# Quantum optics practice questions

1. An Argon laser beam operates at $\lambda = 514$ nm ($\hbar\omega = 2.41$eV) with a power, $P = 0.1$ pW. (1pW $= 10^{-12}$W, 1eV$\simeq 1.6 \times 10^{-19}$J).

   A. Find the photon flux, $\Phi$.

   B. If the detector is open for 0.1s time interval, with a quantum efficiency, $\eta$, of 20%. Find the average number of photons, $\overline{n}$, detected during the time window.

   C. Knowing that the laser beam can be described as a coherent state with a Poissonian distribution, what is the standard deviation, $\Delta n \equiv \sqrt{\overline{n^2} - \overline{n}^2}$, in the number of photons detected?

2. A 10mW He:Ne laser operating at 632.8 nm (1.96 eV) is detected with a photo-diode of responsivity of $R = \frac{i}{P} = 0.43$ A.W$^{-1}$.

   A. The quantum efficiency of the detector.

   B. The average photo-current.

3. Using the identity $\left[\widehat{A}, \widehat{B}\widehat{C}\right] = \left[\widehat{A}, \widehat{B}\right]\widehat{C} + \widehat{B}\left[\widehat{A}, \widehat{C}\right]$, evaluate the following expressions: : (Reminder: $\widehat{N}_\ell = \widehat{a}_\ell^\dagger \widehat{a}_\ell$ and $\left[\widehat{a}_\ell, \widehat{a}_\ell^\dagger\right] = 1$ )

   A. $\left[\widehat{a}_\ell^\dagger, \widehat{a}_k\right]$

   B. $\left[\widehat{a}_\ell, \widehat{N}_\ell\right]$

   C. $\left[\widehat{a}_\ell^\dagger, \widehat{N}_\ell\right]$

   D. $\left[\widehat{a}_\ell, \widehat{a}_\ell^\dagger \widehat{a}_\ell \widehat{a}_\ell^\dagger\right]$

4. The temporal evolution of an operator in the Heisenberg picture is :

$$\widehat{O}_{\mathrm{H}}(t) = e^{i\widehat{H}t/\hbar}\widehat{O}(0)e^{-i\widehat{H}t/\hbar} , \tag{1}$$

   where $\widehat{O}(0) = \widehat{O}_{\mathrm{S}}$ is the Schrödinger picture operator. The Baker-Hausdorff lemma tells us that we can expand $\widehat{O}_{\mathrm{H}}(t)$ as follows:

$$\begin{aligned}\widehat{O}(t) = \widehat{O}(0) + \frac{it}{\hbar}\left[\widehat{H}, \widehat{O}(0)\right] + \frac{1}{2!}\left(\frac{it}{\hbar}\right)^2\left[\widehat{H}, \left[\widehat{H}, \widehat{O}(0)\right]\right] + \dots \\ + \frac{1}{n!}\left(\frac{it}{\hbar}\right)^n\left[\widehat{H}, \left[\widehat{H}, \left[\widehat{H}, \dots \left[\widehat{H}, \widehat{O}(0)\right]\right]\right]\right] + \dots\end{aligned} \tag{2}$$

   A. Use the equation eq.(2) with $\widehat{H} = \sum_k \widehat{H}_k = \sum_k \hbar\omega_k \widehat{a}_k^\dagger \widehat{a}_k$ to deduce that in the Heisenberg representation that: $\widehat{a}_\ell(t) = \widehat{a}_\ell(0)e^{-i\omega_\ell t}$.

   B. Without calculation, give the time evolution of $\widehat{a}_\ell^\dagger(t)$ in the Heisenberg presentation.

5. Let us consider a single mode quantum state of the following form:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\{|0\rangle + |4\rangle\} \tag{3}$$

   A. Calculate $\overline{n} = \langle\Psi|\,\widehat{N}|\Psi\rangle = \langle\Psi|\,\widehat{a}^\dagger\widehat{a}|\Psi\rangle$.

   B. Calculate $\overline{n^2} = \langle\Psi|\,\widehat{N}^2|\Psi\rangle$.

   C. Calculate $\Delta n \equiv \sqrt{\overline{n^2} - \overline{n}^2}$.

We consider a system with a single radiation mode. The quadrature operators in this mode are:

$$\begin{aligned}\widehat{X}_1 &= \frac{1}{2}\left(\widehat{a}+\widehat{a}^\dagger\right)\\ \widehat{X}_2 &= \frac{1}{2i}\left(\widehat{a}-\widehat{a}^\dagger\right)\end{aligned} \tag{4}$$

6. Determine the commutation relations:

$$\left[\widehat{X}_1,\widehat{X}_1\right] \overset{?}{=} \quad ,\left[\widehat{X}_2,\widehat{X}_2\right] \overset{?}{=} \quad , \quad \left[\widehat{X}_1,\widehat{X}_2\right] \overset{?}{=} \ . \tag{5}$$

7. For the Fock state, $|n\rangle$, and the quadrature operators, $\widehat{X}_1$, and $\widehat{X}_2$ given in (4):
   A. Calculate $\langle X_j\rangle_n = \langle n|\,\widehat{X}_j|n\rangle$ for $j=1,2$.

   B. Calculate $\left\langle X_j^2\right\rangle_n = \langle n|\,\widehat{X}_j^2|n\rangle$ for $j=1,2$.

   C. Calculate $\Delta X_j = \sqrt{\left\langle X_j^2\right\rangle_n - \langle X_j\rangle_n^2}$ for $j=1,2$.

Let us consider a pure single-mode photonic state $|\Psi\rangle$ which is a superposition of a vacuum state, $|0\rangle$, and a state $|1\rangle$ with a single photon. We adopt the 'qubit' notation for this state in order to write :

$$|\Psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle \ . \tag{6}$$

where $\theta \in [0,\pi]$ and $\phi \in [0,2\pi[$ . **n.b.** the states $|0\rangle$ et $|1\rangle$ are related by the usual ladder operations $\widehat{a}^\dagger|0\rangle = |1\rangle$, $\widehat{a}|1\rangle = |0\rangle$. ($|0\rangle$ et $|1\rangle$.

8. For the state $|\Psi\rangle$ of Eq.(6) :
   A. Calculate $\langle X_i\rangle_\Psi = \langle\Psi|\,X_i|\Psi\rangle$ pour $i=1,2$.

   B. Calculate $\left\langle X_i^2\right\rangle_\Psi \equiv \langle\Psi|\,X_i^2|\Psi\rangle$ pour $i=1,2$.

   C. Calculate $\Delta X_i = \sqrt{\langle X_i^2\rangle_\Psi - \langle X_i\rangle_\Psi^2}$ for $i=1,2$.

   D. Plot $\Delta X_1^2$, $\Delta X_2^2$ and $\Delta X_1\Delta X_2$ when $\phi = 0$. What can you say about these graphs considering what you have learned about squeezing ?

For the following questions, we recall that any quantum state, $|\psi\rangle$, in a unique mode $\ell$ can be written as a superposition of number states, $|n\rangle\,|n\rangle =$ (we suppress the index $\ell$ ) :

$$|\psi\rangle = \sum_{n=0}^{\infty} C_n\,|n\rangle \ . \tag{7}$$

The quasi-classical state (Glauber state/coherent state) on the other hand is written explicitly:

$$|\alpha\rangle \equiv e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle \ . \tag{8}$$

9. For a coherent state $|\alpha\rangle$ :

   A. Calculate the average number of photons: $\overline{n} = \left\langle \widehat{N} \right\rangle \equiv \langle\alpha|\,\widehat{a}^\dagger\widehat{a}\,|\alpha\rangle$.

   B. Calculate the fluctuation of the photon number for a coherent state: $\Delta n = \sqrt{\langle n^2\rangle - \langle n\rangle^2}$.

   C. Calculate $\langle X_j\rangle_\alpha = \langle\alpha|\,\widehat{X}_j|\alpha\rangle$ for $j = 1, 2$.

   D. Calculate $\langle X_j^2\rangle_\alpha = \langle\alpha|\,\widehat{X}_j^2|\alpha\rangle$ for $j = 1, 2$.

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Consider a beam-splitter and an arbitrary state, $|\Psi\rangle$, expressed either on the basis of the input channels $|\Psi_{1,2}\rangle$ or in terms of the output channels $|\Psi_{3,4}\rangle$:

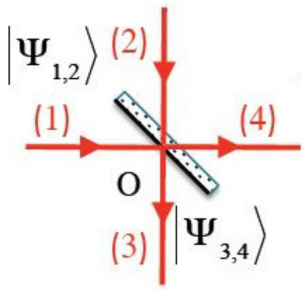


Figure 1: Transformation by a beam-splitter between input beams in channels 1 and 2 and output channels 3 and 4.

The lowering operators transform as follows:

$$\widehat{a}_3 = r\widehat{a}_1 + t\widehat{a}_2$$
$$\widehat{a}_4 = t'\widehat{a}_1 + r'\widehat{a}_2\ ,$$

where $r$ , $r'$, $t$, $t'$ are reflexion and transmission coefficients that in general have complex values that can be obtained by 'classical' electromagnetic calculations (or measurements) of the beam-splitter.

It is often practical to describe this transformation as an $S$-matrix:

$$\begin{bmatrix}\widehat{a}_3\\ \widehat{a}_4\end{bmatrix} = \begin{bmatrix} r & t\\ t' & r'\end{bmatrix}\begin{bmatrix}\widehat{a}_1\\ \widehat{a}_2\end{bmatrix} = [S]\begin{bmatrix}\widehat{a}_1\\ \widehat{a}_2\end{bmatrix}\ . \tag{9}$$

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10. Which of the following $S$-matrices are physically acceptables?

   A. $[S] = \begin{bmatrix} ir & t\\ t & ir\end{bmatrix}$ for $r$ and $t$ real, with $r^2 + t^2 = 1$.

   B. $[S] = \begin{bmatrix} r & it\\ it & r\end{bmatrix}$ for $r$ and $t$ real, with $r^2 + t^2 = 1$.

   C. $[S] = \begin{bmatrix} r & t\\ t & -r\end{bmatrix}$ for $r$ and $t$ real with $r^2 + t^2 = 1$.

   D. $[S] = \begin{bmatrix} r & it\\ -it & r\end{bmatrix}$ for $r$ and $t$ real with $r^2 + t^2 = 1$

   E. $[S] = \begin{bmatrix} r_c & t_c\\ t_c & r_c\end{bmatrix}$ with $r_c$ and $t_c$ as complex coefficients that satisfy $r_c t_c^* = -r_c^* t_c$ and $|r_c|^2 + |t_c|^2 = 1$.

11. Let us consider the state $|n\rangle_1$ with $n$ photons in channel 1. Express $|n\rangle_1$ in terms of $\widehat{a}_1^\dagger$ and the vacuum state $|0\rangle_1$:

   A. $|n\rangle_1 = \frac{1}{\sqrt{n!}} \left(\widehat{a}_1^\dagger\right)^n |0\rangle_1$

   B. $|n\rangle_1 = \frac{1}{\sqrt{(n-1)!}} \left(\widehat{a}_1^\dagger\right)^n |0\rangle_1$

   C. $|n\rangle_1 = \sqrt{n!} \left(\widehat{a}_1^\dagger\right)^n |0\rangle_1$

12. Express $|\Psi\rangle = |n\rangle_1 \otimes |m\rangle_2 = |n, m\rangle_{1,2}$ in terms of $\widehat{a}_1^\dagger$ and $\widehat{a}_2^\dagger$ acting on the two-channel vacuum $|0, 0\rangle$. Note that the vacuum doesn't depend on the choice of basis. (More than one answer may be correct)

   A. $|\Psi\rangle = \frac{1}{\sqrt{n!m!}} \widehat{a}_1^{\dagger n} \widehat{a}_2^{\dagger m} |0, 0\rangle$.

   B. $|\Psi\rangle = \frac{1}{\sqrt{n!m!}} \widehat{a}_1^{\dagger m} \widehat{a}_2^{\dagger n} |0, 0\rangle$

   C. $|\Psi\rangle = \frac{1}{\sqrt{n!m!}} \widehat{a}_2^{\dagger m} \widehat{a}_1^{\dagger n} |0, 0\rangle$

13. Express the above state $|\Psi\rangle = |n, m\rangle_{1,2}$ in terms of the output state $\widehat{a}_3^\dagger$ and $\widehat{a}_4^\dagger$ operators.

   A. $|\Psi\rangle = \frac{1}{\sqrt{n!m!}} \left(r^* \widehat{a}_3^\dagger - t^* \widehat{a}_4^\dagger\right)^n \left(t^* \widehat{a}_3^\dagger + r^* \widehat{a}_4^\dagger\right)^m |0, 0\rangle$

   B. $|\Psi\rangle = \frac{1}{\sqrt{n!m!}} \left(r^* \widehat{a}_3^\dagger + t^* \widehat{a}_4^\dagger\right)^n \left(t^* \widehat{a}_3^\dagger + r^* \widehat{a}_4^\dagger\right)^m |0, 0\rangle$

   C. $|\Psi\rangle = \frac{1}{\sqrt{n!m!}} \left(r \widehat{a}_3^\dagger + t \widehat{a}_4^\dagger\right)^n \left(t \widehat{a}_3^\dagger + r \widehat{a}_4^\dagger\right)^m |0, 0\rangle$

14. Consider state $|\Psi\rangle = |1, 0\rangle_{1,2}$. Give the expression of this state on the basis of the output channels.

   A. $|\Psi\rangle = |1, 0\rangle_{1,2} = r^* |1, 0\rangle_{3,4} + t^* |0, 1\rangle_{3,4}$

   B. $|\Psi\rangle = |1, 0\rangle_{1,2} = t^* |1, 0\rangle_{3,4} + r^* |0, 1\rangle_{3,4}$

   C. $|\Psi\rangle = |1, 0\rangle_{1,2} = r |1, 0\rangle_{3,4} + t |0, 1\rangle_{3,4}$

15. Let us consider the state, $|\Psi\rangle = |1, 1\rangle_{1,2}$, correspond to exactly 1-photon in each input channel. Which is the expression for $|\Psi\rangle = |1, 1\rangle_{1,2}$ basis of output channels ?

   A. $|\Psi\rangle = \sqrt{2}\, rt |2, 0\rangle_{3,4} + \left(t^2 + r^2\right) |1, 1\rangle_{3,4} + \sqrt{2}\, tr |0, 2\rangle_{3,4}$.

   B. $|\Psi\rangle = \sqrt{2}\, r^* t^* |2, 0\rangle_{3,4} + \left[(t^*)^2 + (r^*)^2\right] |1, 1\rangle_{3,4} + \sqrt{2}\, t^* r^* |0, 2\rangle_{3,4}$

   C. $|\Psi\rangle = r^* t^* |2, 0\rangle_{3,4} + \left[(t^*)^2 + (r^*)^2\right] |1, 1\rangle_{3,4} + t^* r^* |0, 2\rangle_{3,4}$

16. Continuing the previous question, the state $|\Psi\rangle = |1, 1\rangle_{1,2}$, corresponds to exactly one photon state in each entry channel arriving at the same time. Let us consider now a symmetric 50/50 beam-splitter with $r = \frac{i}{\sqrt{2}}$ and $t = \frac{1}{\sqrt{2}}$. What is the probability of detecting one photon in each of the output channels?

   A. 0

   B. 1/3

   C. 1/4

   D. 1

For the following questions, let us consider the special (but common) case where $\left[\widehat{A},\widehat{B}\right]\neq 0$ and :

$$\left[\widehat{A},\left[\widehat{A},\widehat{B}\right]\right]=0=\left[\widehat{B},\left[\widehat{A},\widehat{B}\right]\right]\ . \tag{10}$$

17. Under the above conditions, prove the identity: $\left[\widehat{B},\widehat{A}^n\right]=n\widehat{A}^{n-1}\left[\widehat{B},\widehat{A}\right]$.

18. Use the identity from question 17. to show that $\left[\widehat{B},e^{-\widehat{A}x}\right]=-xe^{-\widehat{A}x}\left[\widehat{B},\widehat{A}\right]$.

19. Use the identity from Question 18. to find the following expression:

$$e^{\widehat{A}x}\widehat{B}e^{-\widehat{A}x}=\widehat{B}-x\left[\widehat{B},\widehat{A}\right] \tag{11}$$

20. We define an operator $\widehat{O}(x)\equiv e^{\widehat{A}x}e^{\widehat{B}x}$. Compute the derivative, $\frac{d\widehat{O}}{dx}$, and use the expression from Eq.(11) to show that:

$$\frac{d\widehat{O}}{dx}=\left(\widehat{A}+\widehat{B}-x\left[\widehat{B},\widehat{A}\right]\right)\widehat{O}(x) \tag{12}$$

21. Show that an integral over $x$ of Eq.(12) that :

$$e^{\widehat{A}}e^{\widehat{B}}=e^{\widehat{A}+\widehat{B}}e^{\frac{1}{2}\left[\widehat{A},\widehat{B}\right]}\ . \tag{13}$$

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22. Consider a single-mode displacement operator, $\widehat{D}(\alpha)\equiv\exp\left(\alpha\widehat{a}^\dagger-\alpha^*\widehat{a}\right)$.

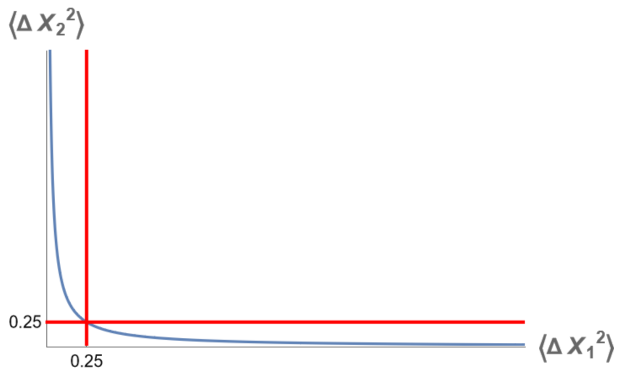
    A. Use the disentangling theorem you derived in eq.(13), i.e. $e^{(\widehat{A}+\widehat{B})}=e^{-\frac{1}{2}[\widehat{A},\widehat{B}]}e^{\widehat{A}}e^{\widehat{B}}$ (valid when $[\widehat{A},[\widehat{A},\widehat{B}]]=[\widehat{B},[\widehat{A},\widehat{B}]]=0$), in order to show that:

    $$\widehat{D}\left(\alpha\right)=e^{-\frac{1}{2}|\alpha|^2}e^{+\alpha\widehat{a}^\dagger}e^{-\alpha^*\widehat{a}}\ , \tag{14}$$

    B. Use eq.(14) to show that the displacement operator, $\widehat{D}(\alpha)$, acting on the vacuum state, $|0\rangle$, generates a coherent state, i.e. $\widehat{D}\left(\alpha\right)|0\rangle=|\alpha\rangle$ (where $\widehat{a}\left|\alpha\right\rangle=\alpha\left|\alpha\right\rangle$ and $\langle\alpha|\alpha\rangle=1$). (For this demonstration, we recall the properties that $\widehat{a}^\dagger\left|n\right\rangle=\sqrt{n+1}\left|n+1\right\rangle$, and $\widehat{a}\left|n\right\rangle=\sqrt{n}\left|n-1\right\rangle$).

23. Let us consider a single-mode electromagnetic field in a squeezed vacuum state with the quadrature operator $\widehat{X}_1$ squeezed.

1. What is the defining property of a squeezed state in terms of quadrature uncertainties?
2. When the uncertainty in $\widehat{X}_1$ is reduced below the vacuum level of the vacuum, what happens to the uncertainty in the conjugate quadrature $\widehat{X}_2$? Explain briefly.
3. Consider the following graph for regions of, $\langle \Delta X_1^2 \rangle$ and $\langle \Delta X_2^2 \rangle$ for quantum states. Indicate which regions correspond to squeezed states (1), which region corresponds to non-squeezed quantum or classical averages (2), and which region (3) corresponds to forbidden quadrature fluctuation values.

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Let us recall that a coherent state $|\alpha\rangle$ has a Poissonian photon number distribution with :

$$(\Delta n)^2 = \langle \widehat{n} \rangle \ . \tag{15}$$

We saw in class that the squeezing operator can be written:

$$\widehat{S}\,(\xi) = \exp\left(\xi^* \widehat{a}^2 - \xi a^{\dagger,2}\right) \tag{16}$$

with the complex number, $\xi = r^{i\theta}$, being the squeezing parameter. A squeezed vacuum is generated by

$$|0,\xi\rangle \equiv \widehat{S}\,(\xi)\,|0\rangle \tag{17}$$

Squeezed lowering and raising operators are then respectively defined as $\widehat{A}_\xi \equiv \widehat{S}\,(\xi)\,\widehat{a}\widehat{S}^\dagger\,(\xi)$ and $\widehat{A}_\xi^\dagger \equiv \widehat{S}\,(\xi)\,\widehat{a}^\dagger\widehat{S}^\dagger\,(\xi)$.

As you saw in class, one can calculate expectation values of quantum operators for a squeezed vacuum using the transformation relations:

$$\begin{aligned} \widehat{a} &= \widehat{A}_\xi \cosh(r) - e^{i\theta}\widehat{A}_\xi^\dagger \sinh(r) \\ \widehat{a}^\dagger &= \widehat{A}_\xi^\dagger \cosh(r) - e^{-i\theta}\widehat{A}_\xi \sinh(r) \end{aligned} \tag{18}$$

and the properties that, $\widehat{A}_\xi\,|0,\xi\rangle = 0$, $\langle 0,\xi|\,\widehat{A}_\xi^\dagger = 0$ and $\left[\widehat{A}_\xi, \widehat{A}_\xi^\dagger\right] = 1$.

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24. **Photon Number Fluctuations of a squeezed state:** You will demonstrate in this exercise that squeezed states, like squeezed vacuum states $|0,\xi\rangle$ exhibit super-Poissonian photon statistics. **Hint:** In order to simplify the following calculations, you can take $\theta = 0$ since the values you will calculate won't depend on its value.

1. Use the above reminders, and in particular Eqs.(17) and (18) to calculate the mean photon number of a squeezed vacuum state (in function of $r$):

$$\overline{n}_{0_\xi} \equiv \langle \widehat{N} \rangle_{0_\xi} \equiv \langle 0,\xi| \, \widehat{a}^\dagger \widehat{a} \, |0,\xi\rangle \tag{19}$$

2. Calculate now $\langle \widehat{N}^2 \rangle_{0_\xi}$ for the squeezed vacuum :

$$\langle \widehat{N}^2 \rangle_{0_\xi} \equiv \langle 0,\xi| \, \widehat{N}^2 \, |0,\xi\rangle \tag{20}$$

3. Use the above results to determine the photon fluctuations, $\langle \Delta \widehat{N}^2 \rangle_{0_\xi} = \langle \widehat{N}^2 \rangle_{0_\xi} - \left( \langle \widehat{N} \rangle_{0_\xi} \right)^2$. Compare your result to the coherent state fluctuations given in Eq.(15), and explain why this is indeed a super-Poissonian distribution.