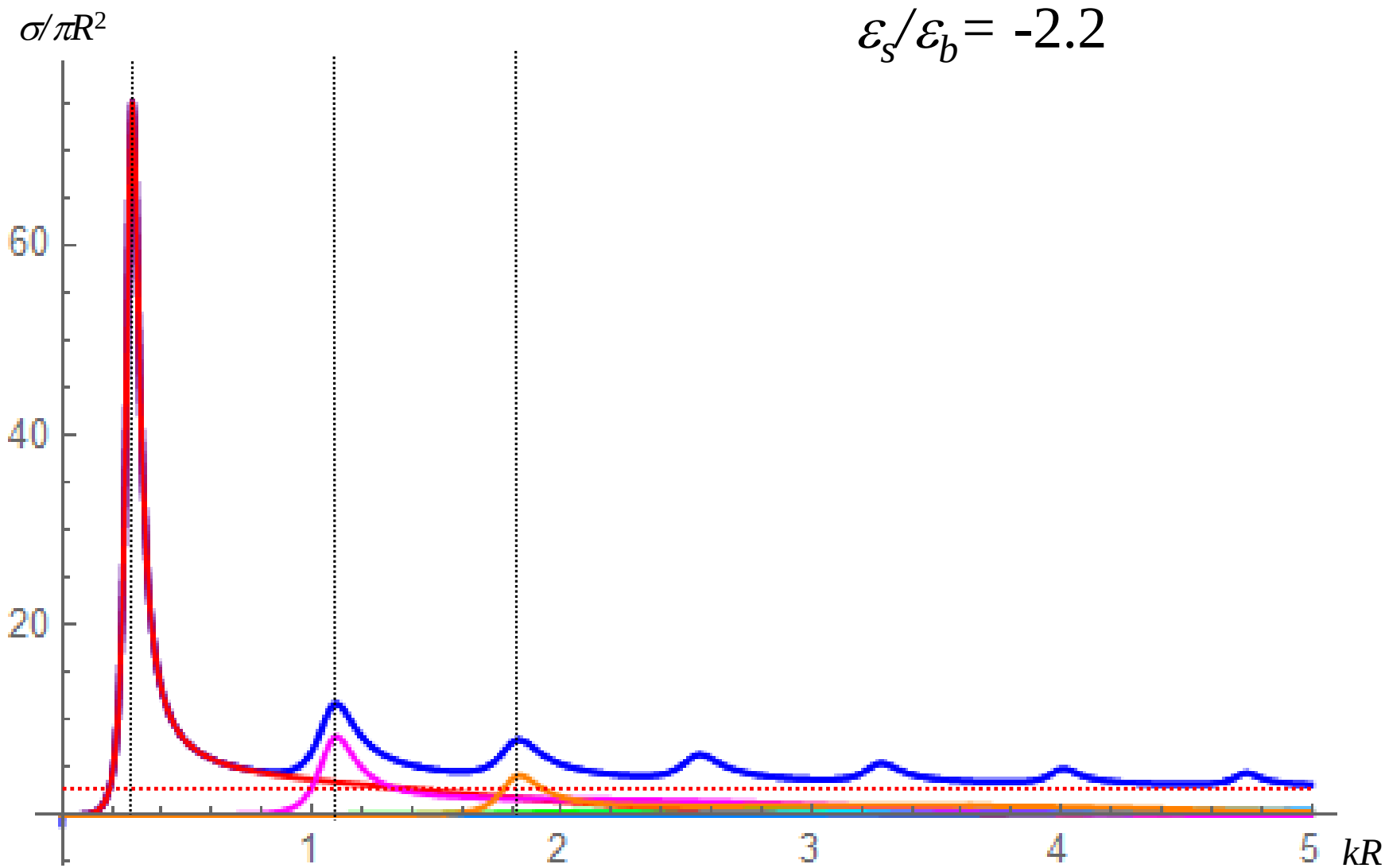
Low contrast dielectric physics : Ripples, Gallery modes and the Extinction paradox



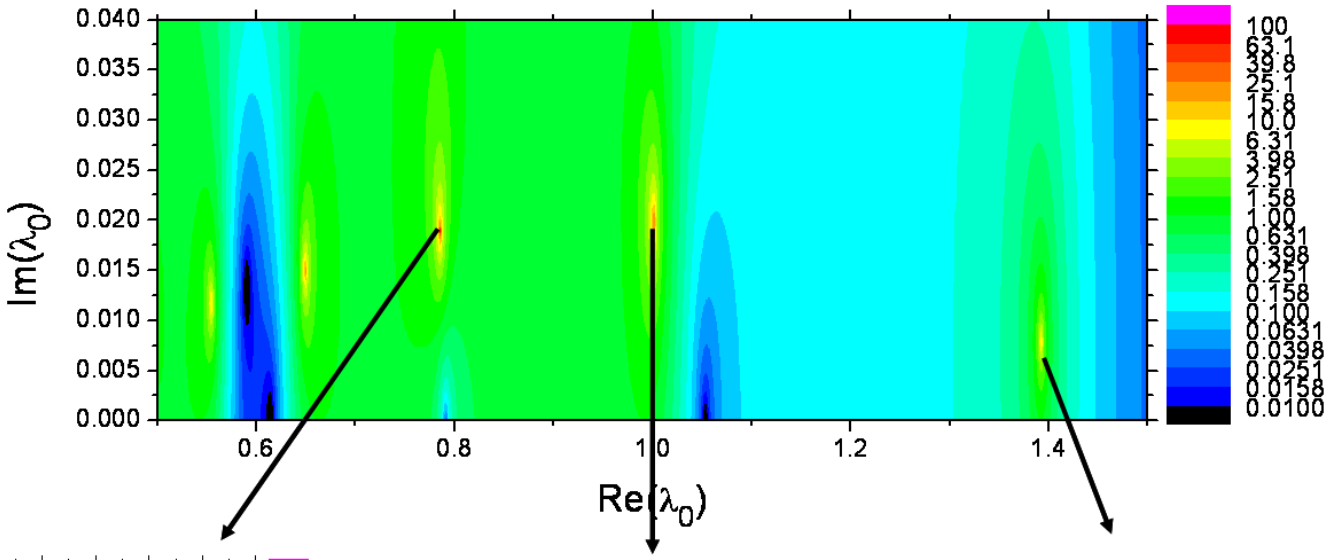
High contrast dielectric physics : Magnetic dipole modes

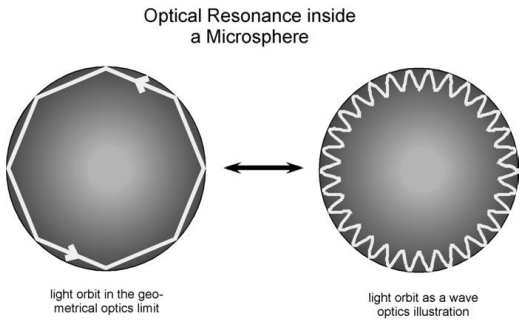


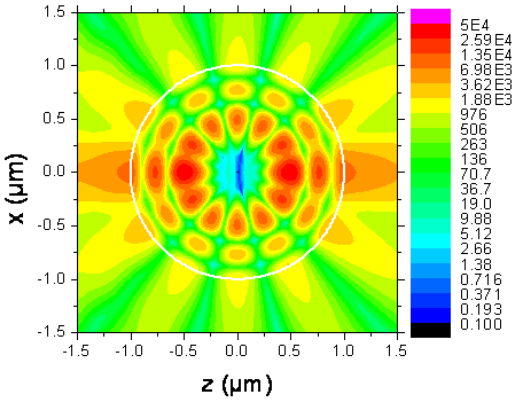
Plasmonic physics : Electric dipole modes

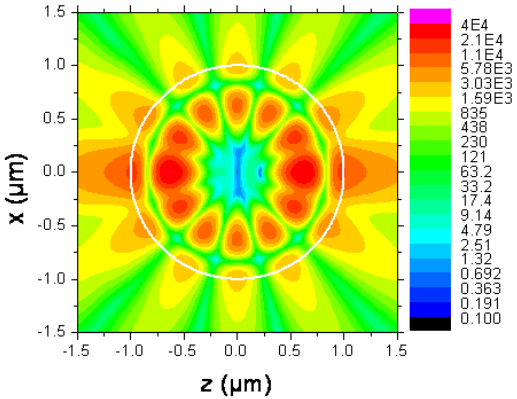


Gallery (*i.e.* surface) modes as “leaky” modes

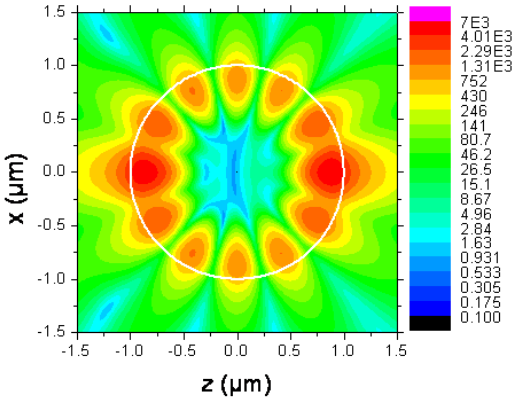






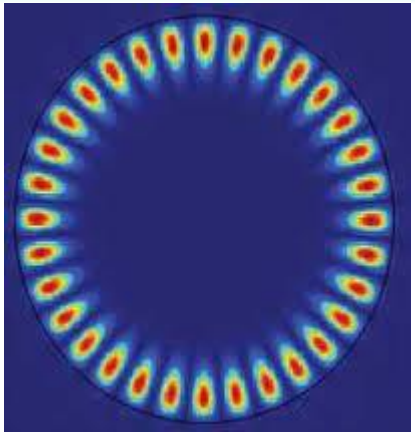


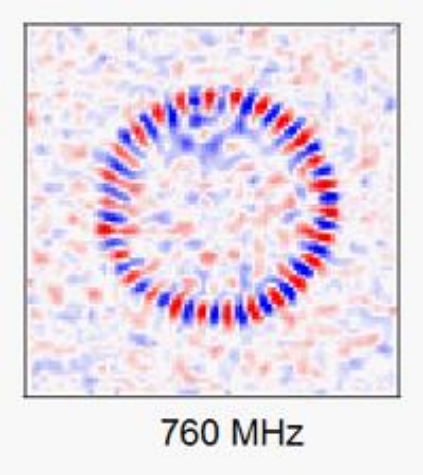
electric n=6

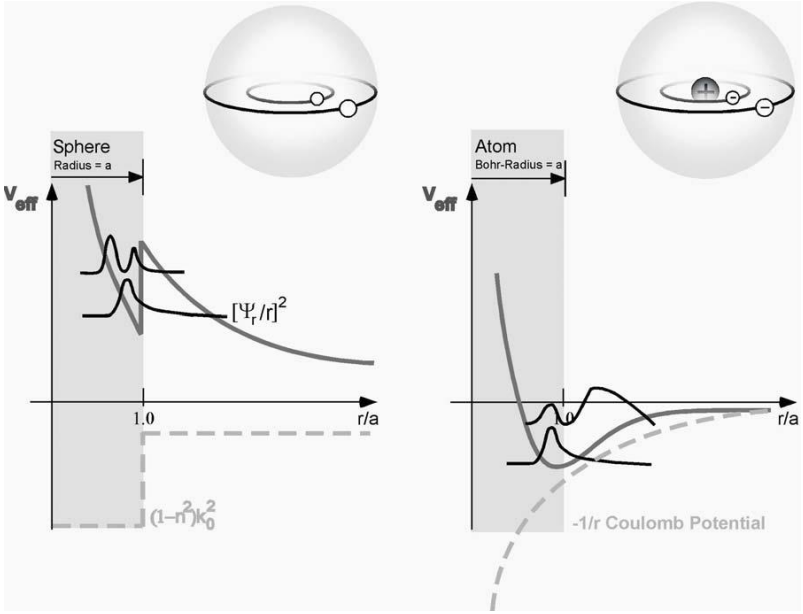


magnetic n=6

Gallery (*i.e.* surface) modes as “leaky” modes







What physics is hidden in the Mie coefficients ?

$$a_n = \frac{\rho_s \psi_n(k_s R)\psi_n'(kR) - (\mu_s/\mu)\psi_n(kR)\psi_n'(k_s R)}{\rho_s \psi_n(k_s R)\xi_n'(kR) - (\mu_s/\mu)\xi_n(kR)\psi_n'(k_s R)}$$

$$b_n = \frac{\rho_s \psi_n(kR)\psi_n'(k_s R) - (\mu_s/\mu)\psi_n(k_s R)\psi_n'(kR)}{\rho_s \xi_n(kR)\psi_n'(k_s R) - (\mu_s/\mu)\psi_n(k_s R)\xi_n'(kR)}$$

Difficult to extract physical understanding from exact solutions !

Mie coefficients are the analogue of the Fresnel reflection coefficients (spherical geometry)

Fresnel reflection coefficients

$$R_\perp = \frac{\frac{\mu_s}{\mu}\alpha(1,\gamma) - \alpha(\rho_s,\gamma)}{\frac{\mu_s}{\mu}\alpha(1,\gamma) + \alpha(\rho_s,\gamma)}$$

$$R_\parallel = \frac{\frac{\varepsilon_s}{\varepsilon}\alpha(1,\gamma) - \alpha(\rho_s,\gamma)}{\frac{\varepsilon_s}{\varepsilon}\alpha(1,\gamma) + \alpha(\rho_s,\gamma)}$$

$$\alpha(\rho,\gamma) \equiv \sqrt{\rho^2 - \gamma^2} \qquad \gamma \equiv \frac{k_\parallel}{k}$$

TE

$$t_n^{(h)} = -\frac{j_n(kR)}{h_n(kR)} \frac{\frac{\mu_s}{\mu}\varphi_n(kR) - \varphi_n(\rho_s kR)}{\frac{\mu_s}{\mu}\varphi_n^{(+)}(kR) - \varphi_n(\rho_s kR)}$$

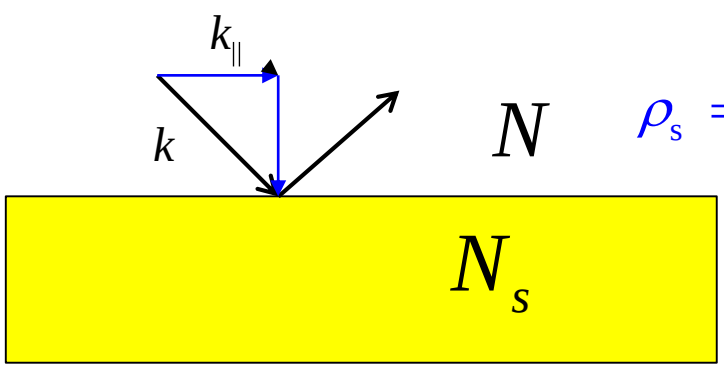
TM

$$t_n^{(e)} = -\frac{j_n(kR)}{h_n(kR)} \frac{\frac{\varepsilon_s}{\varepsilon}\varphi_n(kR) - \varphi_n(\rho_s kR)}{\frac{\varepsilon_s}{\varepsilon}\varphi_n^{(+)}(kR) - \varphi_n(\rho_s kR)}$$

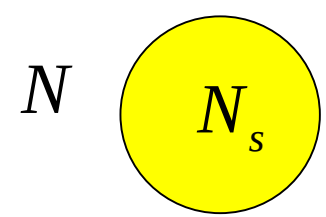
$$\varphi_n^{(1)}(x) \equiv \varphi_n(x) \equiv \frac{[x j_n(x)]'}{j_n(kR)}$$

$$\varphi_n^{(2)}(x) \equiv \frac{[x y_n(x)]'}{y_n(kR)}$$

$$\varphi_n^{(+)}(x) \equiv \frac{\left[x h_n^{(+)}(x)\right]'}{h_n^{(+)}(kR)}$$



$$\rho_s = \frac{k_s}{k} = \frac{N_s}{N}$$



Quasi-static limit : magnetic response ?

$$t_n^{(e)} = -\frac{j_n(kR)}{h_n^{(+)}(kR)} \frac{\frac{\varepsilon_s}{\varepsilon}\varphi_n(kR) - \varphi_n(k_s R)}{\frac{\varepsilon_s}{\varepsilon}\varphi_n^{(+)}(kR) - \varphi_n(k_s R)}$$

$$t_n^{(h)} = -\frac{j_n(kR)}{h_n^{(+)}(kR)} \frac{\frac{\mu_s}{\mu}\varphi_n(kR) - \varphi_n(k_s R)}{\frac{\mu_s}{\mu}\varphi_n^{(+)}(kR) - \varphi_n(k_s R)}$$

$$\lim_{kR\to 0} \quad t_n^{(h)} \to \frac{i(kR)^{2n+1}(n+1)}{(2n-1)!!(2n+1)!!} \frac{\mu_s - \mu_b}{(n+1)\mu_b + n\mu_s} \qquad \text{(magnetic)}$$

$$t_n^{(e)} \to \frac{i(kR)^{2n+1}(n+1)}{(2n-1)!!(2n+1)!!} \frac{\varepsilon_s - \varepsilon_b}{(n+1)\varepsilon_b + n\varepsilon_s} \qquad \text{(electric)}$$

Quasi-static limit – electric dipole

$$\lim_{x\to 0} j_n(x) \to \frac{x^n}{(2n+1)!!}$$

$$\lim_{x\to 0} \varphi_n^{(+)}(x) \to -n$$

$$x = kR$$

$$\lim_{x\to 0} h_n(x) \to \frac{(2n-1)!!}{ix^{n+1}}$$

$$\lim_{x\to 0} \varphi_n(x) \to n+1$$

$$\lim_{x\to 0} t_n^{(e)}(x) = \lim_{x\to 0} -\frac{j_n(x)}{h_n^{(+)}(x)} \frac{\frac{\varepsilon_s}{\varepsilon}\varphi_n(x) - \varphi_n(\rho_s x)}{\frac{\varepsilon_s}{\varepsilon}\varphi_n^{(+)}(x) - \varphi_n(\rho_s x)} \to \frac{ix^{2n+1}(n+1)}{(2n-1)!!\,(2n+1)!!} \frac{\frac{\varepsilon_s}{\varepsilon} - 1}{\frac{\varepsilon_s}{\varepsilon} n + (n+1)}$$

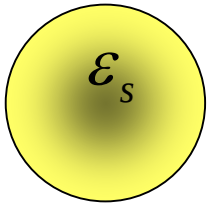
Electric dipole term (n=1) dominates for small particle sizes :

$$\boxed{\lim_{kR\to 0} t_n^{(e)}(kR) \to ikR^3 \frac{2}{3} \frac{\varepsilon_s - \varepsilon}{\varepsilon_s + 2\varepsilon}}$$

Polarizability approach (plasmonics)

Electric dipole moment and polarizability, $\alpha(\omega)$

$$\vec{\mathbf{p}} = \varepsilon_0 \varepsilon \alpha(\omega) \vec{\mathbf{E}}_{\text{exc}}$$



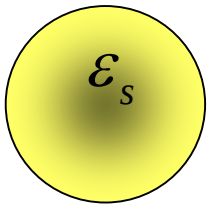
$\alpha(\omega) = 6\pi t_1^{(e)} / ik^3$ Electric dipole polarizability can be deduced from Mie theory !

$$\alpha_0(\omega) \equiv \lim_{kR \to 0} \alpha(\omega) = 4\pi R^3 \frac{\varepsilon_s - \varepsilon}{\varepsilon_s + 2\varepsilon}$$

Polarizability approach

Electric dipole moment and polarizability, $\alpha(\omega)$

$$\vec{\mathbf{p}} = \varepsilon_0 \varepsilon \alpha(\omega) \vec{\mathbf{E}}_{\text{exc}}$$



$$\alpha(\omega) = 6\pi t_1^{(e)} / ik^3$$

$$\alpha_0 = \lim_{kR \to 0} \alpha(\omega) = 4\pi R^3 \frac{\varepsilon_s - \varepsilon_b}{2\varepsilon_b + \varepsilon_s}$$

Infinite ? dielectric response
When $\varepsilon_s = -2\varepsilon$

No !

Unitarity imposes :

$$\alpha(\omega) = \frac{A(\omega)}{1 - i\frac{k^3}{6\pi} A(\omega)} \Rightarrow \lim_{\omega \to 0} \alpha(\omega) = \frac{\alpha_0}{1 - i\frac{k^3}{6\pi}\alpha_0} \qquad |\alpha(\omega)| \leq \frac{6\pi}{k^3}$$

Unitary limit on cross sections

$$\sigma_{\text{ext}} = \frac{2\pi}{k^2}\sum_{n=1}^{\infty}(2n+1)\text{Re}\{a_n + b_n\} \xrightarrow{\text{Unitarity}} \leq \frac{\lambda^2}{2\pi}\sum_{n=1}^{\infty}(2n+1)2$$

For a dipole resonance : $\sigma_d \leq \frac{3\lambda^2}{2\pi}$

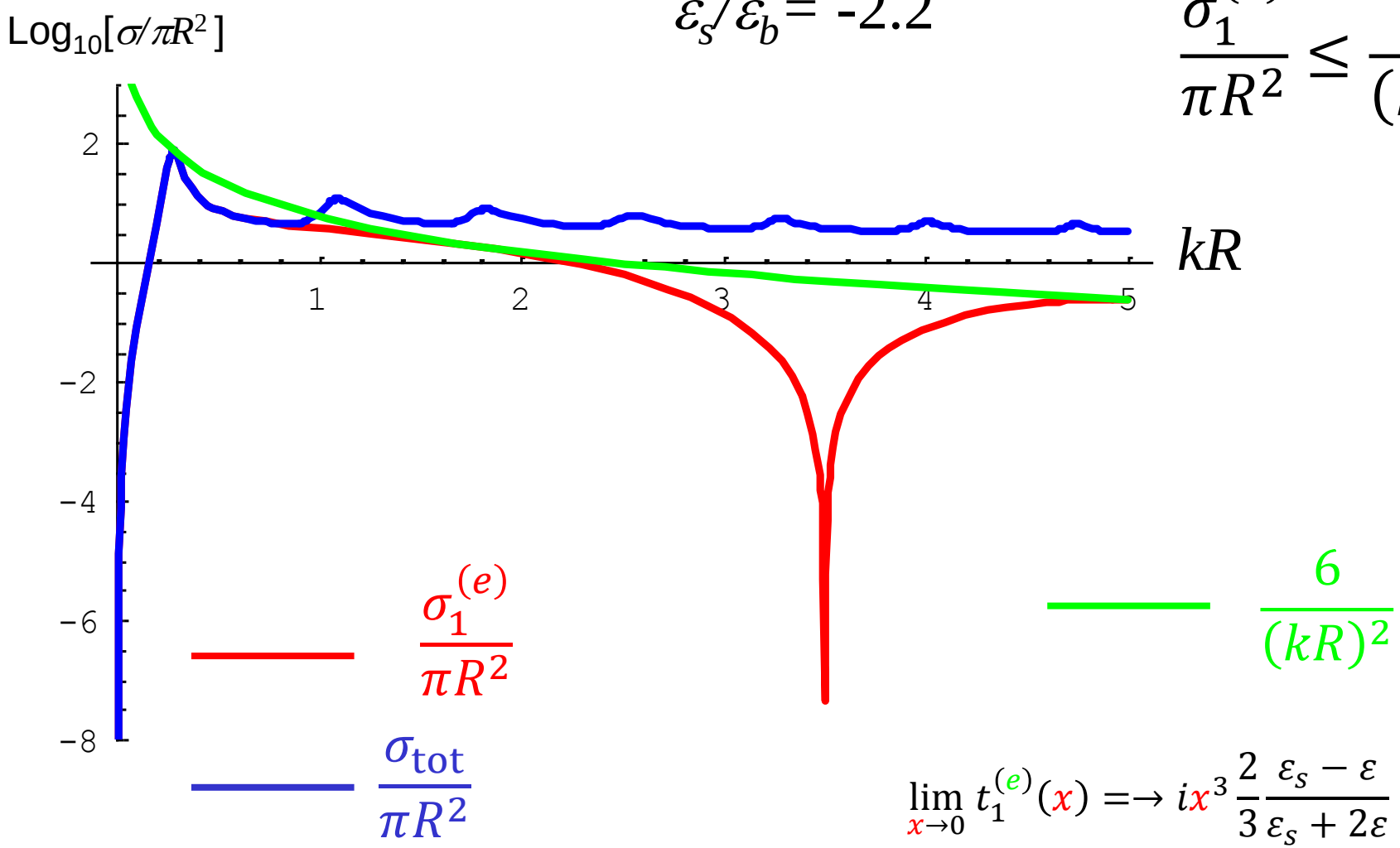
Flux conservation : cross section limit doesn't depend on the properties of the resonator !

$$\frac{\sigma_d^{(e)}}{\pi R^2} \leq \frac{6}{(kR)^2}$$

Plasmonic physics : Electric dipole modes

$\varepsilon_s/\varepsilon_b = -2.2$

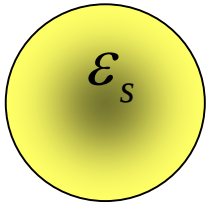
$$\frac{\sigma_1^{(e)}}{\pi R^2} \leq \frac{6}{(kR)^2}$$



$$\lim_{x\to 0} t_1^{(e)}(x) =\to i x^3 \frac{2}{3}\frac{\varepsilon_s - \varepsilon}{\varepsilon_s + 2\varepsilon}$$

Polarizability picture for electric dipoles

$$k = \frac{\omega}{c}\sqrt{\varepsilon}$$



$$\vec{\mathbf{p}} = \varepsilon_0 \varepsilon \alpha(\omega) \vec{\mathbf{E}}_{\text{exc}}$$

$$\lim_{\omega \to 0} \alpha(\omega) = 4\pi R^3 \frac{\varepsilon_s - \varepsilon}{2\varepsilon + \varepsilon_s}$$

Energy conservation :

$$\mathrm{Im}\{\alpha(\omega)\} = \frac{k^3}{6\pi} |\alpha(\omega)|^2$$

Unitary limit :

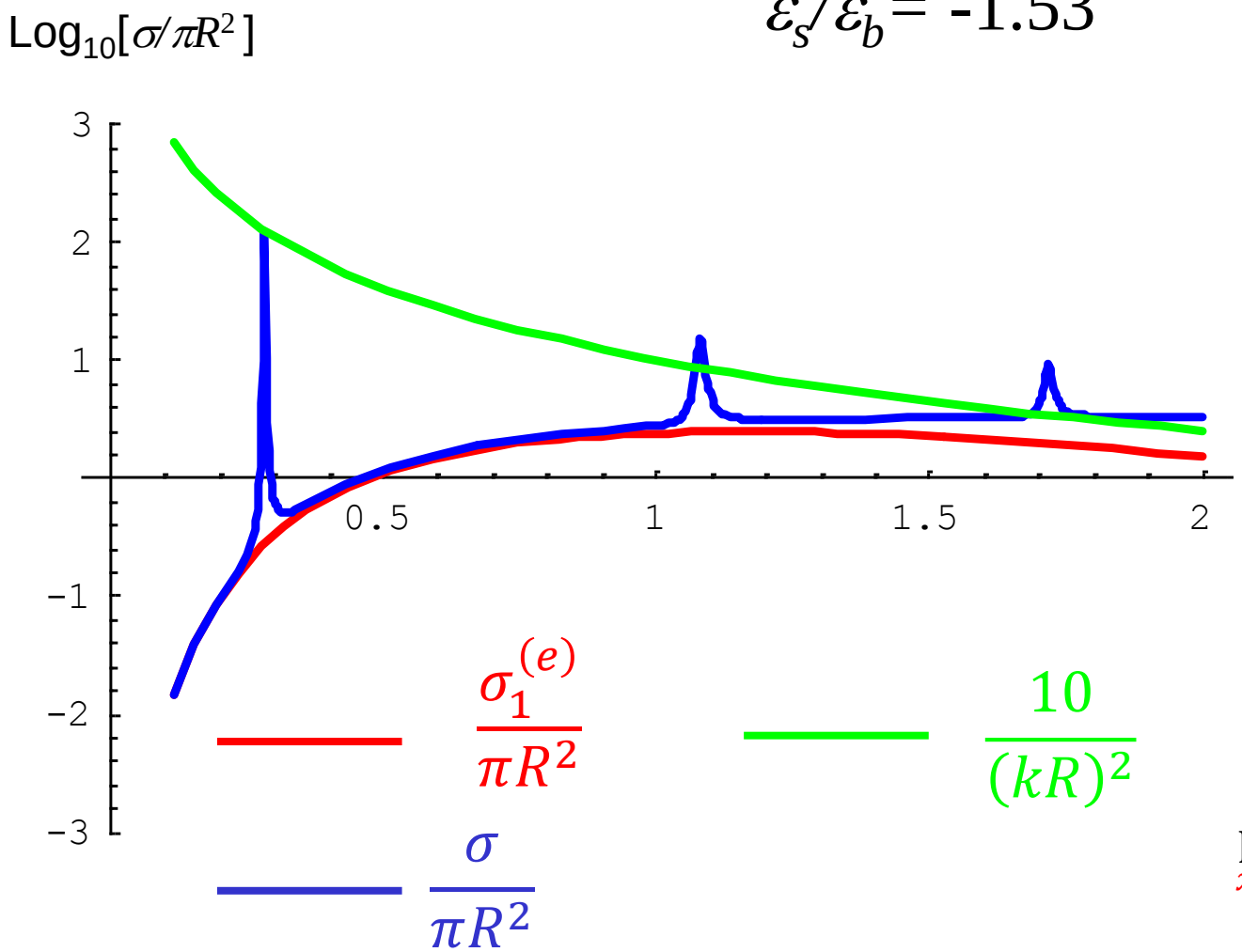
$$|\alpha(\omega)| \leq \frac{6\pi}{k^3}$$

$$\alpha(\omega) \xrightarrow{\text{u.l.}} \frac{6\pi}{ik^3}$$

Plasmonic physics : Electric quadrupole modes ?

$\varepsilon_s/\varepsilon_b = -1.53$

$$\frac{\sigma_2^{(e)}}{\pi R^2} \leq \frac{10}{(kR)^2}$$



$$\lim_{x\to 0} t_2^{(e)}(x) = \to \frac{ix^5}{15}\frac{\varepsilon_s - \varepsilon}{2\varepsilon_s + 3\varepsilon}$$