Quantum optics practice questions

1. An Argon laser beam operates at $\lambda = 514$ nm ($\hbar\omega = 2.41$eV) with a power, $P = 0.1$ pW. (1pW $= 10^{-12}$W, 1eV$\simeq 1.6 \times 10^{-19}$J).

   A. Find the photon flux, Φ.

   Solution :

   $$\Phi = \frac{P}{\hbar\omega} = \frac{10^{-13}\text{W}}{2.41\text{eV}} \frac{1\text{ev}}{1.6 * 10^{-19}\text{J}} = 2.59\,10^5 \text{photons}\,\text{s}^{-1}\,. \tag{1}$$

   B. If the detector is open for 0.1s time interval, with a quantum efficiency, η, of 20%. Find the average number of photons, $\overline{n}$, detected during the time window.

   Solution :

   $$\overline{n} = 0.1 * \Phi * 0.2 = 0.1 * 2.59\,10^5 * 0.2 = 5186\,\text{photons}\,. \tag{2}$$

   C. Knowing that the laser beam can be described as a coherent state with a Poissonian distribution, what is the standard deviation, $\Delta n \equiv \sqrt{\overline{n^2} - \overline{n}^2}$, in the number of photons detected?

   Solution :

   $$\Delta n = \sqrt{\overline{n}} = \sqrt{5186} = 72\,\text{photons}\,. \tag{3}$$

2. A 10mW He:Ne laser operating at 632.8 nm (1.96 eV) is detected with a photo-diode of responsivity of $R = \frac{i}{P} = 0.43$ A.W^{-1}.

   A. The quantum efficiency of the detector.

   Solution :

   $$\eta = \frac{\hbar\omega}{e} \times R = 1.96\text{V} \times 0.43\text{A.W}^{-1} = 0.84 = 84\%$$

   B. The average photo-current.

   Solution :

   $$i\,(\text{A}) = \text{responsivity}(\text{A.W}^{-1}) \times \text{power}(\text{W}) = 0.43 \times 0.01 = 4.3\text{mA}$$

3. Using the identity $\left[\widehat{A}, \widehat{B}\widehat{C}\right] = \left[\widehat{A}, \widehat{B}\right]\widehat{C} + \widehat{B}\left[\widehat{A}, \widehat{C}\right]$, evaluate the following expressions: : (Reminder: $\widehat{N}_\ell = \widehat{a}_\ell^\dagger \widehat{a}_\ell$ and $\left[\widehat{a}_\ell, \widehat{a}_\ell^\dagger\right] = 1$)

   A. $\left[\widehat{a}_\ell^\dagger, \widehat{a}_k\right]$

   Solution : $\left[\widehat{a}_\ell^\dagger, \widehat{a}_k\right] = -\left[\widehat{a}_k, \widehat{a}_\ell^\dagger\right] = -\delta_{\ell,k}$

   B. $\left[\widehat{a}_\ell, \widehat{N}_\ell\right]$

   Solution :

   $$\left[\widehat{a}_\ell, \widehat{N}_\ell\right] = \left[\widehat{a}_\ell, \widehat{a}_\ell^\dagger \widehat{a}_\ell\right] = \left[\widehat{a}_\ell, \widehat{a}_\ell^\dagger\right]\widehat{a}_\ell + \widehat{a}_\ell^\dagger\left[\widehat{a}_\ell, \widehat{a}_\ell\right] = \widehat{a}_\ell$$

   C. $\left[\widehat{a}_\ell^\dagger, \widehat{N}_\ell\right]$

   Solution :

   $$\left[\widehat{a}_\ell^\dagger, \widehat{N}_\ell\right] = \left[\widehat{a}_\ell^\dagger, \widehat{a}_\ell^\dagger \widehat{a}_\ell\right] = \left[\widehat{a}_\ell^\dagger, \widehat{a}_\ell^\dagger\right]\widehat{a}_\ell + \widehat{a}_\ell^\dagger\left[\widehat{a}_\ell^\dagger, \widehat{a}_\ell\right] = -\widehat{a}_\ell^\dagger$$

D. $\left[\widehat{a}_\ell, \widehat{a}_\ell^\dagger \widehat{a}_\ell \widehat{a}_\ell^\dagger\right]$

Solution :

$$\begin{aligned}\left[\widehat{a}_\ell, \widehat{a}_\ell^\dagger \widehat{a}_\ell \widehat{a}_\ell^\dagger\right] &= \left[\widehat{a}_\ell, \widehat{N}_\ell\right] \widehat{a}_\ell^\dagger + \widehat{N}_\ell \left[\widehat{a}_\ell, \widehat{a}_\ell^\dagger\right] = \widehat{a}_\ell \widehat{a}_\ell^\dagger + \widehat{N}_\ell \\ &= 1 + 2\widehat{a}_\ell^\dagger \widehat{a}_\ell = 1 + 2\widehat{N}_\ell \ .\end{aligned}$$

4. The temporal evolution of an operator in the Heisenberg picture is :

$$\widehat{O}_{\mathrm{H}}(t) = e^{i\widehat{H}t/\hbar} \widehat{O}(0) e^{-i\widehat{H}t/\hbar} \ , \tag{4}$$

where $\widehat{O}(0) = \widehat{O}_{\mathrm{S}}$ is the Schrödinger picture operator. The Baker-Hausdorff lemma tells us that we can expand $\widehat{O}_{\mathrm{H}}(t)$ as follows:

$$\begin{aligned}\widehat{O}(t) = \widehat{O}(0) + \frac{it}{\hbar}\left[\widehat{H}, \widehat{O}(0)\right] + \frac{1}{2!}\left(\frac{it}{\hbar}\right)^2 \left[\widehat{H}, \left[\widehat{H}, \widehat{O}(0)\right]\right] + \dots \\ + \frac{1}{n!}\left(\frac{it}{\hbar}\right)^n \left[\widehat{H}, \left[\widehat{H}, \left[\widehat{H}, \dots \left[\widehat{H}, \widehat{O}(0)\right]\right]\right]\right] + \dots\end{aligned} \tag{5}$$

A. Use the equation eq.(5) with $\widehat{H} = \sum_k \widehat{H}_k = \sum_k \hbar\omega_k \widehat{a}_k^\dagger \widehat{a}_k$ to deduce that in the Heisenberg representation that: $\widehat{a}_\ell(t) = \widehat{a}_\ell(0) e^{-i\omega_\ell t}$.

Solution : From the formula, $\left[\widehat{N}_k, \widehat{a}_\ell\right] = -\delta_{\ell,k}\widehat{a}_\ell$

$$\left[\widehat{H}, \widehat{a}_\ell\right] = \left[\widehat{H}_\ell, \widehat{a}_\ell\right] = \hbar\omega_\ell \left[\widehat{N}_\ell, \widehat{a}_\ell\right] = -\hbar\omega_\ell \widehat{a}_\ell \tag{6}$$

Taking $\widehat{O} = \widehat{a}_\ell$ and the the above result we obtain :

$$\begin{aligned}\widehat{a}_\ell(t) &= \widehat{a}_\ell + \frac{it}{\hbar}\left[\widehat{H}, \widehat{a}_\ell\right] + \frac{1}{2!}\left(\frac{it}{\hbar}\right)^2 \left[\widehat{H}, \left[\widehat{H}, \widehat{a}_\ell\right]\right] + \dots \\ &\quad + \frac{1}{n!}\left(\frac{it}{\hbar}\right)^n \left[\widehat{H}, \left[\widehat{H}, \left[\widehat{H}, \dots \left[\widehat{H}, \widehat{a}_\ell\right]\right]\right]\right] + \dots \\ &= \left\{1 - i\omega_\ell t + \frac{(-i\omega t)^2}{2!} + \dots + \frac{(-i\omega t)^n}{n!} + \dots\right\} \widehat{a}_\ell \\ &= e^{-i\omega_\ell t}\widehat{a}_\ell(0) = e^{-i\omega_\ell t}\widehat{a}_\ell \ .\end{aligned} \tag{7}$$

B. Without calculation, give the time evolution of $\widehat{a}_\ell^\dagger(t)$ in the Heisenberg presentation.

Solution : One can simply take the adjoint of Eq. 7 : $\widehat{a}_\ell^\dagger(t) = e^{i\omega_\ell t}\widehat{a}_\ell^\dagger(0) = e^{i\omega_\ell t}\widehat{a}_\ell^\dagger$. Of course we could have obtained this result by the same procedure as the previous question, but this is no longer necessary.

5. Let us consider a single mode quantum state of the following form:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left\{|0\rangle + |4\rangle\right\} \tag{8}$$

A. Calculate $\overline{n} = \langle\Psi|\,\widehat{N}|\Psi\rangle = \langle\Psi|\,\widehat{a}^\dagger\widehat{a}|\Psi\rangle$.

Solution :

$$\overline{n} = \langle\Psi|\,\widehat{N}|\Psi\rangle = \frac{1}{2}\left\{\langle 4| + \langle 0|\right\}\widehat{a}^\dagger\widehat{a}\left\{|0\rangle + |4\rangle\right\} = 2$$

B. Calculate $\overline{n^2} = \langle\Psi|\,\widehat{N}^2|\Psi\rangle$.

Solution :

$$\overline{n^2} = \langle\Psi|\,\widehat{N}^2|\Psi\rangle = \frac{1}{2}\left\{\langle 4| + \langle 0|\right\}\widehat{a}^\dagger\widehat{a}\widehat{a}^\dagger\widehat{a}\left\{|0_\ell\rangle + |4\rangle\right\} = 8\;.$$

C. Calculate $\Delta n \equiv \sqrt{\overline{n^2} - \overline{n}^2}$.

Solution :

$$\Delta n_\ell = \sqrt{\overline{n^2} - \overline{n}^2} = 2\;.$$

We consider a system with a single radiation mode. The quadrature operators in this mode are:

$$\begin{aligned}\widehat{X}_1 &= \frac{1}{2}\left(\widehat{a} + \widehat{a}^\dagger\right)\\ \widehat{X}_2 &= \frac{1}{2i}\left(\widehat{a} - \widehat{a}^\dagger\right)\end{aligned} \tag{9}$$

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6. Determine the commutation relations:

$$\left[\widehat{X}_1, \widehat{X}_1\right] \stackrel{?}{=} \quad, \left[\widehat{X}_2, \widehat{X}_2\right] \stackrel{?}{=} \quad , \quad \left[\widehat{X}_1, \widehat{X}_2\right] \stackrel{?}{=} \;. \tag{10}$$

Solution : Since $\widehat{X}_1$ and $\widehat{X}_2$ are bosonic operators, their commutators with themselves are 0 :

$$\left[\widehat{X}_1, \widehat{X}_1\right] = \left[\widehat{X}_2, \widehat{X}_2\right] = 0$$

We calculate the commutators of the canonically conjugate variables $\widehat{X}_1$ and $\widehat{X}_2$ via their expressions in terms of the raising and lowering operators :

$$\left[\widehat{X}_1, \widehat{X}_2\right] = \frac{1}{4i}\left[\left(\widehat{a} + \widehat{a}^\dagger\right), \left(\widehat{a} - \widehat{a}^\dagger\right)\right] = -\frac{1}{4i}\left[\widehat{a}, \widehat{a}^\dagger\right] + \frac{1}{4i}\left[\widehat{a}^\dagger, \widehat{a}\right] = \frac{i}{2} = -\left[\widehat{X}_2, \widehat{X}_1\right]\;.$$

7. For the Fock state, $|n\rangle$, and the quadrature operators, $\widehat{X}_1$, and $\widehat{X}_2$ given in (9):

A. Calculate $\langle X_j\rangle_n = \langle n|\,\widehat{X}_j|n\rangle$ for $j = 1, 2$.

Solution :

$$\begin{aligned}\langle X_1\rangle_n &= \frac{1}{2}\langle n|\left(\widehat{a} + \widehat{a}^\dagger\right)|n\rangle = 0\\ \langle X_2\rangle_n &= \frac{1}{2i}\langle n|\left(\widehat{a} - \widehat{a}^\dagger\right)|n\rangle = 0\end{aligned}$$

B. Calculate $\left\langle X_j^2\right\rangle_n = \langle n|\,\widehat{X}_j^2|n\rangle$ for $j = 1, 2$.

Solution :

$$\begin{aligned}\widehat{X}_1^2 &= \frac{1}{4}\left(\widehat{a}\widehat{a} + \widehat{a}^\dagger\widehat{a}^\dagger + 2\widehat{a}^\dagger\widehat{a} + 1\right)\\ \widehat{X}_2^2 &= -\frac{1}{4}\left(\widehat{a}\widehat{a} + \widehat{a}^\dagger\widehat{a}^\dagger - 2\widehat{a}^\dagger\widehat{a} - 1\right)\end{aligned} \tag{11}$$

$$\Longrightarrow \langle n|\widehat{X}_1^2|n\rangle = \langle n|\widehat{X}_2^2|n\rangle = \frac{2n+1}{4}$$

C. Calculate $\Delta X_j = \sqrt{\langle X_j^2 \rangle_n - \langle X_j \rangle_n^2}$ for $j = 1, 2$.

Solution : On a $\Delta X_1 = \Delta X_2 = \frac{1}{2}\sqrt{2n+1}$.

Let us consider a pure single-mode photonic state $|\Psi\rangle$ which is a superposition of a vacuum state, $|0\rangle$, and a state $|1\rangle$ with a single photon. We adopt the 'qubit' notation for this state in order to write :

$$|\Psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle \ . \tag{12}$$

where $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi[$. **n.b.** the states $|0\rangle$ et $|1\rangle$ are related by the usual ladder operations $\widehat{a}^\dagger|0\rangle = |1\rangle$, $\widehat{a}|1\rangle = |0\rangle$. ($|0\rangle$ et $|1\rangle$.

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8. For the state $|\Psi\rangle$ of Eq.(12) :

A. Calculate $\langle X_i \rangle_\Psi = \langle \Psi | X_i | \Psi \rangle$ pour $i = 1, 2$.

Solution :

$$\begin{aligned}\langle X_1 \rangle_\Psi &= \frac{1}{2}\left\{\langle 0|\cos(\theta/2) + \langle 1|e^{-i\phi}\sin(\theta/2)\right\}\left(\widehat{a} + \widehat{a}^\dagger\right)\left\{\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle\right\} \\ &= \sin(\theta/2)\cos(\theta/2)\frac{(e^{i\phi} + e^{-i\phi})}{2} = \sin(\theta/2)\cos(\theta/2)\cos\phi\end{aligned}$$

$$\begin{aligned}\langle X_2 \rangle_\Psi &= \frac{1}{2i}\left\{\langle 0|\cos(\theta/2) + \langle 1|e^{-i\phi}\sin(\theta/2)\right\}\left(\widehat{a} - \widehat{a}^\dagger\right)\left\{\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle\right\} \\ &= \sin(\theta/2)\cos(\theta/2)\frac{(e^{i\phi} - e^{-i\phi})}{2i} = \sin(\theta/2)\cos(\theta/2)\sin\phi\end{aligned}$$

B. Calculate $\langle X_i^2 \rangle_\Psi \equiv \langle \Psi | X_i^2 | \Psi \rangle$ pour $i = 1, 2$.

Solution :

$$\begin{aligned}\langle X_1^2 \rangle_\Psi &= \frac{1}{4}\left\{\langle 0|\cos(\theta/2) + \langle 1|e^{-i\phi}\sin(\theta/2)\right\}\left(\widehat{a}\widehat{a} + \widehat{a}^\dagger\widehat{a}^\dagger + 2\widehat{a}^\dagger\widehat{a} + 1\right)\left\{\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle\right\} \\ &= \frac{1}{2}\sin^2(\theta/2) + \frac{1}{4}\end{aligned}$$

$$\begin{aligned}\langle X_2^2 \rangle_\Psi &= \frac{1}{4}\left\{\langle 0|\cos(\theta/2) + \langle 1|e^{-i\phi}\sin(\theta/2)\right\}\left(-\widehat{a}\widehat{a} - \widehat{a}^\dagger\widehat{a}^\dagger + 2\widehat{a}^\dagger\widehat{a} + 1\right)\left\{\cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle\right\} \\ &= \frac{1}{2}\sin^2(\theta/2) + \frac{1}{4}\end{aligned}$$

C. Calculate $\Delta X_i = \sqrt{\langle X_i^2 \rangle_\Psi - \langle X_i \rangle_\Psi^2}$ for $i = 1, 2$.

Solution :

$$\begin{aligned}\Delta X_1 &= \sqrt{\frac{1}{2}\sin^2\left(\frac{\theta}{2}\right)\left(1 - 2\cos^2\left(\frac{\theta}{2}\right)\cos^2(\phi)\right) + \frac{1}{4}} \\ \Delta X_2 &= \sqrt{\frac{1}{2}\sin^2\left(\frac{\theta}{2}\right)\left(1 - 2\cos^2\left(\frac{\theta}{2}\right)\sin^2(\phi)\right) + \frac{1}{4}}\end{aligned}$$

D. Plot ΔX_1^2, ΔX_2^2 and $\Delta X_1 \Delta X_2$ when $\phi = 0$. What can you say about these graphs considering what you have learned about squeezing ?

Solution : Squeezed states are typically obtained in a form where they appear as a superposition of an infinite number of Fock states, in a manner similar to coherent states (Considered below). Nevertheless, the defining feature of *squeezed states* is that $\Delta X_i^2 < \frac{1}{4}$ for some i. Consequently, in regions where the blue curve, corresponding to ΔX_1^2 is below the dashed green line at 0.25, the state, $|\Psi\rangle$ of eq.(12), can justifiably be referred to as being "squeezed" in the X_1 quadrature. One observes that just as in the case of more conventional squeezing, considered later, that a region where where $\Delta X_1^2 < \frac{1}{4}$ is always accompanied by $\Delta X_2^2 > \frac{1}{4}$ such that the quantum uncertainty relation $\Delta X_1 \Delta X_2 \geq \frac{1}{4}$ for conjugate variables is always satisfied.

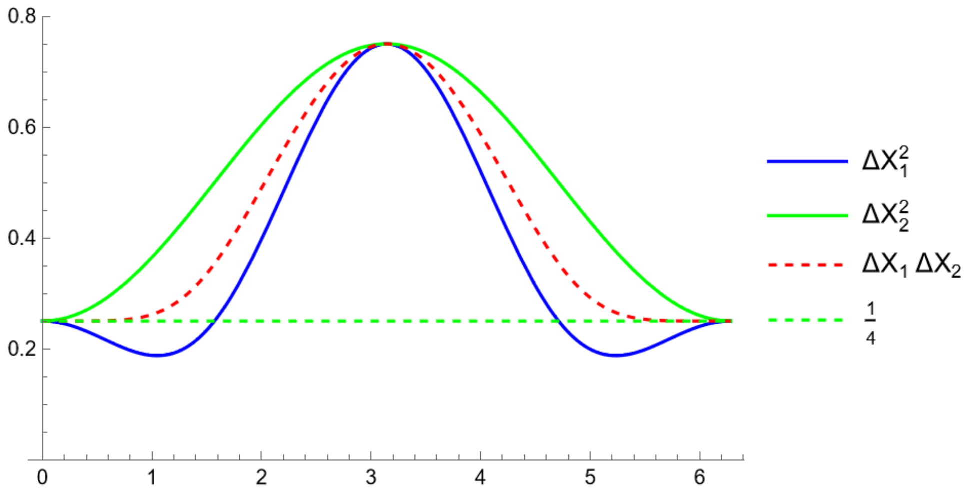

Figure 1: ΔX_1^2, ΔX_2^2 and $\Delta X_1 \Delta X_2$ plotted as functions of θ when $\phi = 0$.

For the following questions, we recall that any quantum state, $|\psi\rangle$, in a unique mode ℓ can be written as a superposition of number states, $|n\rangle\, |n\rangle =$ (we suppress the index ℓ) :

$$|\psi\rangle = \sum_{n=0}^{\infty} C_n |n\rangle \ . \tag{13}$$

The quasi-classical state (Glauber state/coherent state) on the other hand is written explicitly:

$$|\alpha\rangle \equiv e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \ . \tag{14}$$

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9. For a coherent state $|\alpha\rangle$:

A. Calculate the average number of photons: $\overline{n} = \left\langle \widehat{N} \right\rangle \equiv \langle\alpha| \widehat{a}^\dagger \widehat{a} |\alpha\rangle$.

Solution :

$$\begin{aligned}\overline{n} = \left\langle \widehat{N} \right\rangle &\equiv \langle\alpha| \widehat{a}^\dagger \widehat{a} |\alpha\rangle = \langle\alpha| \alpha^* \alpha |\alpha\rangle = \langle\alpha| |\alpha|^2 |\alpha\rangle \\ &= |\alpha|^2 \ .\end{aligned}$$

B. Calculate the fluctuation of the photon number for a coherent state: $\Delta n = \sqrt{\langle n^2\rangle - \langle n\rangle^2}$.

Solution :

$$\langle n^2 \rangle = \langle \widehat{N}^2 \rangle = \langle \alpha | \, \widehat{a}^\dagger \widehat{a} \widehat{a}^\dagger \widehat{a} \, | \alpha \rangle = \langle \alpha | \, \widehat{a}^\dagger \widehat{a}^\dagger \widehat{a} \widehat{a} \, | \alpha \rangle + \langle \alpha | \, \widehat{a}^\dagger \widehat{a} \, | \alpha \rangle$$
$$= |\alpha|^4 + |\alpha|^2 \ .$$

The photon fluctuations are then :

$$\Delta n = \sqrt{\langle n^2 \rangle - \langle n \rangle^2} = \sqrt{|\alpha|^2} = |\alpha|$$

It is good to keep in mind that fluctuations of both coherent states and vacuum states are at the quantum limit, $(\Delta n)^2 = \langle \widehat{n} \rangle$.

C. Calculate $\langle X_j \rangle_\alpha = \langle \alpha | \, \widehat{X}_j | \alpha \rangle$ for $j = 1, 2$.

Solution :

$$\langle \alpha | X_1 | \alpha \rangle = \frac{1}{2} \langle \alpha | \left(\widehat{a} + \widehat{a}^\dagger \right) | \alpha \rangle = \frac{1}{2} \left(\alpha + \alpha^* \right) = \mathrm{Re}\left\{ \alpha \right\}$$
$$\langle \alpha | X_2 | \alpha \rangle = \frac{1}{2i} \langle \alpha | \left(\widehat{a} - \widehat{a}^\dagger \right) | \alpha \rangle = \frac{1}{2i} \left(\alpha - \alpha^* \right) = \mathrm{Im}\left\{ \alpha \right\}$$

D. Calculate $\left\langle X_j^2 \right\rangle_\alpha = \langle \alpha | \, \widehat{X}_j^2 | \alpha \rangle$ for $j = 1, 2$.

Solution :

$$\left\langle \widehat{X}_1^2 \right\rangle = \frac{1}{4} \langle \alpha | \, \widehat{a}\widehat{a} + \widehat{a}^\dagger \widehat{a}^\dagger + 2\widehat{a}^\dagger \widehat{a} + 1 \, | \alpha \rangle = \frac{1}{4} \left\{ \alpha^2 + (\alpha^*)^2 + 2\alpha^* \alpha + 1 \right\}$$
$$= \frac{1}{4} \left(\alpha + \alpha^* \right)^2 + \frac{1}{4} = \left(\mathrm{Re}\left\{ \alpha \right\} \right)^2 + \frac{1}{4} \ ,$$

where we have used

$$\left[\widehat{a}, \widehat{a}^\dagger \right] = 1 \Longrightarrow \widehat{a}\widehat{a}^\dagger = 1 + \widehat{a}^\dagger \widehat{a}$$

Similarly :

$$\left\langle \widehat{X}_2^2 \right\rangle = -\frac{1}{4} \langle \alpha | \, \widehat{a}\widehat{a} + \widehat{a}^\dagger \widehat{a}^\dagger - 2\widehat{a}^\dagger \widehat{a} - 1 \, | \alpha \rangle = -\frac{1}{4} \left\{ \alpha^2 + (\alpha^*)^2 - 2\alpha^* \alpha - 1 \right\}$$
$$= \frac{1}{4} \left(\frac{\alpha - \alpha^*}{i} \right)^2 + \frac{1}{4} = \left(\mathrm{Im}\left\{ \alpha \right\} \right)^2 + \frac{1}{4} \ ,$$

Consider a beam-splitter and an arbitrary state, $|\Psi\rangle$, expressed either on the basis of the input channels $|\Psi_{1,2}\rangle$ or in terms of the output channels $|\Psi_{3,4}\rangle$:

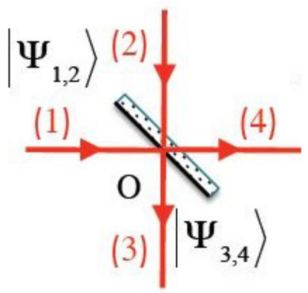

Figure 2: Transformation by a beam-splitter between input beams in channels 1 and 2 and output channels 3 and 4.

The lowering operators transform as follows:

$$\widehat{a}_3 = r\widehat{a}_1 + t\widehat{a}_2$$
$$\widehat{a}_4 = t'\widehat{a}_1 + r'\widehat{a}_2 \ ,$$

where r , r', t, t' are reflexion and transmission coefficients that in general have complex values that can be obtained by 'classical' electromagnetic calculations (or measurements) of the beam-splitter.

It is often practical to describe this transformation as an S-matrix:

$$\begin{bmatrix}\widehat{a}_3\\ \widehat{a}_4\end{bmatrix} = \begin{bmatrix} r & t \\ t' & r'\end{bmatrix}\begin{bmatrix}\widehat{a}_1\\ \widehat{a}_2\end{bmatrix} = [S]\begin{bmatrix}\widehat{a}_1\\ \widehat{a}_2\end{bmatrix}\ . \tag{15}$$

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10. Which of the following S-matrices are physically acceptables?

    A. $[S] = \begin{bmatrix} ir & t \\ t & ir\end{bmatrix}$ for r and t real, with $r^2 + t^2 = 1$.

    Correct : Since $[S]$ is unitary, *i.e.* $[S]^\dagger[S] = \mathbb{1}$

    B. $[S] = \begin{bmatrix} r & it \\ it & r\end{bmatrix}$ for r and t real, with $r^2 + t^2 = 1$.

    Correct : Since $[S]^\dagger[S] = \mathbb{1}$

    C. $[S] = \begin{bmatrix} r & t \\ t & -r\end{bmatrix}$ for r and t real with $r^2 + t^2 = 1$.

    Correct : Unlike the previous proposed $[S]$ matrices, this one is Hermitian, *i.e.* $[S]^\dagger = [S]$, but it is still ok because it is also unitary, $[S]^\dagger[S] = \mathbb{1}$.

    D. $[S] = \begin{bmatrix} r & it \\ -it & r\end{bmatrix}$ for r and t real with $r^2 + t^2 = 1$

    Incorrect This matrix is Hermitian, like in the previous case, but this time it is not unitairy :

    $$[S]^\dagger[S] = \begin{bmatrix} r & it \\ -it & r\end{bmatrix}\begin{bmatrix} r & it \\ -it & r\end{bmatrix} = \begin{bmatrix} r^2+t^2 & 2irt \\ -2irt & r^2+t^2\end{bmatrix} = \begin{bmatrix} 1 & 2irt \\ -2irt & 1\end{bmatrix} \neq \mathbb{1} \tag{16}$$

    E. $[S] = \begin{bmatrix} r_c & t_c \\ t_c & r_c\end{bmatrix}$ with r_c and t_c as complex coefficients that satisfy $r_c t_c^* = -r_c^* t_c$ and $|r_c|^2 + |t_c|^2 = 1$.

    Correct : An electromagnetic calculation with a symmetric beam-splitter consisting of a dielectric plate would give us a matrix of this form. It is important to take into account the complex nature of the coefficients if we use an S-matrix of this form. For the purposes of a quantum optics calculation, at least concerning behaviour of the beam splitters, we would obtain the same results with any of the unitary matrices from this exercise (but not with the non-unitary matrix of D.).

11. Let us consider the state $|n\rangle_1$ with n photons in channel 1. Express $|n\rangle_1$ in terms of $\widehat{a}_1^\dagger$ and the vacuum state $|0\rangle_1$:

    A. $|n\rangle_1 = \frac{1}{\sqrt{n!}}\left(\widehat{a}_1^\dagger\right)^n|0\rangle_1$ **Correct Choice**

    B. $|n\rangle_1 = \frac{1}{\sqrt{(n-1)!}}\left(\widehat{a}_1^\dagger\right)^n|0\rangle_1$

    C. $|n\rangle_1 = \sqrt{n!}\left(\widehat{a}_1^\dagger\right)^n|0\rangle_1$

12. Express $|\Psi\rangle = |n\rangle_1 \otimes |m\rangle_2 = |n,m\rangle_{1,2}$ in terms of $\widehat{a}_1^\dagger$ and $\widehat{a}_2^\dagger$ acting on the two-channel vacuum $|0,0\rangle$. Note that the vacuum doesn't depend on the choice of basis. (More than one answer may be correct)

    A. $|\Psi\rangle = \frac{1}{\sqrt{n!m!}}\widehat{a}_1^{\dagger n}\widehat{a}_2^{\dagger m}|0,0\rangle$. **This is correct**

    B. $|\Psi\rangle = \frac{1}{\sqrt{n!m!}}\widehat{a}_1^{\dagger m}\widehat{a}_2^{\dagger n}|0,0\rangle$

    C. $|\Psi\rangle = \frac{1}{\sqrt{n!m!}}\widehat{a}_2^{\dagger m}\widehat{a}_1^{\dagger n}|0,0\rangle$ **This is also correct**

13. Express the above state $|\Psi\rangle = |n,m\rangle_{1,2}$ in terms of the output state $\widehat{a}_3^\dagger$ and $\widehat{a}_4^\dagger$ operators.

A. $|\Psi\rangle = \frac{1}{\sqrt{n!m!}}\left(r^*\hat{a}_3^\dagger - t^*\hat{a}_4^\dagger\right)^n\left(t^*\hat{a}_3^\dagger + r^*\hat{a}_4^\dagger\right)^m|0,0\rangle$

B. $|\Psi\rangle = \frac{1}{\sqrt{n!m!}}\left(r^*\hat{a}_3^\dagger + t^*\hat{a}_4^\dagger\right)^n\left(t^*\hat{a}_3^\dagger + r^*\hat{a}_4^\dagger\right)^m|0,0\rangle$

C. $|\Psi\rangle = \frac{1}{\sqrt{n!m!}}\left(r\hat{a}_3^\dagger + t\hat{a}_4^\dagger\right)^n\left(t\hat{a}_3^\dagger + r\hat{a}_4^\dagger\right)^m|0,0\rangle$

Correct choice explanation : Since the S-matrix is necessarily unitairy, $S^\dagger = S^{-1}$, Eq.(15), tells us that:

$$\begin{bmatrix}\hat{a}_1\\ \hat{a}_2\end{bmatrix} = [S]^\dagger\begin{bmatrix}\hat{a}_3\\ \hat{a}_4\end{bmatrix} = \begin{bmatrix}r^* & t^*\\ t^* & r^*\end{bmatrix}\begin{bmatrix}\hat{a}_3\\ \hat{a}_4\end{bmatrix},$$

which gives us

$$\begin{aligned}\hat{a}_1 &= r^*\hat{a}_3 + t^*\hat{a}_4\\ \hat{a}_2 &= t^*\hat{a}_3 + r^*\hat{a}_4\ .\end{aligned} \tag{17}$$

Taking the adjoint of the equations (17), we have (as seen in the course) that:

$$\begin{aligned}\hat{a}_1^\dagger &= r\hat{a}_3^\dagger + t\hat{a}_4^\dagger\\ \hat{a}_2^\dagger &= t\hat{a}_3^\dagger + r\hat{a}_4^\dagger;.\end{aligned} \tag{18}$$

so the expression in C is found by substituting in these expressions into $|\Psi\rangle = \frac{1}{\sqrt{n!m!}}\hat{a}_1^{\dagger n}\hat{a}_2^{\dagger m}|0,0\rangle$.

14. Consider state $|\Psi\rangle = |1,0\rangle_{1,2}$. Give the expression of this state on the basis of the output channels.

A. $|\Psi\rangle = |1,0\rangle_{1,2} = r^*|1,0\rangle_{3,4} + t^*|0,1\rangle_{3,4}$

B. $|\Psi\rangle = |1,0\rangle_{1,2} = t^*|1,0\rangle_{3,4} + r^*|0,1\rangle_{3,4}$

C. $|\Psi\rangle = |1,0\rangle_{1,2} = r|1,0\rangle_{3,4} + t|0,1\rangle_{3,4}$

Solution : This is a direct application of 13.C with $n_1 = 1$ and $n_2 = 0$:

$$|1,0\rangle_{1,2} = \hat{a}_1^\dagger|0,0\rangle = \left(r\hat{a}_3^\dagger + t\hat{a}_4^\dagger\right)|0,0\rangle\ .$$

15. Let us consider the state, $|\Psi\rangle = |1,1\rangle_{1,2}$, correspond to exactly 1-photon in each input channel. Which is the expression for $|\Psi\rangle = |1,1\rangle_{1,2}$ basis of output channels ?

A. $|\Psi\rangle = \sqrt{2}\,rt|2,0\rangle_{3,4} + \left(t^2+r^2\right)|1,1\rangle_{3,4} + \sqrt{2}\,tr|0,2\rangle_{3,4}$.

Solution : This result follows from 13.C, because one of:

$$\begin{aligned}|\Psi\rangle = |1,1\rangle_{1,2} &= \left(r\hat{a}_3^\dagger + t\hat{a}_4^\dagger\right)\left(t\hat{a}_3^\dagger + r\hat{a}_4^\dagger\right)|0,0\rangle\\ &= rt\hat{a}_3^\dagger\hat{a}_3^\dagger|0,0\rangle + r^2\hat{a}_3^\dagger\hat{a}_4^\dagger|0,0\rangle + t^2\hat{a}_4^\dagger\hat{a}_3^\dagger|0,0\rangle + tr\hat{a}_4^\dagger\hat{a}_4^\dagger|0,0\rangle\\ &= \sqrt{2}\,rt|2,0\rangle_{3,4} + \left(t^2+r^2\right)|1,1\rangle_{3,4} + \sqrt{2}\,tr|0,2\rangle_{3,4}\end{aligned}$$

B. $|\Psi\rangle = \sqrt{2}\,r^*t^*|2,0\rangle_{3,4} + \left[(t^*)^2 + (r^*)^2\right]|1,1\rangle_{3,4} + \sqrt{2}\,t^*r^*|0,2\rangle_{3,4}$

C. $|\Psi\rangle = r^*t^*|2,0\rangle_{3,4} + \left[(t^*)^2 + (r^*)^2\right]|1,1\rangle_{3,4} + t^*r^*|0,2\rangle_{3,4}$

16. Continuing the previous question, the state $|\Psi\rangle = |1,1\rangle_{1,2}$, corresponds to exactly one photon state in each entry channel arriving at the same time. Let us consider now a symmetric 50/50 beam-splitter with $r = \frac{i}{\sqrt{2}}$ and $t = \frac{1}{\sqrt{2}}$. What is the probability of detecting one photon in each of the output channels?

A. 0

Solution : We indeed obtain the rather surprising result that there is no propabaility for the 2 photons to separate out into 2 different channels $P(1,1) = |{}_{4,3}\langle 1,1|\Psi\rangle|^2 = |{}_{4,3}\langle 1,1|1,1\rangle_{1,2}|^2 = \left|{}_{4,3}\langle 1,1|\left(t^2+r^2\right)|1,1\rangle_{3,4}\right|^2 = 0$ One says that this is an example of photon "grouping" since the photons only emerge together into either detector 3 or 4, but not separately.

B. 1/3
C. 1/4
D. 1

For the following questions, let us consider the special (but common) case where $\left[\widehat{A},\widehat{B}\right]\neq 0$ and :

$$\left[\widehat{A},\left[\widehat{A},\widehat{B}\right]\right]=0=\left[\widehat{B},\left[\widehat{A},\widehat{B}\right]\right]\ . \tag{19}$$

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17. Under the above conditions, prove the identity: $\left[\widehat{B},\widehat{A}^n\right]=n\widehat{A}^{n-1}\left[\widehat{B},\widehat{A}\right]$.

Solution : We can prove this by induction. When $n=1$:

$$\left[\widehat{B},\widehat{A}^{n=1}\right]=\widehat{A}^0\left[\widehat{B},\widehat{A}\right]=\left[\widehat{B},\widehat{A}\right]$$

Now, let us find the expression for $\left[\widehat{B},\widehat{A}^{n+1}\right]$ using the identity from Question 3. and knowing that $\left[\widehat{B},\widehat{A}^n\right]=n\widehat{A}^n\left[\widehat{B},\widehat{A}\right]$

$$\begin{aligned}\left[\widehat{B},\widehat{A}^{n+1}\right]&=\left[\widehat{B},\widehat{A}^n\widehat{A}\right]=\left[\widehat{B},\widehat{A}^n\right]\widehat{A}+\widehat{A}^n\left[\widehat{A},\widehat{B}\right]\\&=n\widehat{A}^{n-1}\left[\widehat{B},\widehat{A}\right]\widehat{A}+\widehat{A}^n\left[\widehat{B},\widehat{A}\right]\end{aligned} \tag{20}$$

Given the condition $\left[\widehat{A},\left[\widehat{A},\widehat{B}\right]\right]=0$, we know that:

$$\left[\widehat{A},\left[\widehat{B},\widehat{A}\right]\right]=0\Longrightarrow\left[\widehat{B},\widehat{A}\right]\widehat{A}=\widehat{A}\left[\widehat{B},\widehat{A}\right] \tag{21}$$

Substituting eq. (21) in eq. (20) we get

$$\begin{aligned}\left[\widehat{B},\widehat{A}^{n+1}\right]&=\left[\widehat{B},\widehat{A}^n\widehat{A}\right]=\left[\widehat{B},\widehat{A}^n\right]\widehat{A}+\widehat{A}^n\left[\widehat{B},\widehat{A}\right]\\&=n\widehat{A}^{n-1}\widehat{A}\left[\widehat{B},\widehat{A}\right]+\widehat{A}^n\left[\widehat{B},\widehat{A}\right]\\&=(n+1)\,\widehat{A}^n\left[\widehat{B},\widehat{A}\right]\qquad\text{Q.E.D.}\end{aligned} \tag{22}$$

18. Use the identity from question 17. to show that $\left[\widehat{B},e^{-\widehat{A}x}\right]=-xe^{-\widehat{A}x}\left[\widehat{B},\widehat{A}\right]$.

Solution :

$$\begin{aligned}\left[\widehat{B},e^{-\widehat{A}x}\right]&=\sum_{m=0}\frac{1}{m!}\left[\widehat{B},\left(-\widehat{A}x\right)^m\right]=\sum_{n=1}\frac{x^m(-1)^m}{m!}\left[\widehat{B},\widehat{A}^m\right]\\&=\sum_{n=1}\frac{x^m(-1)^m}{m!}m\widehat{A}^{m-1}\left[\widehat{B},\widehat{A}\right]\\&=\sum_{m=1}(-x)\frac{(-1)^{m-1}x^{m-1}\widehat{A}^{m-1}}{(m-1)!}\left[\widehat{B},\widehat{A}\right]=\sum_{n=0}(-x)\frac{\left(-x\widehat{A}\right)^n}{n!}\left[\widehat{B},\widehat{A}\right]\\&=-xe^{-\widehat{A}x}\left[\widehat{B},\widehat{A}\right]\ ,\end{aligned}$$

where we invoked the identity from Question 17. in the second line, and at the end of the third line, we made the index change $n\equiv m-1$.

19. Use the identity from Question 18. to find the following expression:

$$e^{\widehat{A}x}\widehat{B}e^{-\widehat{A}x} = \widehat{B} - x\left[\widehat{B},\widehat{A}\right] \tag{23}$$

Solution :

$$\left[\widehat{B}, e^{-\widehat{A}x}\right] = \widehat{B}e^{-\widehat{A}x} - e^{-\widehat{A}x}\widehat{B} = -xe^{-\widehat{A}x}\left[\widehat{B},\widehat{A}\right]$$

Multiply both sides from the left by $e^{\widehat{A}x}$ to find:

$$e^{\widehat{A}x}\widehat{B}e^{-\widehat{A}x} - \widehat{B} = -x\left[\widehat{B},\widehat{A}\right] = x\left[\widehat{A},\widehat{B}\right]$$
$$\implies e^{\widehat{A}x}\widehat{B}e^{-\widehat{A}x} = \widehat{B} + x\left[\widehat{A},\widehat{B}\right]$$

20. We define an operator $\widehat{O}(x) \equiv e^{\widehat{A}x}e^{\widehat{B}x}$. Compute the derivative, $\frac{d\widehat{O}}{dx}$, and use the expression from Eq.(23) to show that:

$$\frac{d\widehat{O}}{dx} = \left(\widehat{A} + \widehat{B} - x\left[\widehat{B},\widehat{A}\right]\right)\widehat{O}(x) \tag{24}$$

Solution :

$$\begin{aligned}\frac{d}{dx}\widehat{O}(x) = \frac{d}{dx}e^{\widehat{A}x}e^{\widehat{B}x} &= \widehat{A}e^{\widehat{A}x}e^{\widehat{B}x} + e^{\widehat{A}x}\widehat{B}e^{\widehat{B}x} \\ &= \left(\widehat{A} + e^{\widehat{A}x}\widehat{B}e^{-\widehat{A}x}\right)e^{\widehat{A}x}e^{\widehat{B}x} \\ &= \left(\widehat{A} + \widehat{B} - x\left[\widehat{B},\widehat{A}\right]\right)\widehat{O}(x)\end{aligned} \tag{25}$$

21. Show that an integral over x of Eq.(24) that :

$$e^{\widehat{A}}e^{\widehat{B}} = e^{\widehat{A}+\widehat{B}}e^{\frac{1}{2}[\widehat{A},\widehat{B}]}\,. \tag{26}$$

Solution : We first divide both sides of Eq.(24) by $\widehat{O}(x)$ and multiplying each side by dx and then:

$$\int_0^1 \frac{d\widehat{O}}{\widehat{O}} = \int_0^1 d\ln\widehat{O} = \int_0^1 \left(\widehat{A} + \widehat{B} - x\left[\widehat{B},\widehat{A}\right]\right)dx$$

$$\implies \ln\widehat{O}(x=1) - \ln\widehat{O}(x=0) = \left(\widehat{A} + \widehat{B} - \frac{1}{2}\left[\widehat{B},\widehat{A}\right]\right)$$
$$\implies \ln e^{\widehat{A}}e^{\widehat{B}} = \left(\widehat{A} + \widehat{B} - \frac{1}{2}\left[\widehat{B},\widehat{A}\right]\right) \implies e^{\widehat{A}}e^{\widehat{B}} = e^{\widehat{A}+\widehat{B}}e^{\frac{1}{2}[\widehat{A},\widehat{B}]}$$

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22. Consider a single-mode displacement operator, $\widehat{D}(\alpha) \equiv \exp\bigl(\alpha\widehat{a}^\dagger - \alpha^*\widehat{a}\bigr)$.

A. Use the disentangling theorem you derived in eq.(26), i.e. $e^{(\widehat{A}+\widehat{B})} = e^{-\frac{1}{2}[\widehat{A},\widehat{B}]}e^{\widehat{A}}e^{\widehat{B}}$ (valid when $[\widehat{A},[\widehat{A},\widehat{B}]] = [\widehat{B},[\widehat{A},\widehat{B}]] = 0$), in order to show that:

$$\widehat{D}\left(\alpha\right) = e^{-\frac{1}{2}|\alpha|^2}e^{+\alpha\widehat{a}^\dagger}e^{-\alpha^*\widehat{a}}\,, \tag{27}$$

Solution : We simply make the replacements $\widehat{A} \equiv \alpha\widehat{a}^\dagger$, $\widehat{B} \equiv -\alpha^*\widehat{a}$ and use the commutation relation, $[\widehat{a},\widehat{a}^\dagger]$ to find that:

$$[\widehat{A},\widehat{B}] = -[\alpha\widehat{a}^\dagger, \alpha^*\widehat{a}] = -|\alpha|^2[\widehat{a}^\dagger,\widehat{a}] = |\alpha|^2$$

.

B. Use eq.(27) to show that the displacement operator, $\widehat{D}(\alpha)$, acting on the vacuum state, $|0\rangle$, generates a coherent state, i.e. $\widehat{D}(\alpha)|0\rangle = |\alpha\rangle$ (where $\widehat{a}|\alpha\rangle = \alpha|\alpha\rangle$ and $\langle\alpha|\alpha\rangle = 1$). (For this demonstration, we recall the properties that $\widehat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$, and $\widehat{a}|n\rangle = \sqrt{n}|n-1\rangle$).

Solution :

$$e^{-\alpha^*\widehat{a}} = \sum_{n=0}^{\infty}\frac{(-\alpha^*\widehat{a})^n}{n!}$$

but since $\widehat{a}^n|0\rangle = 0$

$$e^{-\alpha^*\widehat{a}}|0\rangle = |0\rangle \ .$$

Developing out the exponential

$$e^{\alpha\widehat{a}^\dagger} = \sum_{n=0}^{\infty}\frac{\left(\alpha\widehat{a}^\dagger\right)^n}{n!}$$

We then calculate

$$e^{-|\alpha|^2/2}e^{\alpha\widehat{a}^\dagger}|0\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle = |\alpha\rangle \ ,$$

where we used $\left(\widehat{a}^\dagger\right)^n|0\rangle = \sqrt{n!}\,|n\rangle$. This is indeed the coherent state since we can verify that (we can ignore the normalization in this demonstration) :

$$\widehat{a}|\alpha\rangle = \sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\widehat{a}|n\rangle = \sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\sqrt{n}|n-1\rangle h = \alpha\sum_{n=1}^{\infty}\frac{\alpha^{n-1}}{\sqrt{(n-1)!}}|n-1\rangle$$
$$= \alpha\sum_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{m!}}|m\rangle = \alpha|\alpha\rangle$$

23. Let us consider a single-mode electromagnetic field in a squeezed vacuum state with the quadrature operator $\widehat{X}_1$ squeezed.

1. What is the defining property of a squeezed state in terms of quadrature uncertainties?
Solution :

$$\Delta X_i < \frac{1}{2} \tag{28}$$

for either $i = 1$ or $i = 2$ (but never both).

2. When the uncertainty in $\widehat{X}_1$ is reduced below the vacuum level of the vacuum, what happens to the uncertainty in the conjugate quadrature $\widehat{X}_2$? Explain briefly.
Solution :

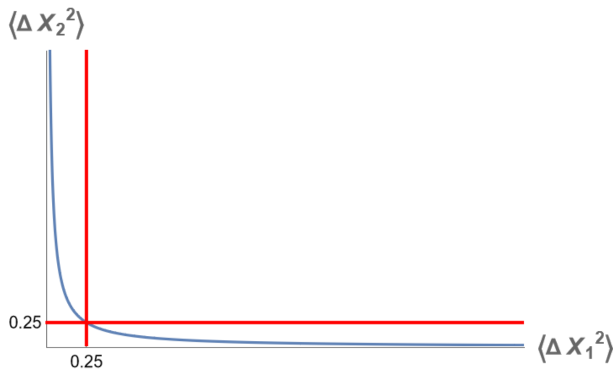
$$\Delta X_i < \frac{1}{2} \tag{29}$$

for either $i = 1$ or $i = 2$ (but never both).

3. Consider the following graph for regions of, $\langle\Delta X_1^2\rangle$ and $\langle\Delta X_2^2\rangle$ for quantum states. Indicate which regions correspond to squeezed states (1), which region corresponds to non-squeezed quantum or classical averages (2), and which region (3) corresponds to forbidden quadrature fluctuation values.

Let us recall that a coherent state $|\alpha\rangle$ has a Poissonian photon number distribution with :

$$(\Delta n)^2 = \langle\widehat{n}\rangle \ . \tag{30}$$



We saw in class that the squeezing operator can be written:

$$\widehat{S}(\xi) = \exp\left(\xi^* \widehat{a}^2 - \xi a^{\dagger,2}\right) \tag{31}$$

with the complex number, $\xi = r^{i\theta}$, being the squeezing parameter. A squeezed vacuum is generated by

$$|0,\xi\rangle \equiv \widehat{S}(\xi)|0\rangle \tag{32}$$

Squeezed lowering and raising operators are then respectively defined as $\widehat{A}_\xi \equiv \widehat{S}(\xi)\widehat{a}\widehat{S}^\dagger(\xi)$ and $\widehat{A}_\xi^\dagger \equiv \widehat{S}(\xi)\widehat{a}^\dagger\widehat{S}^\dagger(\xi)$.

As you saw in class, one can calculate expectation values of quantum operators for a squeezed vacuum using the transformation relations:

$$\begin{aligned} \widehat{a} &= \widehat{A}_\xi \cosh(r) - e^{i\theta}\widehat{A}_\xi^\dagger \sinh(r) \\ \widehat{a}^\dagger &= \widehat{A}_\xi^\dagger \cosh(r) - e^{-i\theta}\widehat{A}_\xi \sinh(r) \end{aligned} \tag{33}$$

and the properties that, $\widehat{A}_\xi|0,\xi\rangle = 0$, $\langle 0,\xi|\widehat{A}_\xi^\dagger = 0$ and $\left[\widehat{A}_\xi, \widehat{A}_\xi^\dagger\right] = 1$.

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24. **Photon Number Fluctuations of a squeezed state:** You will demonstrate in this exercise that squeezed states, like squeezed vacuum states $|0,\xi\rangle$ exhibit super-Poissonian photon statistics. **Hint:** In order to simplify the following calculations, you can take $\theta = 0$ since the values you will calculate won't depend on its value.

    1. Use the above reminders, and in particular Eqs.(32) and (33) to calculate the mean photon number of a squeezed vacuum state (in function of r):

    $$\overline{n}_{0_\xi} \equiv \langle\widehat{N}\rangle_{0_\xi} \equiv \langle 0,\xi|\widehat{a}^\dagger\widehat{a}|0,\xi\rangle \tag{34}$$

    Solution :

    $$\begin{aligned} \langle\widehat{N}\rangle_{0_\xi} &= \langle 0_s|\hat{a}^\dagger\hat{a}|0_s\rangle = \langle 0_s|\left(\hat{A}^\dagger\cosh r - e^{-i\theta}\hat{A}\sinh r\right)\left(\hat{A}\cosh r - e^{i\theta}\hat{A}^\dagger\sinh r\right)|0_s\rangle \\ &= \langle 0_s|\left(-e^{-i\theta}\hat{A}\sinh r\right)\left(-e^{i\theta}\hat{A}^\dagger\sinh r\right)|0_s\rangle \\ &= \sinh^2 r\,\langle 0_s|\hat{A}\hat{A}^\dagger|0_s\rangle = \sinh^2 r \end{aligned}$$

    Since

    $$\langle 0_s|\hat{A}\hat{A}^\dagger|0_s\rangle = \langle 1_s|1_s\rangle = 1\ ,$$

    alternatively one can use the commutator to put the operators in normal order:

    $$\langle 0_s|\hat{A}\hat{A}^\dagger|0_s\rangle = \langle 0_s|\left(1 + \hat{A}^\dagger\hat{A}\right)|0_s\rangle = 1\ .$$

    $$\boxed{\langle\widehat{N}\rangle_{0_\xi} = \sinh^2 r}$$

2. Calculate now $\langle \widehat{N}^2 \rangle_{0_\xi}$ for the squeezed vacuum :

$$\langle \widehat{N}^2 \rangle_{0_\xi} \equiv \langle 0, \xi | \, \widehat{N}^2 \, | 0, \xi \rangle \tag{35}$$

Solution :

$$\langle \widehat{N}^2 \rangle_{0_\xi} = \langle 0_\xi | \, \widehat{a}^\dagger \widehat{a} \widehat{a}^\dagger \widehat{a} \, | 0, \xi \rangle = \langle 0_\xi | \, \widehat{a}^\dagger \widehat{a}^\dagger \widehat{a} \widehat{a} \, | 0, \xi \rangle + \langle 0_\xi | \, \widehat{a}^\dagger \widehat{a} \, | 0, \xi \rangle \tag{36}$$

The rest of this solution can be found in the final pages of course notes

3. Use the above results to determine the photon fluctuations, $\langle \Delta \widehat{N}^2 \rangle_{0_\xi} = \langle \widehat{N}^2 \rangle_{0_\xi} - \left(\langle \widehat{N} \rangle_{0_\xi} \right)^2$. Compare your result to the coherent state fluctuations given in Eq.(30), and explain why this is indeed a super-Poissonian distribution.