Opérateurs Différentiels

Coordonnées sphériques

$$d\boldsymbol{l} = dr\,\hat{\mathbf{r}} + rd\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta d\varphi\,\hat{\boldsymbol{\varphi}} \equiv AdX\,\hat{\mathbf{X}} + BdY\,\hat{\mathbf{Y}} + CdZ\,\hat{\mathbf{Z}}$$

$$A = 1,\ B = r,\ C = r\sin\theta \qquad dX = dr,\ dY = d\theta,\ dZ = d\varphi \qquad \hat{\mathbf{X}} = \hat{\mathbf{r}},\ \hat{\mathbf{Y}} = \hat{\boldsymbol{\theta}},\ \hat{\mathbf{Z}} = \hat{\boldsymbol{\varphi}}$$

$$d\boldsymbol{S} = BCdYdZ\,\hat{\mathbf{X}} + ACdXdZ\,\hat{\mathbf{Y}} + ABdXdY\,\hat{\mathbf{Z}} = r^2\sin\theta d\theta d\varphi\,\hat{\mathbf{r}} + r\sin\theta dr d\varphi\,\hat{\boldsymbol{\theta}} + r dr d\theta\,\hat{\boldsymbol{\varphi}}$$

$$dV = ABCdXdYdZ = r^2\sin\theta dr d\theta d\varphi$$

$$\overrightarrow{grad}\ f = \frac{\hat{\mathbf{X}}}{A}\frac{\partial f}{\partial X} + \frac{\hat{\mathbf{Y}}}{B}\frac{\partial f}{\partial Y} + \frac{\hat{\mathbf{Z}}}{C}\frac{\partial f}{\partial Z} = \hat{\mathbf{r}}\frac{\partial f}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial f}{\partial \theta} + \frac{\hat{\boldsymbol{\varphi}}}{r\sin\theta}\frac{\partial f}{\partial \varphi}$$

$$\operatorname{div}\mathbf{F} = \frac{1}{ABC}\left[\frac{\partial}{\partial X}(BCF_X) + \frac{\partial}{\partial Y}(ACF_Y) + \frac{\partial}{\partial Z}(ABF_Z)\right]$$
$$= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta F_r) + \frac{\partial}{\partial \theta}(r\sin\theta F_\theta) + \frac{\partial}{\partial \varphi}(rF_\varphi)\right]$$
$$= \left[\frac{2}{r}F_r + \frac{\partial F_r}{\partial r} + \frac{\cot\theta}{r}F_\theta + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial \varphi}\right]$$

$$\overrightarrow{\mathrm{rot}}\,\mathbf{F} = \frac{1}{ABC}\begin{vmatrix} A\hat{\mathbf{X}} & B\hat{\mathbf{Y}} & C\hat{\mathbf{Z}} \\ \frac{\partial}{\partial X} & \frac{\partial}{\partial Y} & \frac{\partial}{\partial Z} \\ AF_X & BF_Y & CF_Z \end{vmatrix} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\varphi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ F_r & rF_\theta & r\sin\theta F_\varphi \end{vmatrix}$$

$$\Delta f \equiv \nabla^2 f = \frac{1}{ABC}\left[\frac{\partial}{\partial X}\left(\frac{BC}{A}\frac{\partial f}{\partial X}\right) + \frac{\partial}{\partial Y}\left(\frac{AC}{B}\frac{\partial f}{\partial Y}\right) + \frac{\partial}{\partial Z}\left(\frac{AB}{C}\frac{\partial f}{\partial Z}\right)\right]$$
$$= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{\partial}{\partial \varphi}\left(\frac{1}{\sin\theta}\frac{\partial f}{\partial \varphi}\right)\right]$$
$$= \left[\frac{2}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2} + \frac{\cot\theta}{r^2}\frac{\partial f}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2}\right]$$