Thesis subject

Laboratory : Fresnel Institute

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Title of the thesis subject :

Extrinsic statistics for complex covariance matrices and partially polarized light.

Description of the thesis subject :

The polarization state of a partially polarized light can be represented by a 2×2 complex Hermitian positive definite matrix Σ, called the polarization matrix, representing the covariance of the electric field. A linear interaction between the light and a material can be represented by a 2×2 complex Jones matrix J. Its action $J.\Sigma$ on the polarization state is given by

$$\Sigma \mapsto J\Sigma J^*$$

It is also common to consider the covariance up to the energy by considering a reduced parameter which we will call Σ_R: $\Sigma_R = \frac{\Sigma}{Trace(\Sigma)}$. By renormalizing afterwards, the action of Jones matrices on covariance matrices can be turned into an action on reduced parameters $J.\Sigma_R$.

Most standard statistical tools are defined for unconstrained data in $\mathbb{R}^n$, and commute with Euclidean transformations (translations and rotations). When considering the mean of polarization matrices, we are more interested in the commutation with the action of Jones matrices than in the commutation with Euclidean transformations. However, it can be checked that for most Σ_R, Σ'_R and J, the action of J on the arithmetic mean of Σ_R and Σ'_R does not coincide with the arithmetic mean of $J.\Sigma_R$ and $J.\Sigma'_R$, see Fig.1. The goal of this thesis is to propose and study statistical tools that are adapted to the particular nature of polarization matrices and their reduced versions. For instance, a mean compatible with the action of Jones matrices should at least verify that

$$\text{Mean}(J.X) = J.\text{Mean}(X),$$

where X is the random matrix Σ or Σ_R. Typical tools which could be studied in this PhD include the notion of mean, of spread around the mean, and the notion of Gaussian distributions.
In order to construct these statistical tools, several approaches can be considered. One of them is to start from a Riemannian distance function invariant under the action of Jones matrices. In the last two decades, much effort has been devoted to developing statistics on Riemannian manifolds, see for instance [7]. It has been considered for polarization matrices in [3]. The drawbacks of this approach are, firstly, that it requires the introduction of the formalism of Riemannian manifolds, and secondly, that the statistical tools do not allow for explicit expressions. When considering statistics on a Riemannian manifolds M, a way of avoiding the computational complexity of intrinsic tools is to search for an isometric embedding $\varphi : M \to \mathbb{R}^n$. Given this embedding, the statistical analysis can be performed efficiently in $\mathbb{R}^n$ and projected back to the manifold. Though, this approach fails for covariances matrices, due to the lack of convenient isometric embedding.

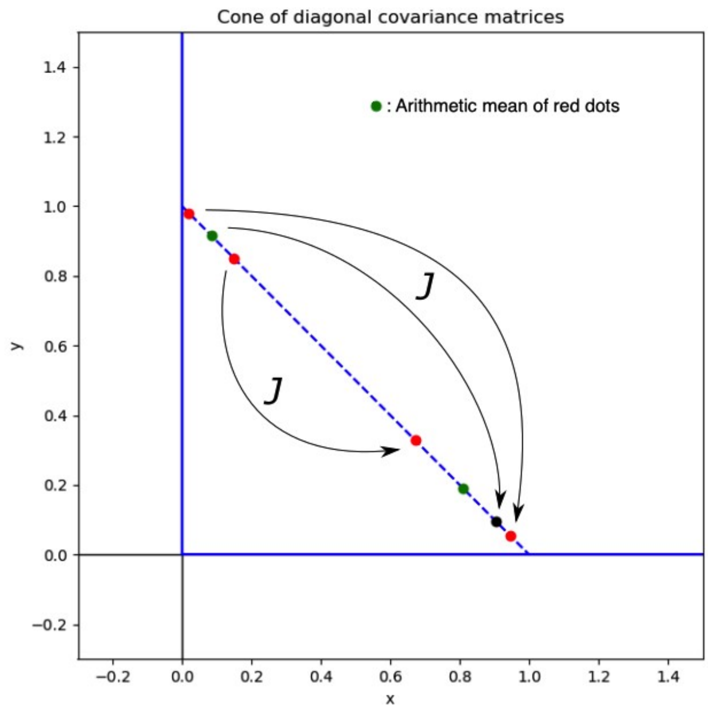

Figure 1 – This plot depicts the cone of 2 × 2 diagonal covariance matrices. The dot line represents reduced parameters, corresponding to the segment $[S_1 = -1, S_1 = 1]$ in the Poincaré sphere. The red points on the upper part of the plot, as well as their arithmetic mean in green, are transformed using the action of the Jones matrix

$$J = \begin{pmatrix} 10 & 0 \\ 0 & 1/10 \end{pmatrix}$$

on reduced parameters. The transported arithmetic mean, shown in black, and new arithmetic mean, shown in green, do not coincide.

An interesting alternative is to consider embeddings in $\mathbb{R}^n$ which are not isometric, but only such that the isometry group G of the space M embeds in the linear group $GL(n)$. In that case, the operations based on the vector structure of the embedding lead to quantities which are guaranteed to commute with the action of G.
In this PhD, the candidate will study such embeddings for polarization matrices and their reduced versions, and more generally for $n \times n$ complex covariance matrices. A special attention will also be given to block-Toeplitz covariance matrices arising in radar signal processing, see [1, 5]. Based on the existing literature, see for instance [2, 4, 6], the candidate will further study the theoretical properties of existing statistical tools, propose new ones, and apply them to problems arising from random electromagnetic waves.

References :

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