Opérateurs Différentiels

Coordonnées cylindriques

$$d\mathbf{l} = d\rho\hat{\boldsymbol{\rho}} + \rho d\varphi\hat{\boldsymbol{\varphi}} + dz\hat{\mathbf{z}} \equiv AdX\,\widehat{\mathbf{X}} + BdY\,\widehat{\mathbf{Y}} + CdZ\,\widehat{\mathbf{Z}}$$

$$A = 1,\ B = \rho,\ C = 1 \qquad dX = d\rho,\ dY = d\varphi,\ dZ = dz \qquad \widehat{\mathbf{X}} = \hat{\boldsymbol{\rho}},\ \widehat{\mathbf{Y}} = \hat{\boldsymbol{\varphi}},\ \widehat{\mathbf{Z}} = \hat{\mathbf{z}}$$

$$d\mathbf{S} = BCdYdZ\,\widehat{\mathbf{X}} + ACdXdZ\,\widehat{\mathbf{Y}} + ABdXdY\,\widehat{\mathbf{Z}} = \rho d\varphi dz\hat{\boldsymbol{\rho}} + d\rho dz\hat{\boldsymbol{\varphi}} + \rho d\rho d\varphi\hat{\mathbf{z}}$$

$$dV = ABCdXdYdZ = \rho d\rho d\varphi dz$$

$$\overrightarrow{grad}\ f = \frac{\widehat{\mathbf{X}}}{A}\frac{\partial f}{\partial X} + \frac{\widehat{\mathbf{Y}}}{B}\frac{\partial f}{\partial Y} + \frac{\widehat{\mathbf{Z}}}{C}\frac{\partial f}{\partial Z}$$
$$= \hat{\boldsymbol{\rho}}\frac{\partial f}{\partial \rho} + \frac{\hat{\boldsymbol{\varphi}}}{\rho}\frac{\partial f}{\partial \varphi} + \hat{\mathbf{z}}\frac{\partial f}{\partial z}$$

$$\operatorname{div}\mathbf{F} = \frac{1}{ABC}\left[\frac{\partial}{\partial X}\left(BCF_X\right) + \frac{\partial}{\partial Y}\left(ACF_Y\right) + \frac{\partial}{\partial Z}\left(ABF_Z\right)\right]$$
$$= \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(\rho F_\rho\right) + \frac{\partial}{\partial \varphi}\left(F_\varphi\right) + \frac{\partial}{\partial z}\left(\rho F_z\right)\right]$$
$$= \frac{1}{\rho}\frac{\partial\left(\rho F_\rho\right)}{\partial \rho} + \frac{1}{\rho}\frac{\partial F_\varphi}{\partial \varphi} + \frac{\partial F_z}{\partial z}$$

$$\overrightarrow{\mathrm{rot}}\,\mathbf{F} = \frac{1}{ABC}\begin{vmatrix} A\widehat{\mathbf{X}} & B\widehat{\mathbf{Y}} & C\widehat{\mathbf{Z}} \\ \dfrac{\partial}{\partial X} & \dfrac{\partial}{\partial Y} & \dfrac{\partial}{\partial Z} \\ AF_X & BF_Y & CF_Z \end{vmatrix} = \frac{1}{\rho}\begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\varphi}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \varphi} & \dfrac{\partial}{\partial z} \\ F_\rho & \rho F_\varphi & F_z \end{vmatrix}$$

$$\Delta f \equiv \nabla^2 f = \frac{1}{ABC}\left[\frac{\partial}{\partial X}\left(\frac{BC}{A}\frac{\partial f}{\partial X}\right) + \frac{\partial}{\partial Y}\left(\frac{AC}{B}\frac{\partial f}{\partial Y}\right) + \frac{\partial}{\partial Z}\left(\frac{AB}{C}\frac{\partial f}{\partial Z}\right)\right]$$
$$= \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{\partial}{\partial \varphi}\left(\frac{1}{\rho}\frac{\partial f}{\partial \varphi}\right) + \frac{\partial}{\partial z}\left(\rho\frac{\partial f}{\partial z}\right)\right]$$
$$= \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$$