

Part 3 :

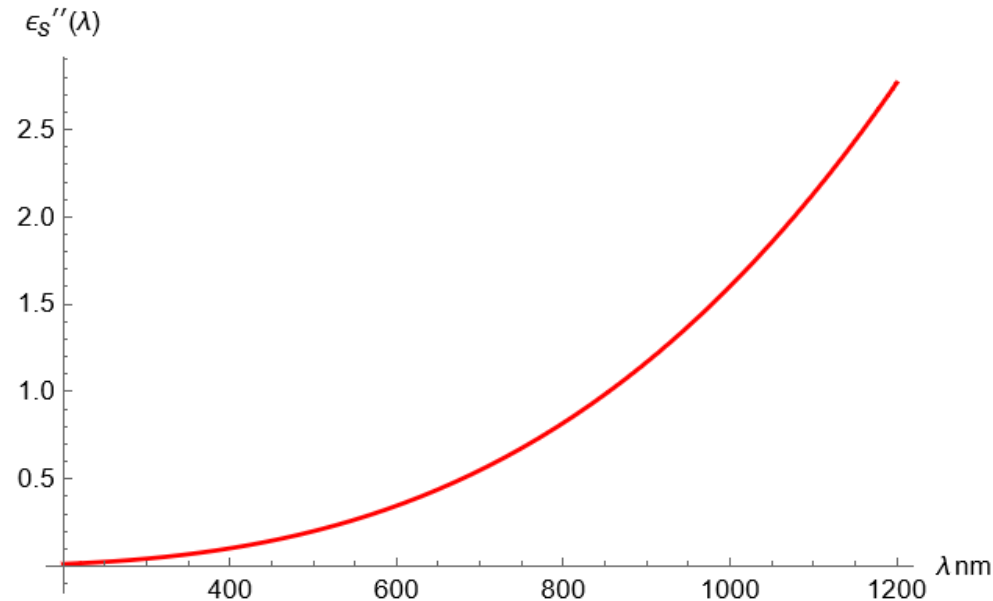
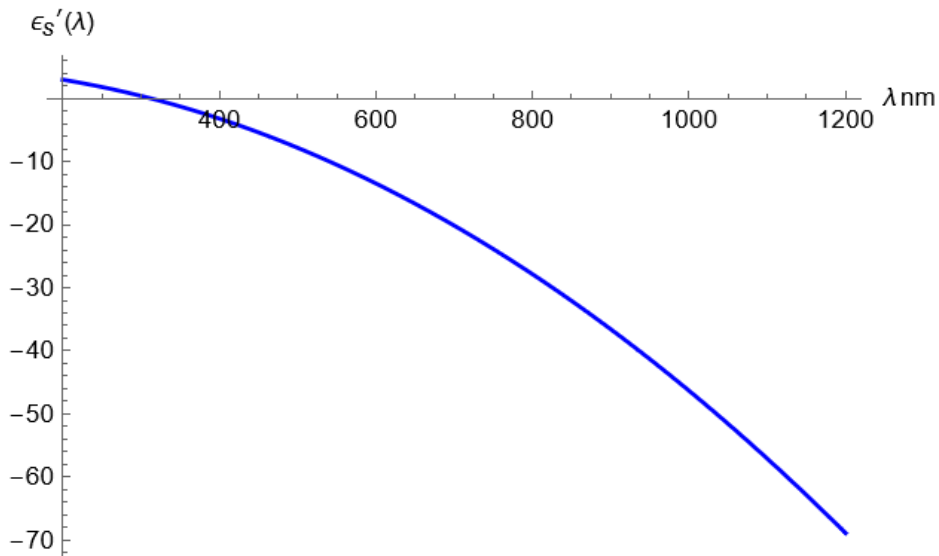
- Light matter interactions in terms of response functions
- Scattering theory :
 - Applications and basics of Mie theory
 - Electric polarizability theory for small scatterers
 - Radiation reactions
- Multipole theory :
 - Multipole basis functions
 - Expansions of Green's functions
 - Purcell factor/Decay rate modifications

Drude-Lorentz model of material media

Frequency *dispersion* of permittivity

$$\varepsilon(\omega) = \varepsilon_{\text{DC}} - \frac{\omega_{\text{pl}}^2}{\omega(\omega + i\Gamma)} - \sum_{\alpha=1}^N \frac{\omega_{\text{pT},\alpha}^2}{\omega^2 + 2i\gamma_{\alpha}\omega - \omega_{T,\alpha}^2}$$

$$\varepsilon(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega)$$

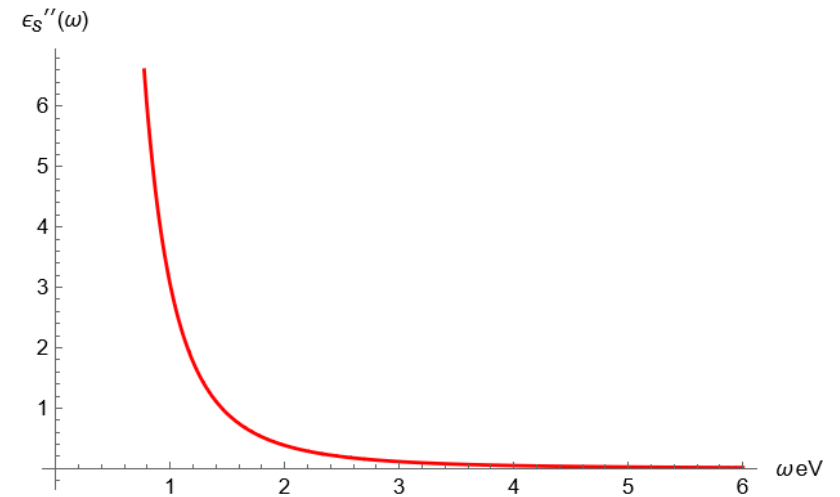
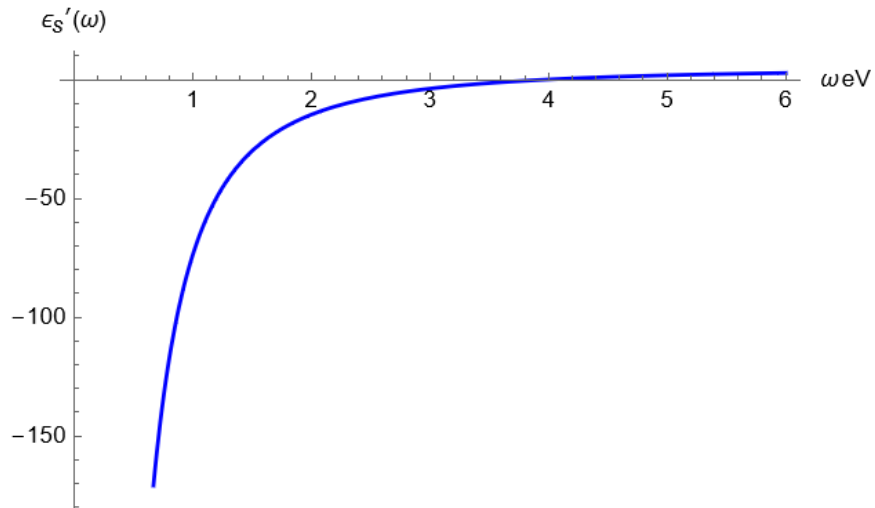


Drude model for the permittivity of silver (Ag)

Drude-Lorentz model of material media

$$\varepsilon(\omega) = \varepsilon_{\text{DC}} - \frac{\omega_{\text{pl}}^2}{\omega(\omega + i\Gamma)} - \sum_{\alpha=1}^N \frac{\omega_{\text{pT},\alpha}^2}{\omega^2 + 2i\gamma_{\alpha}\omega - \omega_{T,\alpha}^2}$$

$$\varepsilon(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega)$$



Drude model for the permittivity of silver (Ag)

Electric constitutive relations (adimensioned units)

$$\mathbf{D}(\mathbf{r}, t) = \int_{-\infty}^t \vec{\varepsilon}(\mathbf{r}, t - t') \cdot \mathbf{E}(\mathbf{r}, t') dt'$$

Homogeneous media



$$\mathbf{D}(\mathbf{r}, t) = \int_{-\infty}^t \vec{\varepsilon}(t - t') \cdot \mathbf{E}(\mathbf{r}, t') dt'$$

Isotropic media



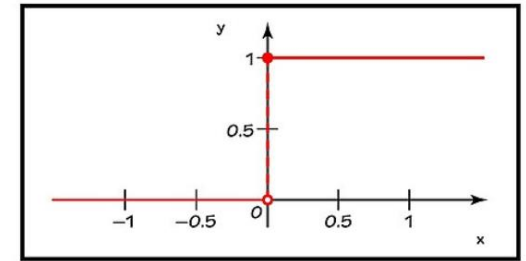
$$\mathbf{D}(\mathbf{r}, t) = \int_{-\infty}^t \varepsilon(t - t') \mathbf{E}(\mathbf{r}, t') dt'$$

Convolution integral
(with Heaviside $\Theta(t)$)



$$\mathbf{D}(\mathbf{r}, t) = \int_{-\infty}^{\infty} \Theta(t) \varepsilon(t - t') \mathbf{E}(\mathbf{r}, t') dt'$$

$\Theta(t)$



The Heaviside Function

Fourier transform

$$\tilde{\varepsilon}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \Theta(t) \varepsilon(t)$$

$$\tilde{\mathbf{D}}(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \mathbf{D}(\mathbf{r}, t)$$

$$\tilde{\mathbf{E}}(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \mathbf{E}(\mathbf{r}, t)$$

$$\mathbf{D}(t) = \int_{-\infty}^{\infty} \Theta(t) \varepsilon(t - t') \mathbf{E}(\mathbf{r}, t') dt' \quad \longrightarrow \quad \tilde{\mathbf{D}}(\mathbf{r}, \omega) = \tilde{\varepsilon}(\omega) \tilde{\mathbf{E}}(\mathbf{r}, \omega) = \{1 + \tilde{\chi}_e(\omega)\} \tilde{\mathbf{E}}(\mathbf{r}, \omega)$$

Susceptibility : $\tilde{\chi}_e(\omega)$ $\tilde{\epsilon}(\omega) = 1 + \tilde{\chi}_e(\omega)$

Fourier transform

$$\tilde{\chi}_e(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \Theta(t) \chi_e(t)$$

$$\tilde{\mathbf{D}}(\mathbf{r}, \omega) = \tilde{\epsilon}(\omega) \tilde{\mathbf{E}}(\mathbf{r}, \omega) = \{1 + \tilde{\chi}_e(\omega)\} \tilde{\mathbf{E}}(\mathbf{r}, \omega)$$

$\chi_e(t)$ is a real – valued function  $\tilde{\chi}_e(-\omega^*) = \tilde{\chi}_e^*(\omega)$

Inverse Fourier transform

$$\chi_e(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} \tilde{\chi}_e(\omega)$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{-i\omega t} \tilde{\mathbf{E}}(\mathbf{r}, \omega)$$

Time harmonic formalism : $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0(\mathbf{r}) \cos(\omega t - \varphi) = \frac{1}{2} \{ \mathbf{E}_0(\mathbf{r}) e^{i\varphi} e^{-i\omega t} + \mathbf{E}_0(\mathbf{r}) e^{-i\varphi} e^{i\omega t} \}$
 $= \text{Re} \{ \mathbf{E}_0(\mathbf{r}) e^{i\varphi} e^{-i\omega t} \}$

Magnetic constitutive relations

(adimensioned units)

$$\mathbf{B}(\mathbf{r}, t) = \int_{-\infty}^t \vec{\mu}(\mathbf{r}, t - t') \cdot \mathbf{H}(\mathbf{r}, t') dt'$$

Homogeneous media



$$\mathbf{B}(\mathbf{r}, t) = \int_{-\infty}^t \vec{\mu}(t - t') \cdot \mathbf{H}(\mathbf{r}, t') dt'$$

Isotropic media

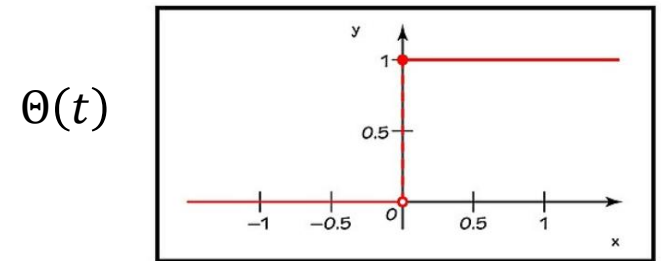


$$\mathbf{B}(\mathbf{r}, t) = \int_{-\infty}^t \mu(t - t') \mathbf{H}(\mathbf{r}, t') dt'$$

Convolution integral
(with Heaviside $\Theta(t)$)



$$\mathbf{B}(\mathbf{r}, t) = \int_{-\infty}^{\infty} \Theta(t) \mu(t - t') \mathbf{H}(\mathbf{r}, t') dt'$$



The Heaviside Function

Fourier transform

$$\tilde{\mu}(\omega) = \int_{-\infty}^{\infty} d\omega e^{i\omega t} \Theta(t) \mu(t)$$

$$\tilde{\mathbf{B}}(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} d\omega e^{i\omega t} \mathbf{B}(\mathbf{r}, t)$$

$$\tilde{\mathbf{H}}(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} d\omega e^{i\omega t} \mathbf{H}(\mathbf{r}, t)$$

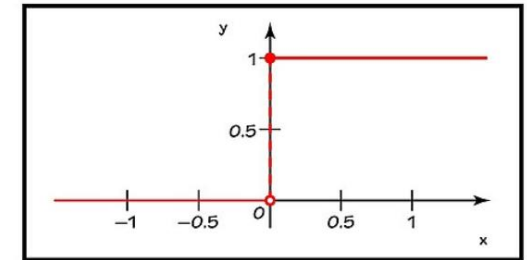
$$\tilde{\mathbf{B}}(\mathbf{r}, \omega) = \tilde{\mu}(\omega) \tilde{\mathbf{H}}(\mathbf{r}, \omega) = \{1 + \tilde{\chi}_m(\omega)\} \tilde{\mathbf{H}}(\mathbf{r}, \omega)$$

In most materials at optical frequencies $\chi_m \sim 0$

Kramers-Krönig relations : causality

Theory of distributions

$$\Theta(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \Theta(t) = \int_0^{\infty} dt e^{i\omega t} \Theta(t) = \pi\delta(\omega) + \text{P.V.} \frac{i}{\omega}$$



The Heaviside Function
 $\Theta(t)$

Causality: $\chi(t) = \chi(t)\Theta(t) \longrightarrow \chi(\omega) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \chi(\omega')\Theta(\omega - \omega')$

$\longrightarrow \chi(\omega) = \frac{1}{i\pi} \text{P.V.} \int_{-\infty}^{\infty} \frac{\chi(\omega')}{\omega' - \omega} d\omega'$

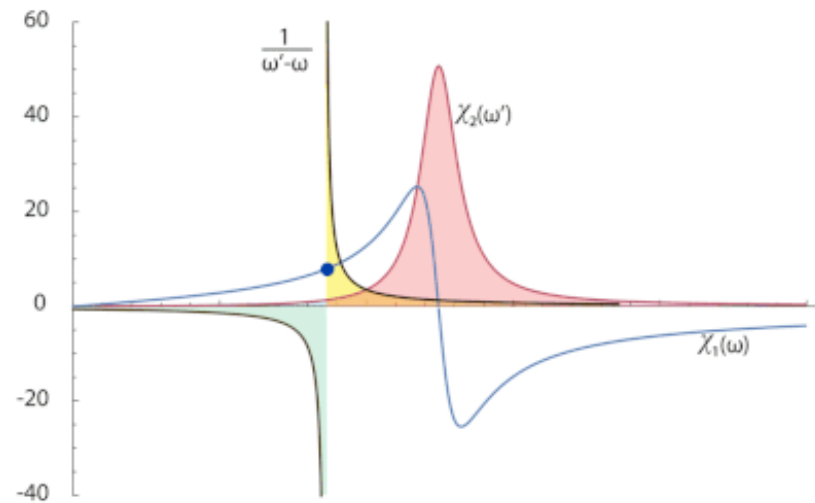
$$\chi(\omega) = \chi'(\omega) + i\chi''(\omega)$$

Kramers-Krönig relations : causality

$$\varepsilon(\omega) = 1 + \chi(\omega) \quad \chi(\omega) = \chi'(\omega) + i\chi''(\omega) = \chi_1(\omega) + i\chi_2(\omega)$$

$$\chi'(\omega) = \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\chi''(\omega')}{\omega' - \omega} d\omega'$$

$$\chi''(\omega) = -\frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\chi'(\omega')}{\omega' - \omega} d\omega'$$

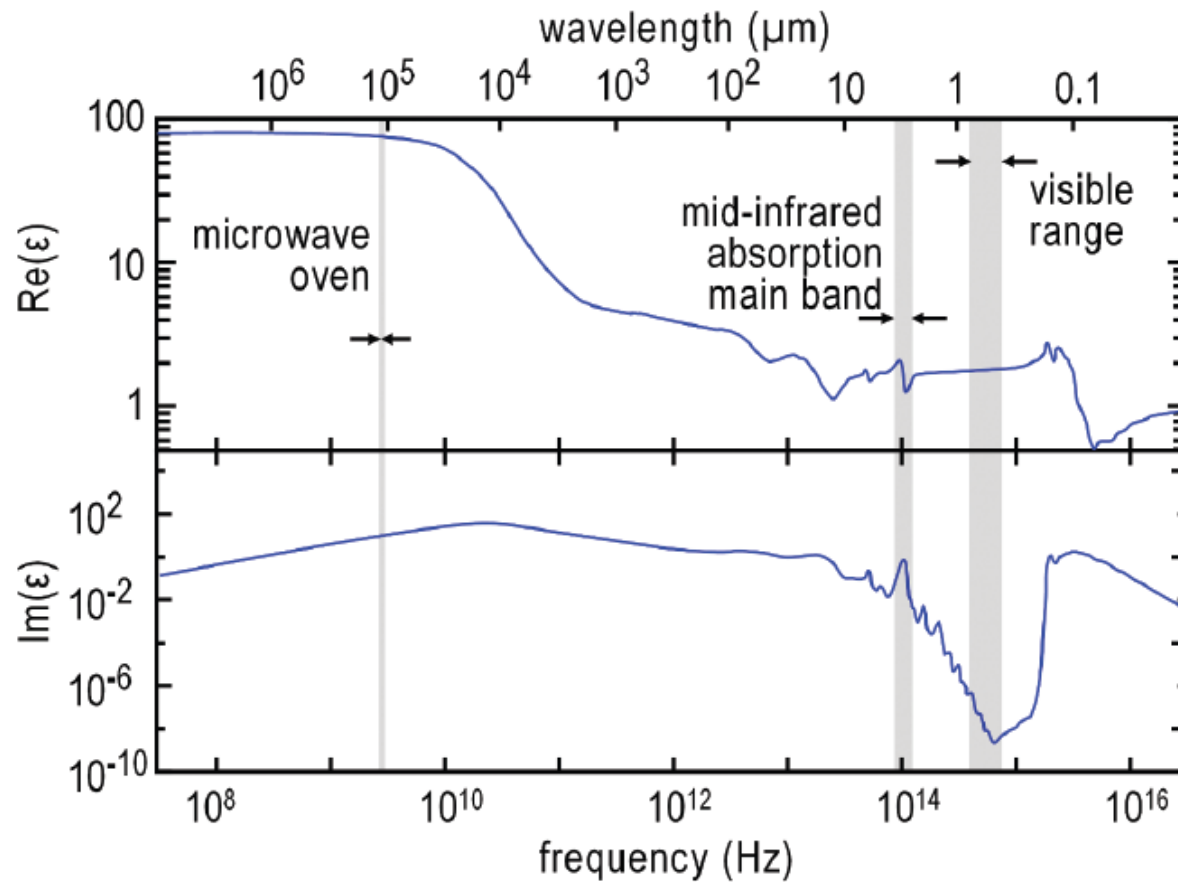


$$\tilde{\chi}_e(-\omega^*) = \tilde{\chi}_e^*(\omega) \quad \longrightarrow$$

$$\chi'(\omega) = \frac{2}{\pi} \mathcal{P} \int_0^{\infty} \frac{\omega' \chi''(\omega')}{\omega'^2 - \omega^2} d\omega'$$

$$\chi''(\omega) = -\frac{2\omega}{\pi} \mathcal{P} \int_0^{\infty} \frac{\chi'(\omega')}{\omega'^2 - \omega^2} d\omega'$$

Permittivity of actual materials is complicated



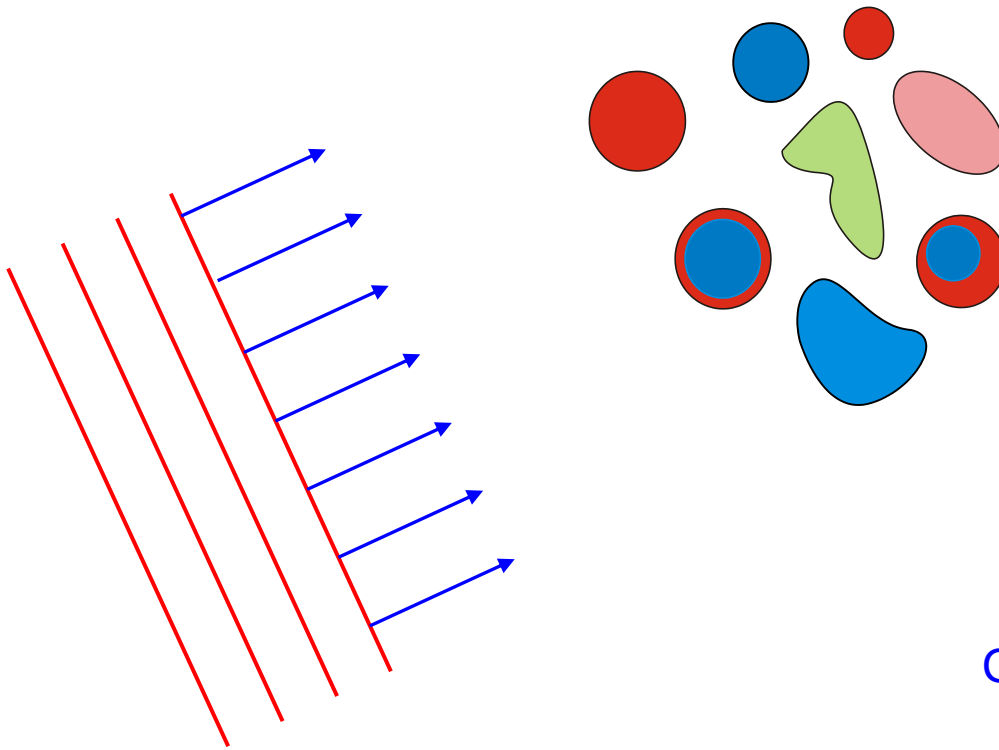
permittivity of water

How do simulate light scattering in the “real world” ?



“Seeing is believing and all we see is scattered light” J.C. Stover

Simplification : piecewise continuous media



Constitutive parameters

$$\mathbf{D}_{\omega}(\vec{\mathbf{r}}) = \vec{\epsilon}(\omega) \cdot \mathbf{E}_{\omega}(\vec{\mathbf{r}})$$

$$\mathbf{H}_{\omega}(\vec{\mathbf{r}}) = \vec{\mu}(\omega) \cdot \mathbf{B}_{\omega}(\vec{\mathbf{r}})$$

Scattering by a spherically symmetric object

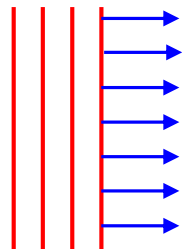
Lorenz - Mie - Debye theory

(1890)

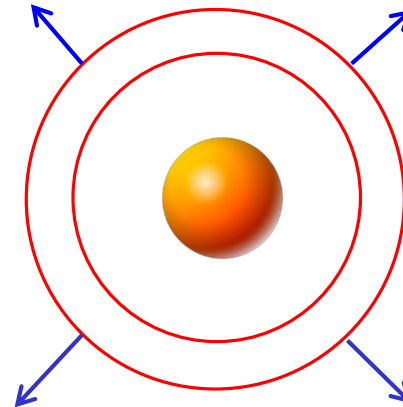
(1908)

(1909)

Incident - 'excitation' field :



Outgoing scattered field



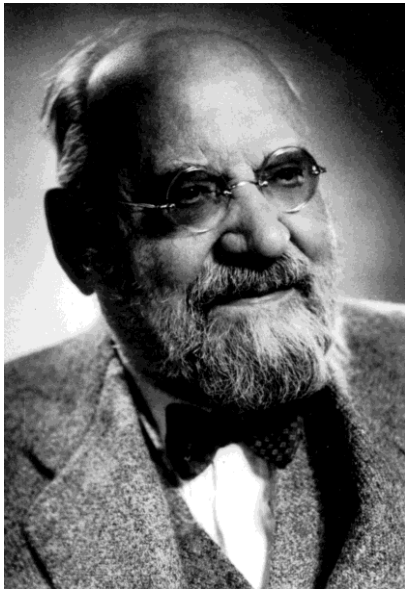
Exact solution !



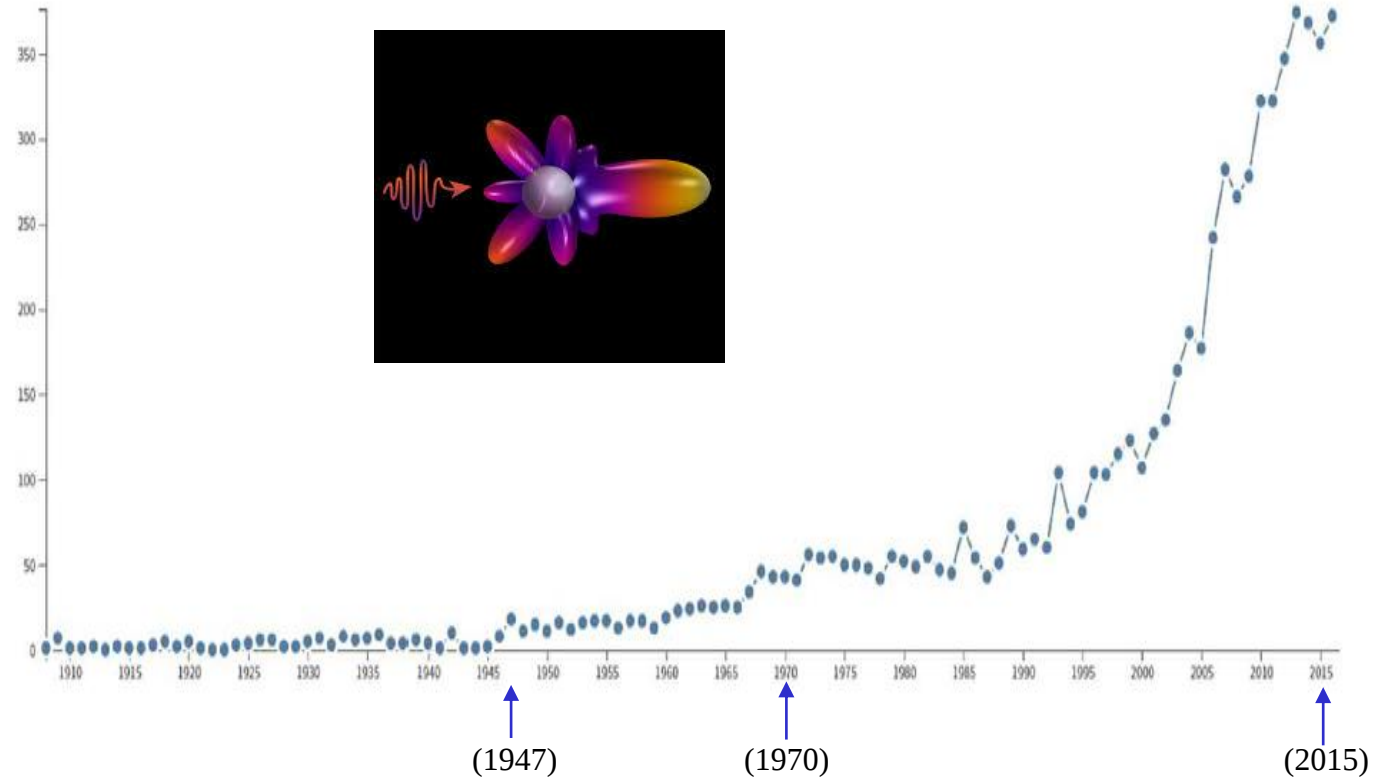
Lorenz(1890)-Mie(1908)-Debye(1909) theory ?



Citations per year



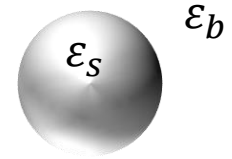
Gustav Mie



Gustav Mie (1868-1957) “**Contributions to the Optics of Turbid Media, particularly of colloidal metal solutions**”
Translation (Royal Aircraft Establishment (1976). (1908)

Ludvig Lorenz (1829–91) “**Light scattering and reflection by a transparent sphere (surface)**”
in Oeuvres scientifiques de L. Lorenz. 1898, p 403-529 (1890).

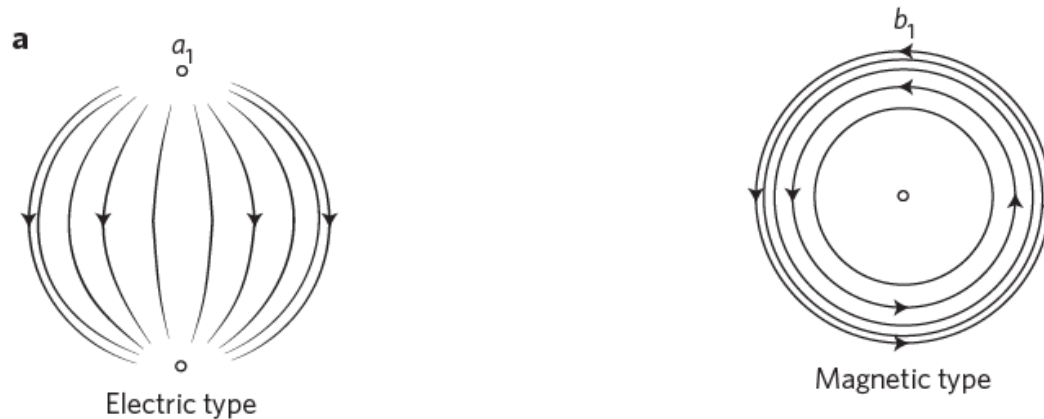
Mie theory : multipole expansion



Dielectric scatterers are often called '**Mie resonators**' but Mie theory is applicable to arbitrary temporal dispersion.

(In fact, Gustav Mie developed his theory to describe dispersive particles)

Fields expanded on the **multipolar** basis



Mie, G. *Ann. Phys.* **25**, 377 (1908).

A lot of physics is hidden in the Mie coefficients

Long winded to derive but easy to use!

$$a_n = \frac{\frac{\epsilon_s}{\epsilon_b} j_n(k_s R) \psi'_n(kR) - \psi'_n(k_s R) j_n(kR)}{\frac{\epsilon_s}{\epsilon_b} j_n(k_s R) \xi'_n(kR) - \psi'_n(k_s R) h_n^{(1)}(kR)}$$

$$b_n = \frac{\frac{\mu_s}{\mu_b} j_n(k_s R) \psi'_n(kR) - \psi'_n(kR) j_n(kR)}{\frac{\mu_s}{\mu_b} \psi_n(k_s R) \xi'_n(kR) - \psi'_n(k_s R) h_n^{(1)}(kR)}$$

$$\psi_n(x) \equiv x j_n(x)$$

$$\xi_n(x) \equiv x h_n^{(1)}(x)$$

$$\frac{k_s}{k} \equiv \frac{N_s}{N}$$

Cross sections : σ

Extinction :

$$\sigma_e = \frac{2\pi}{k^2} \sum_{n=1}^{\infty} (2n+1) \text{Re}\{a_n + b_n\}$$

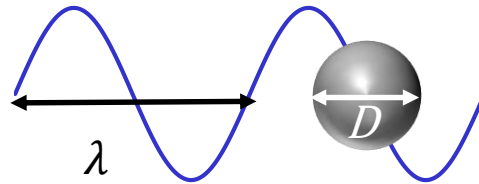
Scattering :

$$\sigma_s = \frac{2\pi}{k^2} \sum_{n=1}^{\infty} (2n+1) (|a_n|^2 + |b_n|^2)$$

Absorption :

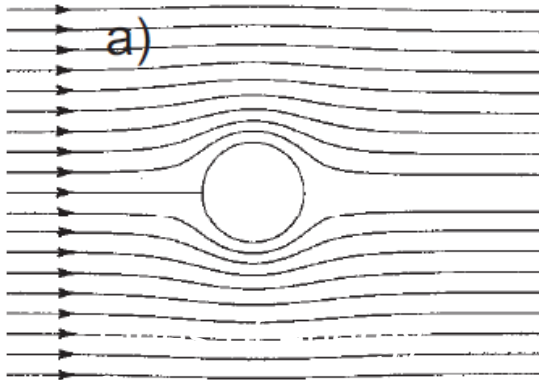
$$\sigma_a = \sigma_e - \sigma_s$$

Interaction of light with subwavelength structures



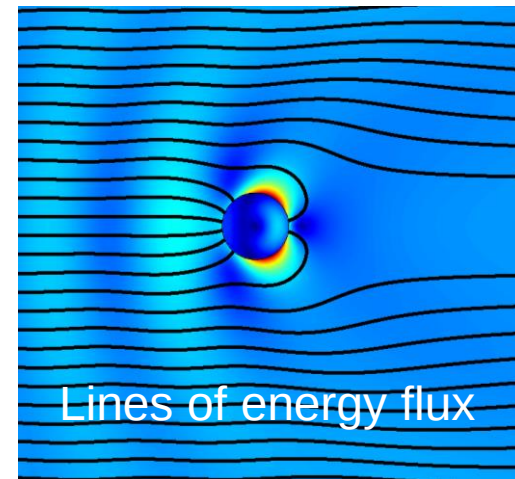
$$D < \lambda$$

Weak interaction



Bohren, C. F., & Huffman, D. R. (2008). *Absorption and scattering of light by small particles*. John Wiley & Sons.

Resonant optical interaction



Extinction cross sections

$$S_{\text{ext}} + S_{\text{inc}} = S_{\text{tot}} - S_{\text{scat}}$$

Mie theory from nano to milli-metric particles

Rayleigh scattering



Glory – backscattering



$$\sigma_p = \sigma_e + \frac{4\pi}{k^2} \left[\frac{n(n+2)}{n+1} \operatorname{Re}\{a_n a_{n+1}^* + b_n b_{n+1}^*\} + \frac{2n+1}{n(n+1)} \operatorname{Re}\{a_n b_n^*\} \right]$$

Radiation pressure



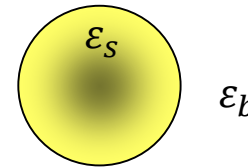
Rainbows, clouds



Photonics : polarizability approach light interacting with a small material sphere)

Electric dipole moment and polarizability, $\alpha(\omega)$

$$\vec{p} = \epsilon_0 \epsilon_b \alpha(\omega) \vec{E}_{\text{exc}}$$



Quasi-static polarizability

$$\alpha_0(\omega) \equiv \lim_{kR \rightarrow 0} \alpha(\omega) = 4\pi R^3 \frac{\epsilon_s - \epsilon_b}{\epsilon_s + 2\epsilon_b}$$

Quasi-static limit of the electric dipole in Mie theory

$$a_n = -t_n^{(e)} \quad b_n = -t_n^{(h)} \quad x = kR$$

$$\lim_{x \rightarrow 0} a_n(x) = - \frac{i x^{2n+1} (n+1)}{(2n-1)!! (2n+1)!!} \frac{\frac{\epsilon_s}{\epsilon_b} - 1}{\frac{\epsilon_s}{\epsilon_b} n + (n+1)}$$

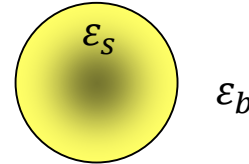
Electric dipole term ($n=1$) dominates for small particle sizes :

$$\begin{aligned} \lim_{kR \rightarrow 0} t_n^{(e)}(kR) &\rightarrow i(kR)^3 \frac{2}{3} \frac{\epsilon_s - \epsilon_b}{\epsilon_s + 2\epsilon_b} = \frac{ik^3}{4\pi} \frac{2}{3} 4\pi R^3 \frac{\epsilon_s - \epsilon_b}{\epsilon_s + 2\epsilon_b} \\ &= \frac{ik^3}{4\pi} \frac{2}{3} \alpha(\omega) \end{aligned}$$

Photonics : polarizability approach to cross section (case of a material sphere)

Electric dipole moment and polarizability, $\alpha(\omega)$

$$\mathbf{p} = \epsilon_0 \epsilon_b \alpha(\omega) \mathbf{E}_{\text{exc}}$$



$$k = \frac{2\pi}{\lambda_b} = \frac{\omega}{c_b}$$

$$\alpha_0(\omega) \equiv \lim_{kR \rightarrow 0} \alpha(\omega) = 4\pi R^3 \frac{\epsilon_s - \epsilon_b}{\epsilon_s + 2\epsilon_b}$$

$$P_{s,e,a} = \sigma_{s,e,a} I_{\text{inc}}$$

$$I_{\text{inc}} \propto \|\mathbf{E}_{\text{inc}}\|^2$$

Cross sections :

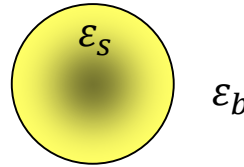
$$\sigma_{\text{ext}} = k \text{Im}\{\alpha(\omega)\} \quad \sigma_{\text{scat}} = \frac{k^4}{6\pi} |\alpha(\omega)|^2 \quad \sigma_{\text{abs}} \equiv \sigma_{\text{ext}} - \sigma_{\text{scat}}$$

There are limits to cross sections !

Radiation reaction

Electric dipole moment and polarizability, $\alpha(\omega)$

$$\mathbf{p} = \epsilon_0 \epsilon_b \alpha(\omega) \mathbf{E}_{\text{exc}}$$



$$\alpha_0(\omega) \equiv \lim_{kR \rightarrow 0} \alpha(\omega) = 4\pi R^3 \frac{\epsilon_s - \epsilon_b}{\epsilon_s + 2\epsilon_b}$$

Infinite ? dielectric response
When $\epsilon_s = -2\epsilon_b$

No !

Unitarity (energy conservation) imposes : $\alpha(\omega) = \frac{A(\omega)}{1 - i \frac{k^3}{6\pi} A(\omega)} \longrightarrow \lim_{\omega \rightarrow 0} \alpha(\omega) = \frac{\alpha_0}{1 - i \frac{k^3}{6\pi} \alpha_0}$

$$k \equiv \frac{2\pi}{\lambda_b} = \frac{\omega}{c_b}$$

$$|\alpha(\omega)| \leq \frac{6\pi}{k^3}$$

$$\sigma_{\text{scat}} = \frac{k^4}{6\pi} |\alpha(\omega)|^2 \leq \frac{6\pi}{k^2} = \frac{3\lambda^2}{2\pi} \cong \frac{\lambda_b^2}{2}$$

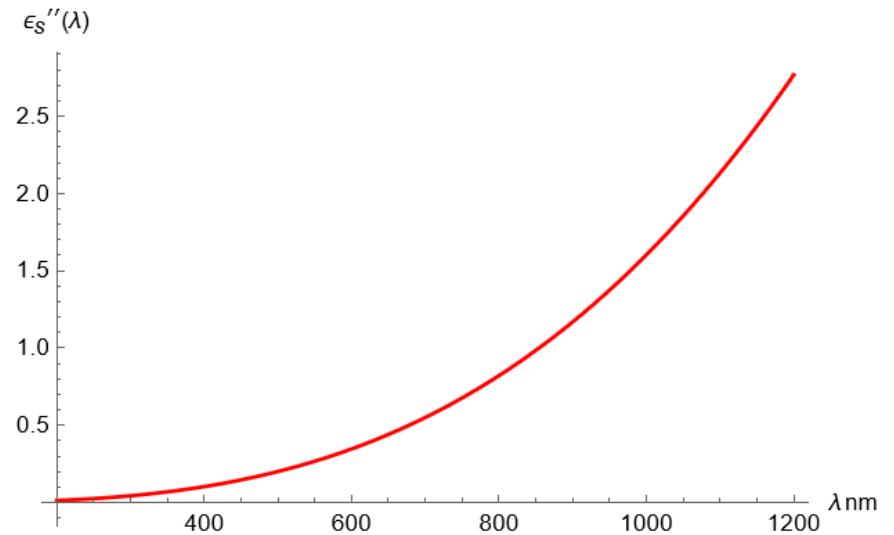
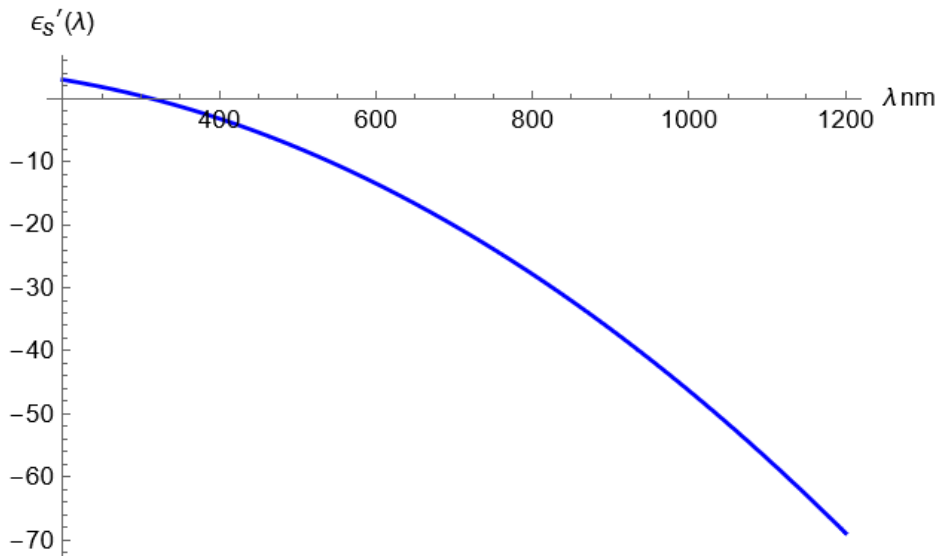
Unitary limit !

Localized surface plasmon resonances (recap)

$$\sigma_{\text{ext}} = k \text{Im}\{\alpha(\omega)\} \quad \sigma_{\text{scat}} = \frac{k^4}{6\pi} |\alpha(\omega)|^2 \quad \sigma_{\text{abs}} \equiv \sigma_{\text{ext}} - \sigma_{\text{scat}}$$

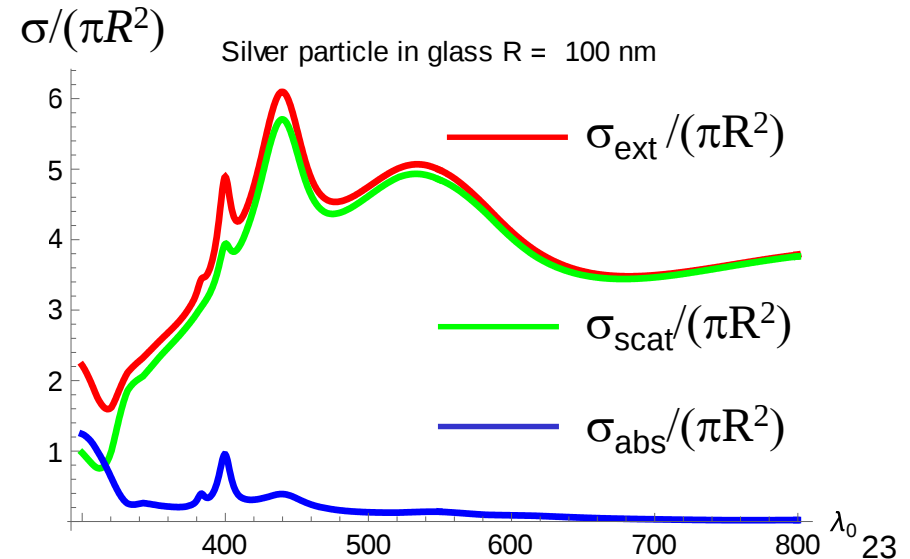
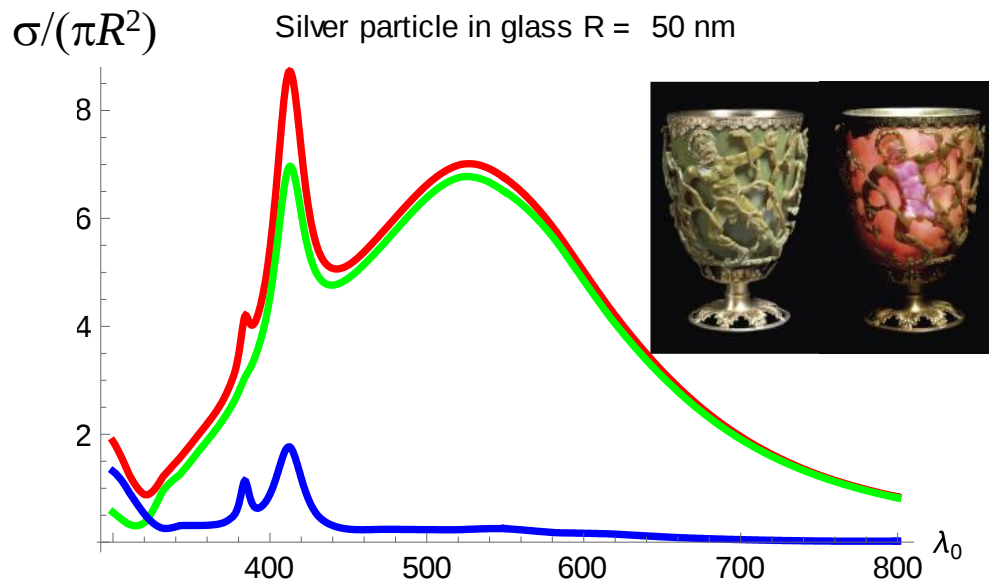
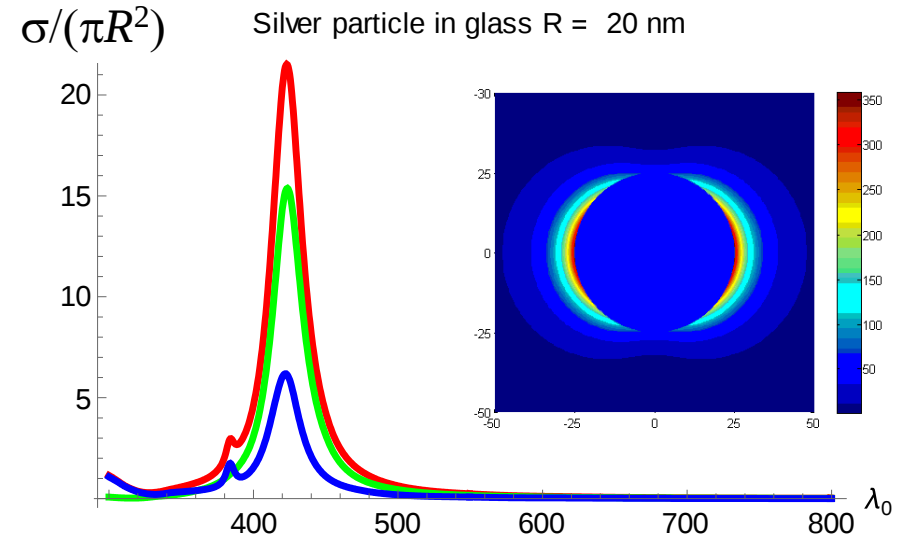
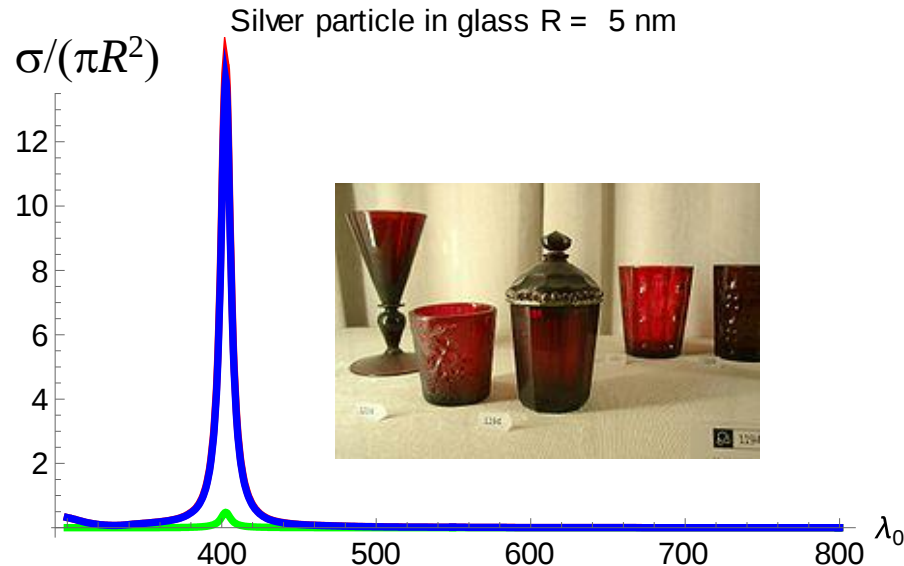
$$\alpha_0(\omega) \equiv \lim_{kR \rightarrow 0} \alpha(\omega) = 4\pi R^3 \frac{\epsilon_s - \epsilon_b}{\epsilon_s + 2\epsilon_b}$$

$$\epsilon_s(\omega) = \epsilon'_s(\omega) + i\epsilon''_s(\omega)$$



Plasmonics : silver spheres in glass

(Localized surface plasmon resonances)



Scattering and absorption by nano-particles (plasmonics)

Mie(1908) - Contributions to the optics of turbid media, particularly of colloidal metal solutions

Stain glass ~ 17th century



Gold “Ruby” glass

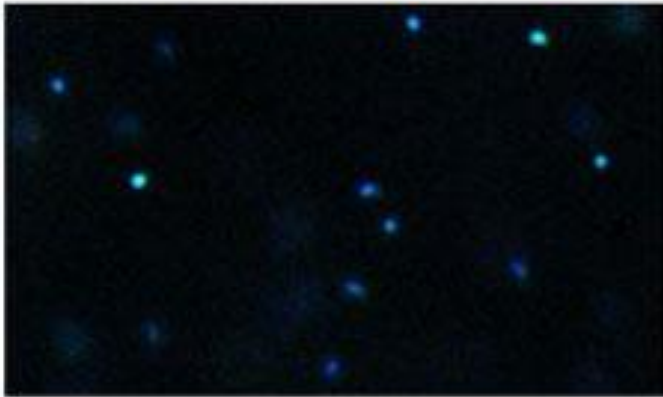


Lycurgus cup ~ 4th century

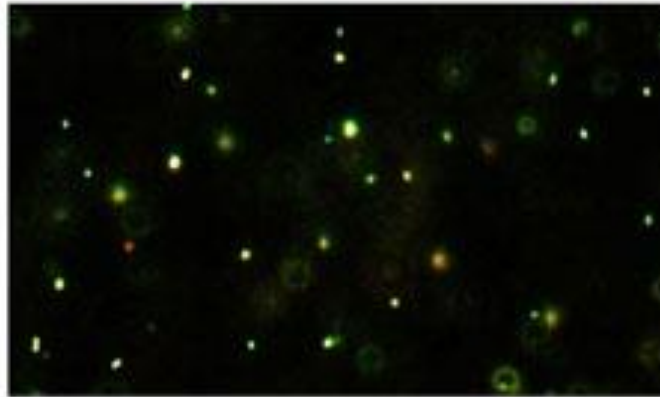


Dark field imaging of plasmonic particles (imaging “below the diffraction limit”)

60nm Silver Nanoparticles



60nm Gold Nanoparticles



100nm Gold NanoUrchins



Definition of the density of states (DOS)

$$\left. \begin{aligned} \Delta\psi + \kappa_b^2(\omega)\psi &= 0 \\ \kappa_b^2(\omega) &\equiv \varepsilon_b \left(\frac{\omega}{c}\right)^2 \end{aligned} \right\} \longrightarrow \boxed{\rho_s(\omega) \equiv \frac{d\kappa_b^2}{d\omega} \int \frac{d^3k}{(2\pi)^3} \delta(k^2 - \kappa_b^2) = \frac{2\omega\varepsilon_b}{c^2} \int \frac{d^3k}{(2\pi)^3} \delta(k^2 - \kappa_b^2)}$$

$$\hat{\mathbb{I}} = \int d^3r |\mathbf{r}\rangle \langle \mathbf{r}| \equiv \int d\mathbf{r} |\mathbf{r}\rangle \langle \mathbf{r}| \qquad \langle \mathbf{r} | \mathbf{k} \rangle = \langle \mathbf{r} | \mathbf{k} \rangle = \frac{e^{i\mathbf{k} \cdot \mathbf{r}}}{(2\pi)^{3/2}} \qquad \text{Classical Theory}$$

$$\hat{\mathbb{I}} = \int d^3k |\mathbf{k}\rangle \langle \mathbf{k}| \equiv \int d\mathbf{k} |\mathbf{k}\rangle \langle \mathbf{k}| \qquad \langle \mathbf{r} | \mathbf{p} \rangle = \frac{e^{i\mathbf{p} \cdot \mathbf{r} / \hbar}}{(2\pi\hbar)^{3/2}} \qquad \text{Quantum Theory}$$

$\mathbf{p} = \hbar \mathbf{k}$

$$\Delta g + \varepsilon_b \left(\frac{\omega}{c}\right)^2 g = -\delta^{(3)}(\mathbf{r}) \rightarrow (k^2 - \kappa_b^2)g(\mathbf{k}) = 1 \rightarrow g(\mathbf{k}) = \frac{1}{k^2 - \kappa_b^2}$$

$$g(\mathbf{r}, \mathbf{r}') = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')}}{k^2 - \kappa_b^2 - i0^+} \qquad \frac{1}{k^2 - \kappa_b^2 - i0^+} = P.V. \left\{ \frac{1}{k^2 - \kappa_b^2} \right\} + i\pi \delta(k^2 - \kappa_b^2)$$

$$\longrightarrow \boxed{\rho_s(\omega) = \frac{2\varepsilon_b\omega}{\pi c^2} \lim_{r \rightarrow 0} \{\text{Im} g(r)\}}$$

LDOS is obtained from $\lim_{r \rightarrow 0} \{\text{Im} g(\mathbf{r})\}$

$$\Delta g(\mathbf{r}) + \kappa_b^2 g(\mathbf{r}) = -\delta^{(3)}(\mathbf{r})$$

$$\kappa_b^2 = \varepsilon_b \left(\frac{\omega}{c}\right)^2 \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_0$$

Density of states (scalar Helmholtz equation) : $\rho_s = \frac{1}{V} \frac{dN}{d\omega} = \frac{\omega^2 \varepsilon_b^{3/2}}{2\pi^2 c^3}$

$$g(\mathbf{r}) = \frac{e^{i\kappa_b r}}{4\pi r} \quad \lim_{r \rightarrow 0} \{\text{Im} g(\mathbf{r})\} = \lim_{r \rightarrow 0} \text{Im} \left(\frac{1 + i\kappa_b r + \dots}{4\pi r} \right) = \frac{\kappa_b}{4\pi}$$

$$\rho_s(\omega) = \frac{2\varepsilon_b \omega}{\pi c^2} \lim_{r \rightarrow 0} \{\text{Im} g(\mathbf{r})\} = \frac{\omega^2 \varepsilon_b^{3/2}}{2\pi^2 c^3}$$

Homogeneous wave equations in 3D (spherically symmetric isotropic potentials)

Schrödinger's equation:

$$\Delta\psi + \frac{2m}{\hbar^2} E\psi - \frac{2m}{\hbar^2} V(r)\psi = 0$$

Acoustic waves:

$$\nabla \cdot \frac{1}{\rho(r)} \nabla \psi - \frac{1}{B(r)} \left(\frac{\omega}{c} \right)^2 \psi = 0$$

Electromagnetic waves:

$$\nabla \times \frac{1}{\mu(r)} \nabla \times \mathbf{E} - \varepsilon(r) \left(\frac{\omega}{c} \right)^2 \mathbf{E} = \vec{0}$$

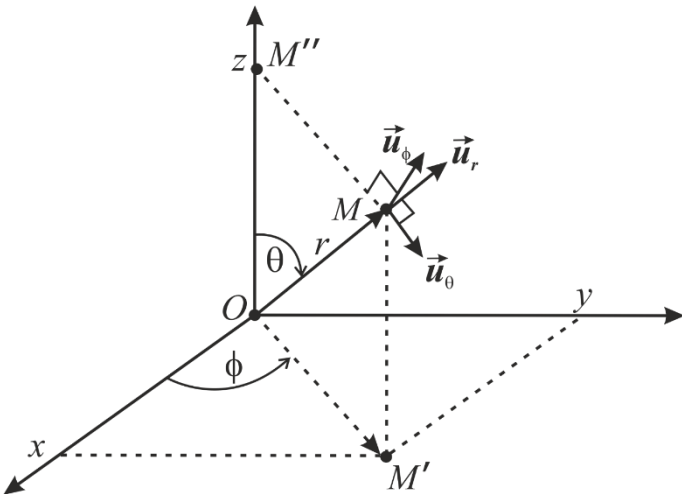
Scalar “light” (acoustics : $\rho = 1, \varepsilon(r) = 1/B(r)$):

$$\Delta\psi + \varepsilon(r) \left(\frac{\omega}{c} \right)^2 \psi = 0$$

Optical fields ($\mu = 1$) :

$$\nabla \times \nabla \times \mathbf{E} - \varepsilon(r) \left(\frac{\omega}{c} \right)^2 \mathbf{E} = \vec{0}$$

Scalar Helmholtz equation in a spherically symmetric potential:



$$\Delta\psi + \varepsilon(r) \left(\frac{\omega}{c}\right)^2 \psi = 0$$

$$\frac{1}{r} \frac{\partial^2(r\psi)}{\partial r^2} + \frac{1}{r^2 \sin^2 \theta} \left[\frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{\partial^2 \psi}{\partial \phi^2} \right] + \left(\frac{\omega}{c}\right)^2 \varepsilon(r) \psi = 0$$

$$\frac{1}{r} \frac{\partial^2(r\psi)}{\partial r^2} - \frac{\vec{\mathbb{L}}_{cl}^2 \psi}{r^2} = 0$$

Separation of variables : $\psi(r, \theta, \phi) = \psi_r(r) \psi_\theta(\theta) \psi_\phi(\phi)$

$$\left[\frac{d^2}{d\phi^2} + a \right] \psi_\phi(\phi) = 0 \Rightarrow a = m^2, \quad \psi_\phi(\phi) \propto e^{im\phi} \Rightarrow \begin{aligned} m &\in \mathbb{Z} \\ m &= -\infty, \dots, -1, 0, 1, \dots, \infty \end{aligned}$$

$$\left[\frac{1}{\sin^2 \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d}{d\theta} \right) - \frac{m^2}{\sin^2 \theta} \right] \psi_\theta(\theta) = -\ell(\ell + 1) \psi_\theta(\theta) \Rightarrow \psi_\theta(\theta) \propto P_\ell^m(\cos \theta) \quad \ell = 0, 1, \dots, |m|$$

Ref : Jackson 3rd edition – chapter 3

Legendre Polynomials : $P_\ell(x)$ Associated Legendre functions : $P_\ell^m(x)$

$$\underbrace{-\left[\frac{1}{\sin^2 \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d}{d\theta} \right) - \frac{m^2}{\sin^2 \theta} \right]}_{\vec{\mathbb{L}}_{\text{cl}}^2} \psi_\theta(\theta) e^{im\phi} = \ell(\ell+1) \psi_\theta(\theta) e^{im\phi}$$

$$\vec{\mathbb{L}}_{\text{cl}}^2 \psi_\theta(\theta) e^{im\phi} = \ell(\ell+1) \psi_\theta(\theta) e^{im\phi} \Rightarrow \psi_\theta(\theta) = P_\ell^m(\cos \theta) \quad \ell = 0, 1, 2, \dots, \infty$$

Legendre Polynomials: $P_\ell(x) = P_\ell^0(x) = \frac{1}{2^n \ell!} \frac{d^\ell}{dx^\ell} [(x^2 - 1)^\ell]$ (Rodrigues' Formula)

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^2 - 3x)$$

$\vdots$

Associated Legendre functions:

$$\begin{aligned} P_\ell^m(x) &= (-1)^m (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x) \\ &= (-1)^m (\sin \theta)^{m/2} \frac{d^m}{dx^m} P_\ell(x) \end{aligned}$$

Scalar Spherical harmonics

$$Y_{\ell,m}(\theta, \phi) = \left[\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!} \right]^{1/2} e^{im\phi} P_{\ell}^m(\cos\theta) \quad \ell = 0, 1, \dots, \infty \quad m = -\ell, \dots, \ell$$

$$\equiv e^{im\phi} \bar{P}_{\ell}^m(\cos\theta)$$

Angular momentum operator: $\vec{L}_{cl} \equiv \frac{1}{i} (\mathbf{r} \times \nabla)$

$$- \left[\frac{1}{\sin^2 \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] Y_{\ell,m}(\theta, \phi)$$

$$\equiv \vec{L}_{cl}^2 Y_{\ell,m}(\theta, \phi) = \ell(\ell+1) Y_{\ell,m}(\theta, \phi)$$

Orthogonal basis:

$$\int_0^{4\pi} Y_{\ell,m}^*(\theta, \phi) Y_{\ell',\mu}(\theta, \phi) d\Omega = \delta_{\ell,\ell'} \delta_{m,\mu}$$

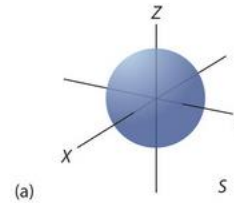
$$n=0 \quad Y_{00}(x) = \frac{1}{\sqrt{4\pi}}$$

$$n=1 \quad \begin{cases} Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi} \\ Y_{10} = \sqrt{\frac{3}{4\pi}} \cos \theta \end{cases}$$

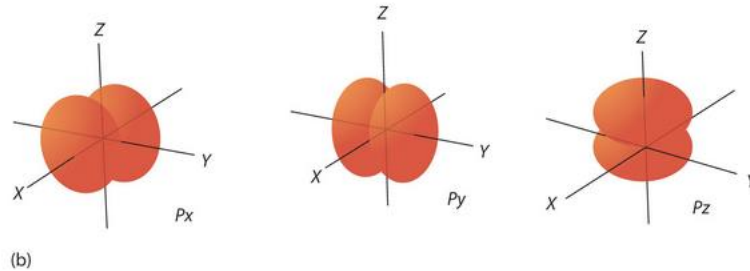
$$n=2 \quad \begin{cases} Y_{22} = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{i2\phi} \\ Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\phi} \\ Y_{20} = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) \end{cases}$$

The scalar harmonics determine the electron 'orbitals'

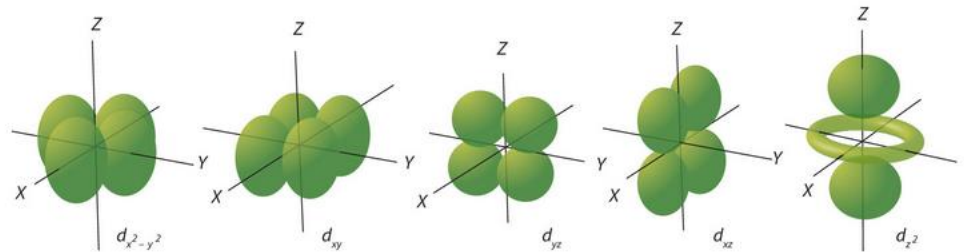
$Y_{0,0}$



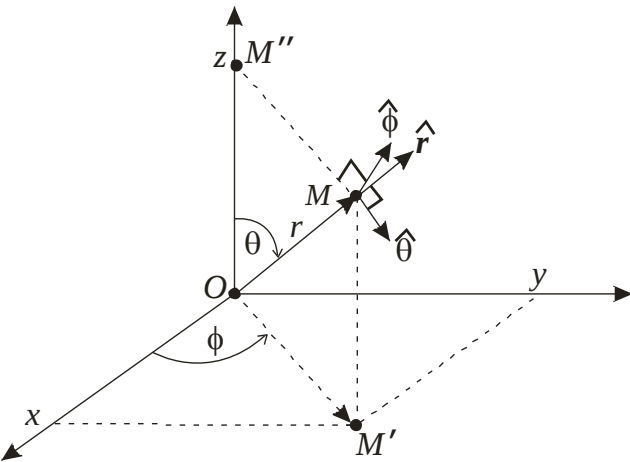
$Y_{1,-1}, Y_{1,0}, Y_{1,1}$



$Y_{2,-2}, Y_{2,-1}, Y_{2,0}, Y_{2,1}, Y_{2,2}$



Scalar Helmholtz equation in a spherically symmetric potential:



$$\Delta\psi + \varepsilon(r) \left(\frac{\omega}{c}\right)^2 \psi = 0$$

$$\underbrace{\frac{1}{r} \frac{\partial^2(r\psi)}{\partial r^2} - \frac{\vec{\mathbb{L}}_{\text{cl}}^2}{r^2} \psi + \varepsilon(r) \left(\frac{\omega}{c}\right)^2 \psi}_{= 0} = 0$$

$$\psi(r, \theta, \phi) \equiv \psi_r(r) Y_{\ell, m}(\theta, \phi) \quad \vec{\mathbb{L}}_{\text{cl}}^2 Y_{\ell, m} = \ell(\ell + 1) Y_{\ell, m}$$

$$Y_{\ell, m}(\theta, \phi) = e^{im\phi} \bar{P}_{\ell}^m(\cos\theta)$$

$$\begin{cases} \ell = 0, 1, 2, \dots, \infty \\ m = -\ell, \dots, \ell \end{cases}$$

$$\frac{d^2 u_{\ell}}{dr^2} - \frac{\ell(\ell + 1)}{r^2} u_{\ell} + \varepsilon(r) \left(\frac{\omega}{c}\right)^2 u_{\ell} = 0$$

$$u_{\ell}(r) \equiv r\psi_r(r)$$

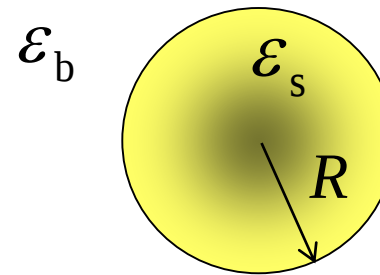
Scattering from a spherically symmetric potential (Quantum mechanical analogy)

Scalar light :

$$\kappa_b^2 \equiv \varepsilon_b \left(\frac{\omega}{c} \right)^2$$

$$\frac{d^2}{dr^2} u_\ell + \kappa_b^2 u_\ell - \left[V_{\text{eff}}^{(\text{opt})}(r, \omega) \right]_\ell u_\ell = 0$$

$$\left[V_{\text{eff}}^{(\text{opt})}(r, \omega) \right]_\ell \equiv \frac{\ell(\ell+1)}{r^2} - (\varepsilon_s - \varepsilon_b) \left(\frac{\omega}{c} \right)^2 \theta(R - r)$$



Schrödinger equation :

$$\frac{d^2}{dr^2} u_\ell + \frac{2m}{\hbar^2} E u_\ell - \left[V_{\text{eff}}^{(\text{Sch})}(r) \right]_\ell u_\ell = 0$$

$$\left[V_{\text{eff}}^{(\text{Sch})}(r) \right]_\ell \equiv \frac{\ell(\ell+1)}{r^2} - \frac{2m}{\hbar^2} \frac{e^2}{4\pi\epsilon_0 r}$$

Effective potential – Photonic “atom”

Scalar light : $\kappa_b^2 \equiv \epsilon_b \left(\frac{\omega}{c}\right)^2$

Whispering Gallery modes $\epsilon_s > \epsilon_b$
 “attractive” potential with radiative loss :

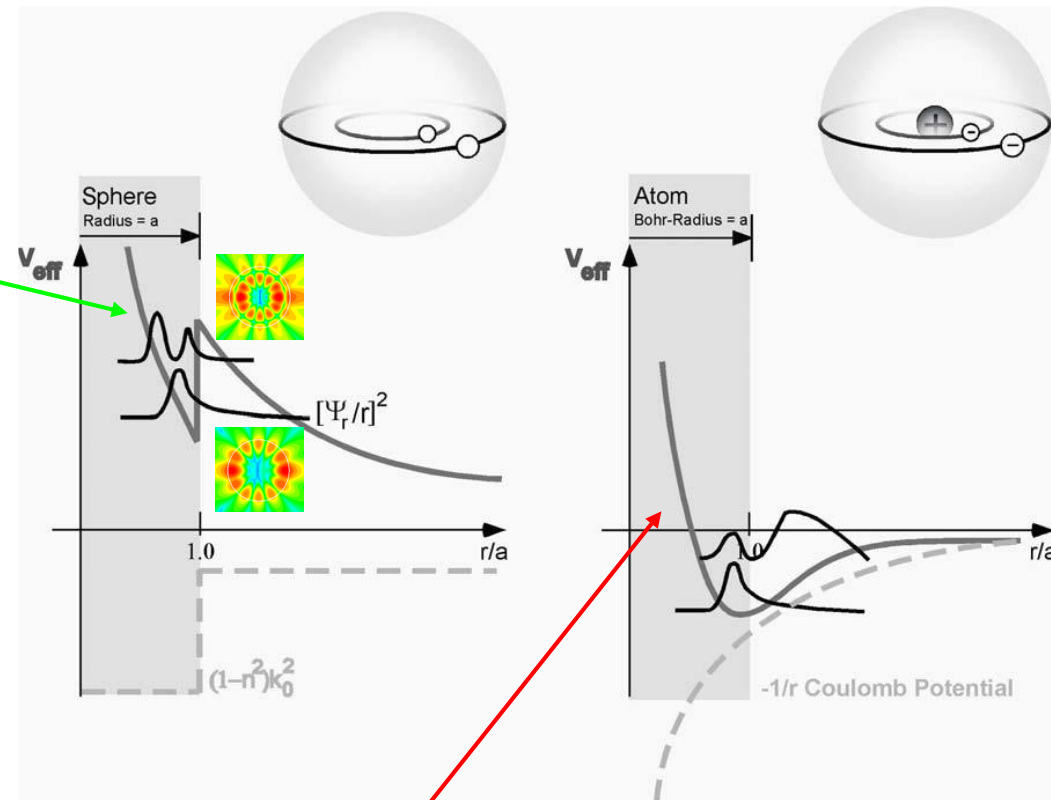
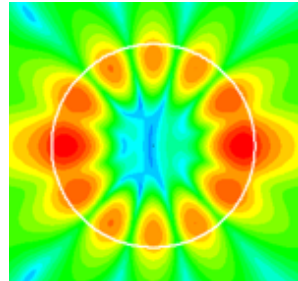
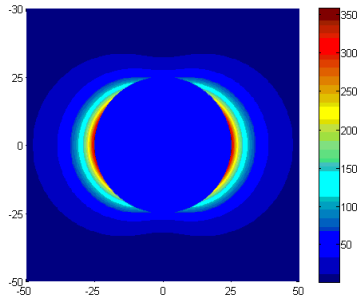
$$\left[V_{\text{eff}}^{(\text{opt})}(r) \right]_{\ell} \equiv \frac{\ell(\ell+1)}{r^2} - (\epsilon_s - \epsilon_b) \left(\frac{\omega}{c}\right)^2 \theta(R-r)$$

Plasmonic resonances
 “repulsive” potential :

$$\epsilon_s < \epsilon_b$$

“Morphologic” resonances
 “attractive” potential :

$$\epsilon_s > \epsilon_b$$



$$\left[V_{\text{eff}}^{(\text{Sch})}(r) \right]_{\ell} \equiv \frac{\ell(\ell+1)}{r^2} - \frac{2m}{\hbar^2} \frac{e^2}{4\pi\epsilon_0 r}$$

Homogeneous media Helmholtz equation

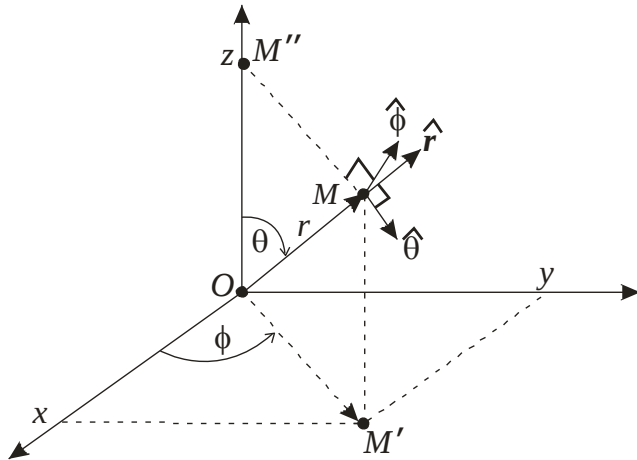
$$\Delta\psi + \kappa^2\psi = 0$$

Separation of variables:

$$\psi(r, \theta, \phi) = \psi_r(r) Y_{\ell, m}(\theta, \phi) \quad \ell = 0, 1, 2, \dots, \infty \quad m = -\ell, \dots, \ell$$

$$\Delta\psi + \kappa^2\psi = 0 \rightarrow \left[\frac{1}{r} \frac{d^2(r\psi_r)}{dr} + \frac{\psi_r \vec{\mathbb{L}}_{\text{cl}}^2}{r^2} \psi_r + \kappa^2 \psi_r \right] Y_{\ell, m} = 0$$

$$\left[r \frac{d^2(r\psi_r)}{dr} + \ell(\ell + 1)\psi_r + \kappa^2 r^2 \psi_r \right] Y_{\ell, m} = 0$$



Change of variables:

$$\frac{z_\ell(r)}{(\kappa r)^{1/2}} \equiv \psi_r(\kappa r) \quad \kappa^2 \equiv \varepsilon_b \left(\frac{\omega}{c} \right)^2$$

Spherical Bessel function equation:

$$r^2 \frac{d^2 z_\ell}{dr^2} + r \frac{dz_\ell}{dr} + \left[\kappa^2 r^2 - \left(\ell + \frac{1}{2} \right)^2 \right] z_\ell = 0$$

Homogeneous Helmholtz equation in spherical coordinates

$$r^2 \frac{d^2 z_\ell}{dr^2} + r \frac{dz_\ell}{dr} + \left[\kappa^2 r^2 - \left(\ell + \frac{1}{2} \right)^2 \right] z_\ell = 0$$

Spherical Bessel, Neumann and Hankel functions:

$$\frac{z_\ell(r)}{(\kappa r)^{1/2}} \equiv \psi_r(\kappa r)$$

$$\psi_r(\kappa r) = \begin{cases} j_\ell(x) \equiv \sqrt{\frac{\pi}{2x}} J_{\ell+1/2}(x) \\ y_\ell(x) \equiv \sqrt{\frac{\pi}{2x}} Y_{\ell+1/2}(x) \\ h_\ell^{(+)}(x) \equiv \sqrt{\frac{\pi}{2x}} H_{\ell+1/2}^{(+)}(x) = j_\ell(x) + iy_\ell(x) \\ h_\ell^{(-)}(x) \equiv \sqrt{\frac{\pi}{2x}} H_{\ell+1/2}^{(-)}(x) = j_\ell(x) - iy_\ell(x) \end{cases}$$

Bessel (1):

$$j_0(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)!} x^{2s} = \frac{\sin x}{x}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

$$\vdots$$

Neumann (2):

$$y_0(x) = -\frac{\cos x}{x}$$

$$y_1(x) = \left(\frac{\cos^2 x}{x^2} - \frac{\sin x}{x} \right)$$

$$\vdots$$

Hankel + (3):

$$h_0^{(+)}(x) = -\frac{i}{x} e^{ix}$$

$$h_1^{(+)}(x) = -e^{ix} \left(\frac{1}{x} + \frac{i}{x^2} \right)$$

$$\vdots$$

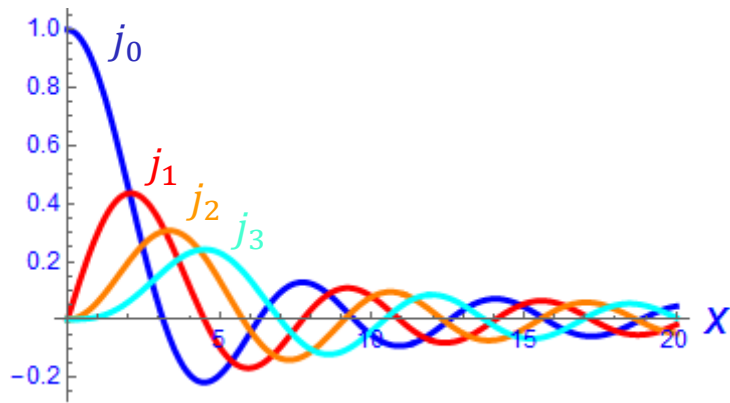
Linearly independent solutions

Spherical Bessel functions (1)

$$j_0(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)!} x^{2s} = \frac{\sin x}{x}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

$$\vdots$$



Outgoing spherical Hankel functions (+)

$$h_0^{(+)}(x) = -\frac{i}{x} e^{ix}$$

$$h_1^{(+)}(x) = -e^{ix} \left(\frac{1}{x} + \frac{i}{x^2} \right)$$

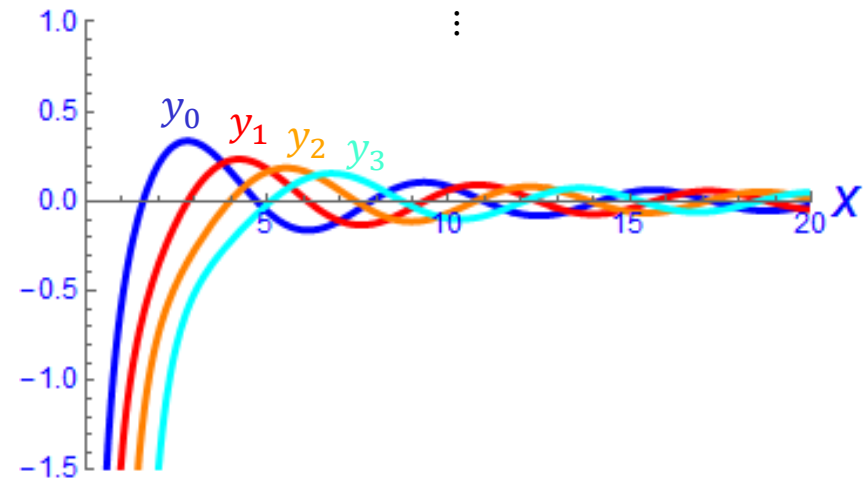
$$\vdots$$

Spherical Neumann functions (2)

$$y_0(x) = -\frac{\cos x}{x}$$

$$y_1(x) = \left(\frac{\cos^2 x}{x^2} - \frac{\sin x}{x} \right)$$

$$\vdots$$



Incoming spherical Hankel functions (-)

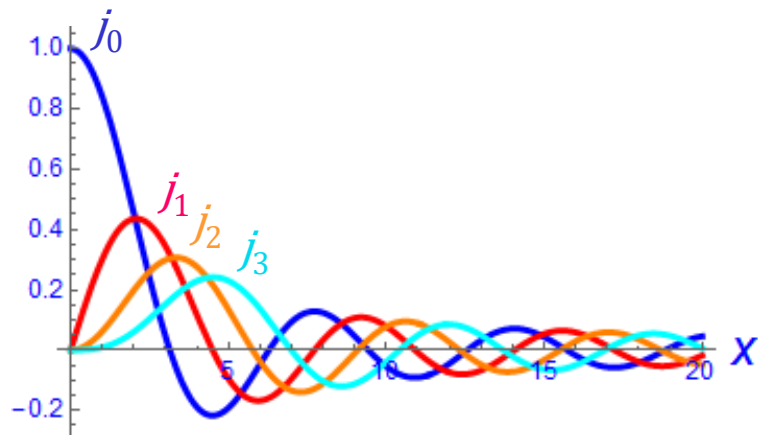
$$h_0^{(-)}(x) = \frac{i}{x} e^{-ix}$$

$$h_1^{(-)}(x) = -e^{-ix} \left(\frac{1}{x} - \frac{i}{x^2} \right)$$

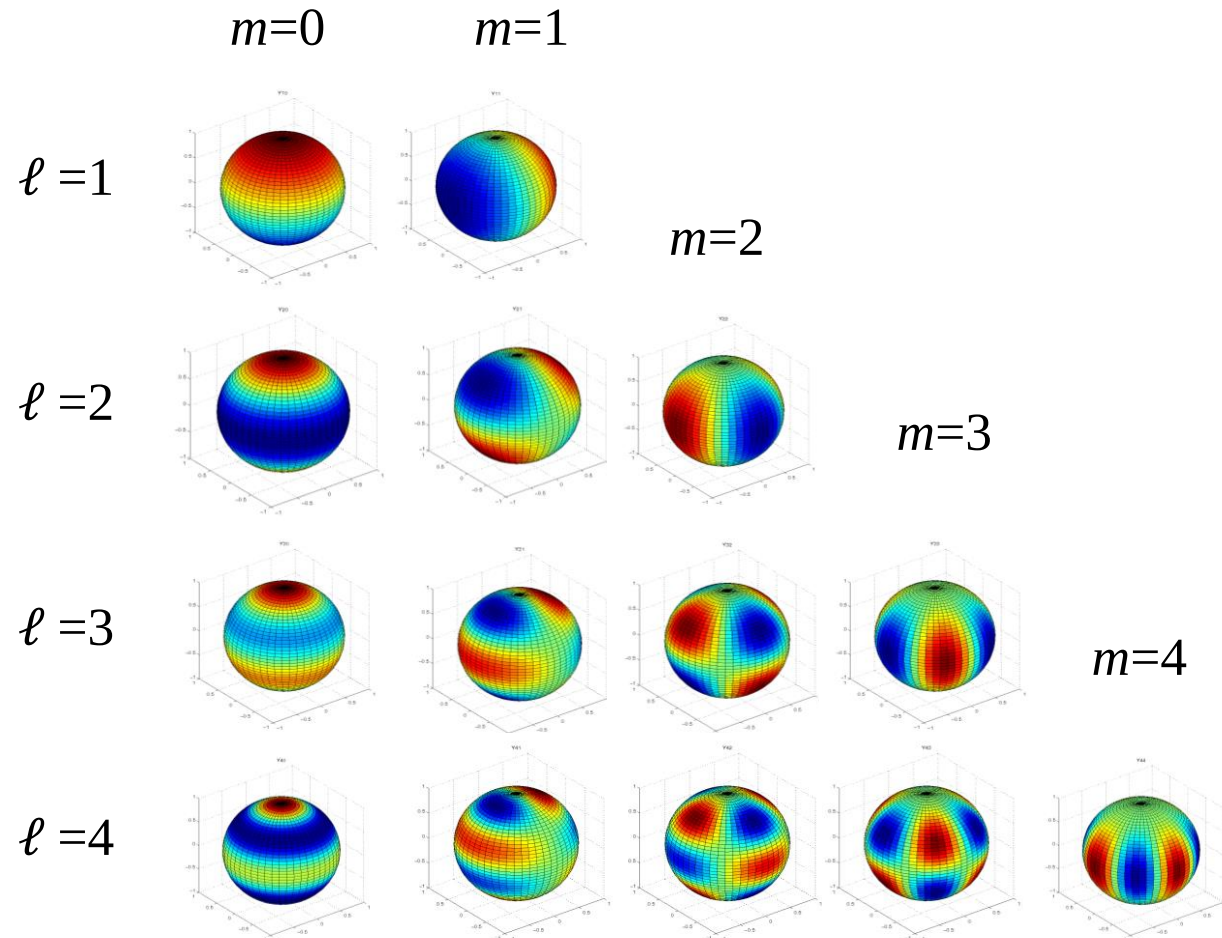
$$\vdots$$

Regular partial wave basis : $\Psi_{\ell,m}(\kappa, \mathbf{r}) \equiv i^\ell \kappa^{3/2} j_\ell(\kappa r) Y_{\ell,m}(\theta, \phi)$

Bessel functions : $j_\ell(\kappa r)$



Spherical Harmonics : $Y_{\ell,m}(\theta, \phi)$



Regular partial waves form a basis for source-free fields (scalar waves)

$$\Psi_{\ell,m}(\kappa, \mathbf{r}) \equiv i^\ell \kappa^{3/2} j_\ell(\kappa r) Y_{\ell,m}(\theta, \phi) \quad \ell = 0, 1, 2, \dots, \infty \quad m = -\ell, \dots, \ell$$

Orthogonality

$$\frac{2}{\pi} \int \Psi_{\ell,m}(\kappa \mathbf{r}) \Psi_{\ell',\mu}(\kappa' \mathbf{r}) d\mathbf{r} = \delta_{\ell,\ell'} \delta_{m,\mu} \delta(\kappa - \kappa')$$

Closure

$$\frac{2}{\pi} \int_0^\infty \sum_{\ell,m=0}^\infty \Psi_{\ell,m}(\kappa \mathbf{r}) \Psi_{\ell,m}(\kappa \mathbf{r}') \frac{d\kappa}{\kappa} = \delta^{(3)}(\mathbf{r} - \mathbf{r}')$$

Partial wave basis (scalar waves)

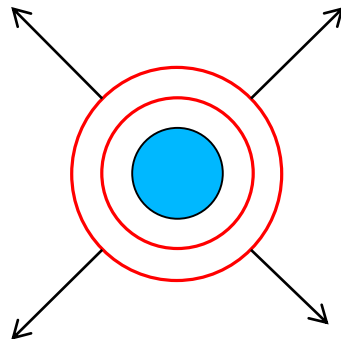
Plane wave expansion :

$$k^{3/2} e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} p_{\ell,m} \Psi_{\ell,m}(k, \mathbf{r})$$

Plane wave Coefficients :

$$p_{\ell,m} = 4\pi Y_{\ell,m}^*(\mathbf{u}_k)$$

$$\mathbf{u}_k \equiv \frac{\mathbf{k}}{k}$$



Regular partial wave :

$$\Psi_{\ell,m}(k, \mathbf{r}) \equiv i^{\ell} \kappa^{3/2} j_{\ell}(kr) Y_{\ell,m}(\theta, \phi)$$

Outgoing “partial” wave :

$$\Psi_{\ell,m}^{(+)}(k, \mathbf{r}) \equiv i^{\ell} k^{3/2} h_{\ell}^{(+)}(kr) Y_{\ell,m}(\theta, \phi)$$

Green's function (scalar waves)

Complete solutions to the wave equations valid for arbitrary sources

We want to solve the inhomogeneous wave equation for an arbitrary source $s(\vec{r})$ in a homogeneous medium :

$$\Delta\psi(\mathbf{r}) + k_b^2\psi(\mathbf{r}) = -s(\mathbf{r})$$

$$k_b^2 = \varepsilon_b \left(\frac{\omega}{c}\right)^2$$

This can be achieved by solving for the Green's function $g(\mathbf{r}, \mathbf{r}')$

$$\Delta g(\mathbf{r}, \mathbf{r}') + k_b^2 g(\mathbf{r}, \mathbf{r}') = -\delta^{(3)}(\mathbf{r} - \mathbf{r}')$$

This solution to the wave equation, $\psi(\vec{r})$ for an arbitrary source, $s(\vec{r})$ is then found by:

$$\psi(\mathbf{r}) = \int g(\mathbf{r}, \mathbf{r}') s(\mathbf{r}') d\mathbf{r}'$$

Green's function in a homogeneous media
can be constructed from the 'partial' waves: $\Psi_{\ell,m}^{(+)}$, $\Psi_{\ell,m}$

$$\Delta g(\mathbf{r}, \mathbf{r}') + k_b^2 g(\mathbf{r}, \mathbf{r}') = \delta^{(3)}(\mathbf{r} - \mathbf{r}') \quad k_b^2 = \varepsilon_b \left(\frac{\omega}{c}\right)^2$$

$$g(\mathbf{r}, \mathbf{r}') = \frac{i}{k_b^2} \sum_{\ell \geq 0, m} (-)^l \Psi_{\ell,m}^{(+)}(k_b, \mathbf{r}_{>}) \Psi_{\ell,m}(k_b, \mathbf{r}_{<})$$

$$|\mathbf{r}| > |\mathbf{r}'| \Rightarrow \{\mathbf{r}_{>} = \mathbf{r} \text{ and } \mathbf{r}_{<} = \vec{\mathbf{r}}'\}$$

$$|\mathbf{r}| < |\mathbf{r}'| \Rightarrow \{\mathbf{r}_{>} = \mathbf{r}' \text{ and } \mathbf{r}_{<} = \mathbf{r}\}$$

Regular partial waves:

$$\Psi_{\ell,m}(\kappa, \mathbf{r}) = i^\ell \kappa^{3/2} j_\ell(\kappa r) Y_{\ell,m}(\theta, \phi)$$

Outgoing "partial" waves:

$$\Psi_{\ell,m}^{(+)}(\kappa, \mathbf{r}) = i^\ell \kappa^{3/2} h_\ell^{(+)}(\kappa r) Y_{\ell,m}(\theta, \phi)$$

Closed form for the Green's function :

The infinite sum can be simplified by letting $\vec{r}' \rightarrow \mathbf{0}$ and remarking that only $\Psi_{0,0}(k\mathbf{0}) = \frac{1}{\sqrt{4\pi}}$ is non-zero in this limit.

The Green's function then simplifies to:

$$\begin{aligned} g(\vec{r}, \vec{0}) &= \frac{i}{\kappa^2} \sum_{\ell \geq 0, m} \Psi_{\ell, m}^{(+)}(\kappa \mathbf{r}_{>}) \Psi_{\ell, m}(\kappa \mathbf{r}_{<}) \\ &\rightarrow \frac{i}{\kappa^2} \Psi_{0,0}^{(+)}(\kappa \mathbf{r}) \Psi_{0,0}(\kappa \mathbf{0}) = \frac{e^{i\kappa r}}{4\pi r} \end{aligned}$$

Famous result for Green's function of the scalar Helmholtz equation:

$$g(\mathbf{r}_1, \mathbf{r}) = \frac{e^{i\kappa|\mathbf{r}_1 - \mathbf{r}_2|}}{4\pi|\mathbf{r}_1 - \mathbf{r}_2|} \equiv \frac{e^{i\kappa r_{12}}}{4\pi r_{12}}$$

Homogenous Maxwell equation in spherical coordinates

$$\nabla \times \nabla \times \mathbf{E} - \kappa^2 \mathbf{E} = \mathbf{0}$$

$$\kappa^2 = \varepsilon\mu \left(\frac{\omega}{c}\right)^2$$

$$\Delta\psi + \kappa^2\psi = 0$$

Transverse vector “partial” waves : $\nabla \cdot \mathbf{M}_{n,m} = \nabla \cdot \mathbf{N}_{n,m} = 0$

Scalar “partial” waves

$$\Psi_{\ell,m}(\kappa\mathbf{r}) \equiv i^\ell \kappa^{3/2} j_\ell(\kappa r) Y_{\ell,m}(\theta, \phi)$$



$$\mathbf{M}_{n,m}(\kappa\mathbf{r}) \equiv \frac{\nabla \times [\mathbf{r} \Psi_{n,m}(\kappa r)]}{\sqrt{n(n+1)}}$$

magnetic

$$\mathbf{N}_{n,m}(\kappa\mathbf{r}) \equiv \frac{\nabla \times \mathbf{M}_{n,m}(\kappa\mathbf{r})}{i\kappa}$$

electric

$$\ell = 0, 1, 2, \dots, \infty \quad m = -\ell, \dots, \ell$$

Longitudinal vector “partial” waves

$$\nabla(\nabla \cdot \mathbf{L}) + \kappa^2 \mathbf{L} = \mathbf{0}$$

Bouwkamp-Casimir (Jackson) approach

$$\mathbf{L}_{\ell,m}(\kappa\mathbf{r}) = \frac{\nabla[\Psi_{\ell,m}(\kappa r)]}{\kappa\sqrt{\ell(\ell+1)}}$$

$$\Delta(\mathbf{r} \cdot \mathbf{H}) + \kappa^2 \mathbf{r} \cdot \mathbf{H} = \mathbf{0}$$

$$\Delta(\mathbf{r} \cdot \mathbf{E}) + \kappa^2 \mathbf{r} \cdot \mathbf{E} = \mathbf{0}$$

$$\Delta \mathbf{A} + \kappa^2 \mathbf{A} = \mathbf{0} \quad \mathbf{A} = \mathbf{L}, \mathbf{M}, \mathbf{N}$$

Vector spherical harmonics

$$Y_{\ell,m}(\theta, \phi) = \left[\frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!} \right]^{1/2} e^{im\phi} P_{\ell}^m(\cos\theta) \equiv e^{im\phi} \bar{P}_{\ell}^m(\cos\theta)$$

Angular momentum operator : $\vec{\mathbb{L}}_{\text{op}} \equiv \frac{1}{i} (\mathbf{r} \times \nabla)$

3 types of vector spherical harmonics :

$$\mathbf{W}_{\ell,m}^{(1)}(\theta, \phi) \equiv \mathbf{Y}_{\ell,m}(\theta, \phi) \equiv \mathbf{u}_r Y_{\ell,m}(\theta, \phi) \quad \ell=0,1,2,\dots,\infty \quad m=-\ell, \dots, \ell$$

$$\mathbf{W}_{n,m}^{(2)}(\theta, \phi) \equiv \mathbf{X}_{n,m}(\theta, \phi) \equiv \frac{\vec{\mathbb{L}}_{\text{op}}}{i} Y_{n,m}(\theta, \phi) = \mathbf{Z}_{n,m}(\theta, \phi) \times \mathbf{u}_r \quad n=1,2,\dots,\infty \quad m=-n, \dots, n$$

$$\mathbf{W}_{n,m}^{(3)}(\theta, \phi) \equiv \mathbf{Z}_{n,m}(\theta, \phi) \equiv \frac{r \vec{\nabla} Y_{n,m}(\theta, \phi)}{\sqrt{n(n+1)}} = \mathbf{u}_r \times \vec{\mathbf{X}}_{n,m}(\theta, \phi)$$

$$\int_0^{4\pi} d\Omega \mathbf{W}_{\nu,\mu}^{(j),*}(\theta, \phi) \cdot \mathbf{W}_{n,m}^{(k)}(\theta, \phi) = \delta_{j,k} \delta_{n,\nu} \delta_{m,\mu}$$

Vector spherical harmonics

$$\mathbf{Y}_{\ell,m}(\theta, \phi) = \mathbf{u}_r Y_{\ell,m}(\theta, \phi) \quad \ell = 0, 1, 2, \dots, \infty \quad m = -\ell, \dots, \ell$$

$$\mathbf{X}_{n,m}(\theta, \phi) = e^{im\phi} [\mathbf{u}_\theta i \bar{u}_n^m(\theta) - \mathbf{u}_\phi \bar{s}_n^m(\theta)]$$

$$\mathbf{Z}_{n,m}(\theta, \phi) = e^{im\phi} [\mathbf{u}_\theta \bar{s}_n^m(\theta) - \mathbf{u}_\phi i \bar{u}_n^m(\theta)] \quad n = 1, 2, \dots, \infty \quad m = -n, \dots, n$$

$$\bar{u}_n^m(\theta) = \frac{1}{\sqrt{n(n+1)}} \frac{m}{\sin \theta} \bar{P}_n^m(\theta)$$

$$\bar{s}_n^m(\theta) = \frac{1}{\sqrt{n(n+1)}} \frac{d}{d\theta} \bar{P}_n^m(\theta)$$

Vector 'partial waves' $\nabla \times \nabla \times \mathbf{E} - \kappa^2 \mathbf{E} = \mathbf{0}$

Ricatti-Bessel functions

$$\psi_n(x) \equiv x j_n(x)$$

$$\psi'_n(x) = [j_n(x) + x j'_n(x)]$$

$$\mathbf{M}_{n,m}(\kappa \mathbf{r}) = i^n \kappa^{3/2} j_n(\kappa r) \mathbf{X}_{n,m}(\theta, \phi)$$

$$\mathbf{N}_{n,m}(\kappa \mathbf{r}) = i^{n-1} \kappa^{3/2} \left[\frac{1}{\kappa r} j_n(\kappa r) \sqrt{n(n+1)} \mathbf{Y}_{n,m}(\theta, \phi) + \psi'_n(\kappa r) \mathbf{Z}_{n,m}(\theta, \phi) \right]$$

$$n=1,2,\dots,\infty \quad -n < m < n$$

$$\nabla \cdot \mathbf{M}_{n,m} = \nabla \cdot \mathbf{N}_{n,m} = 0$$

$$\nabla \times \mathbf{M}_{n,m} = \kappa \mathbf{N}_{n,m}$$

$$\nabla \times \mathbf{N}_{n,m} = \kappa \mathbf{M}_{n,m}$$

$$\Delta \mathbf{M}_{n,m} + \kappa^2 \mathbf{M}_{n,m} = \vec{0}$$

$$\Delta \mathbf{N}_{n,m} + \kappa^2 \mathbf{N}_{n,m} = \mathbf{0}$$

Longitudinal partial waves

$$\nabla(\nabla \cdot \mathbf{L}_{\ell,m}) + \kappa^2 \mathbf{L}_{\ell,m} = \mathbf{0}$$

$$\mathbf{L}_{\ell,m}(\kappa \mathbf{r}) = \frac{\nabla[\Psi_{\ell,m}(\kappa \mathbf{r})]}{i\kappa\sqrt{\ell(\ell+1)}} \quad \nabla \times \mathbf{L}_{\ell,m} = \mathbf{0}$$

$$\mathbf{L}_{\ell,m}(\kappa \mathbf{r}) = i^{\ell-1} \kappa^{3/2} \left[j'_\ell(\kappa r) \mathbf{Y}_{\ell,m}(\theta, \phi) + \sqrt{\ell(\ell+1)} \frac{j_\ell(\kappa r)}{\kappa r} \mathbf{Z}_{\ell,m}(\theta, \phi) \right]$$

$$\ell = 0, 1, 2, \dots, \infty \quad -\ell < m < \ell$$

$$\Delta \mathbf{L}_{\ell,m} + \kappa^2 \mathbf{L}_{\ell,m} = \mathbf{0}$$

Regular partial waves form a basis for source-free fields (vector waves)

Orthogonality

$$\begin{aligned} \int \mathbf{M}_{n,m}(\kappa \mathbf{r}) \cdot \mathbf{M}_{\nu,\mu}(\kappa' \mathbf{r}) d\mathbf{r} &= (-)^n \kappa \delta_{n,\nu} \delta_{m,\mu} \delta(\kappa - \kappa') & n, \nu &= 1, 2, \dots, \infty & m &= -n, \dots, n \\ & & \mu &= -\nu, \dots, \nu \\ \int \mathbf{N}_{n,m}(\kappa \mathbf{r}) \cdot \mathbf{N}_{\nu,\mu}(\kappa' \mathbf{r}) d\mathbf{r} &= (-)^{n-1} \kappa \delta_{n,\nu} \delta_{m,\mu} \delta(\kappa - \kappa') \\ \int \mathbf{L}_{\ell,m}(\kappa \mathbf{r}) \cdot \mathbf{L}_{\ell',\mu}(\kappa' \mathbf{r}) d\mathbf{r} &= (-)^{n-1} \kappa \delta_{\ell,\ell'} \delta_{m,\mu} \delta(\kappa - \kappa') & \ell &= \mathbf{0}, 1, 2, \dots, \infty & m &= -\ell, \dots, \ell \\ & & & & \mu &= -\ell', \dots, \ell' \end{aligned}$$

Closure

$$\begin{aligned} \frac{2}{\pi} \int_0^\infty \sum_{n,m=0}^\infty (-)^n \mathbf{M}_{n,m}(\kappa \mathbf{r}) \mathbf{M}_{n,m}(\kappa \mathbf{r}') \frac{d\kappa}{\kappa} &+ \frac{2}{\pi} \int_0^\infty (-)^{n-1} \sum_{n,m=0}^\infty (-)^n \mathbf{N}_{n,m}(\kappa \mathbf{r}) \mathbf{N}_{n,m}(\kappa \mathbf{r}') \frac{d\kappa}{\kappa} \\ &+ \frac{2}{\pi} \int_0^\infty \sum_{\ell,m=0}^\infty (-)^{\ell-1} \mathbf{L}_{\ell,m}(\kappa \mathbf{r}) \mathbf{L}_{\ell,m}(\kappa \mathbf{r}') \frac{d\kappa}{\kappa} = \mathbb{I} \delta^{(3)}(\mathbf{r} - \mathbf{r}') \end{aligned}$$

Development of a vector plane wave (Vector partial wave basis)

Plane wave expansion :

$$\kappa^{3/2} e^{i\boldsymbol{\kappa}_i \cdot \mathbf{r}} \mathbf{u}_i = \sum_{n=1}^{\infty} \sum_{m=-n}^n \left[p_{n,m}^{(h)} \mathbf{M}_{n,m}(\kappa \mathbf{r}) + p_{n,m}^{(e)} \mathbf{N}_{n,m}(\kappa \mathbf{r}) \right]$$

$$\mathbf{u}_i \equiv \frac{\boldsymbol{\kappa}_i}{\kappa}$$

Plane wave coefficients :

$$p_{n,m}^{(h)} = 4\pi \mathbf{X}_{n,m}^*(\mathbf{u}_i) \cdot \hat{\mathbf{e}}_i \quad p_{n,m}^{(e)} = 4\pi \mathbf{Z}_{n,m}^*(\mathbf{u}_i) \cdot \hat{\mathbf{e}}_i$$

Green's function (vector waves)

Complete solutions to the wave equations valid for arbitrary sources

We want to solve the inhomogeneous wave equation for an arbitrary source $\mathbf{j}(\mathbf{r})$ in a homogeneous medium:

$$\nabla \times \nabla \times \mathbf{E}_\omega(\mathbf{r}) - \frac{\omega^2}{c^2} \epsilon_b \mu_b \mathbf{E}_\omega(\mathbf{r}) = \frac{i\omega}{\epsilon_b \epsilon_0 c^2} \mathbf{j}_\omega(\mathbf{r})$$

This can be achieved by solving for the **dyadic** Green's function $\overleftrightarrow{\mathbf{g}}(\mathbf{r}, \mathbf{r}')$:

$$\nabla \times \frac{1}{\mu_b} \nabla \times \overleftrightarrow{\mathbf{g}}(\mathbf{r}, \mathbf{r}') - \frac{\omega^2}{c^2} \epsilon_b \overleftrightarrow{\mathbf{g}}(\mathbf{r}, \mathbf{r}') = \overleftrightarrow{\mathbb{I}} \delta^{(3)}(\mathbf{r} - \mathbf{r}')$$

This solution to the wave equation, $\overrightarrow{\mathbf{E}}(\mathbf{r})$, for an arbitrary source, $\overrightarrow{\mathbf{j}}(\mathbf{r})$, is then:

$$\mathbf{E}(\mathbf{r}) = \frac{i\omega}{\epsilon_b \epsilon_0 c^2} \int \overleftrightarrow{\mathbf{g}}(\mathbf{r}, \mathbf{r}') \cdot \overrightarrow{\mathbf{j}}(\mathbf{r}') d\mathbf{r}'$$

Green's function dyadic constructed from the vector 'partial' waves:

$$\mathbf{M}_{n,m}^{(+)}, \mathbf{N}_{n,m}^{(+)}, \mathbf{M}_{n,m}, \mathbf{N}_{n,m}$$

Green's function solution satisfying 'outgoing' field conditions

$$\vec{\mathbf{g}}(\mathbf{r}, \mathbf{r}') = \frac{i}{\kappa^2} \sum_{n \geq 1, m} \left\{ \mathbf{M}_{n,m}^{(+)}(\kappa \mathbf{r}_{>}) \mathbf{M}_{n,m}(\kappa \mathbf{r}_{<}) + \mathbf{N}_{n,m}^{(+)}(\kappa \mathbf{r}_{>}) \mathbf{N}_{n,m}(\kappa \mathbf{r}_{<}) \right\} - \frac{\mathbf{u}_r \mathbf{u}_r}{\kappa^2} \delta^{(3)}(\mathbf{r} - \mathbf{r}')$$

$$\mathbf{u}_r \equiv \frac{\vec{\mathbf{r}}}{r}$$

$$|\mathbf{r}| > |\mathbf{r}'| \Rightarrow \{\mathbf{r}_{>} = \mathbf{r} \text{ and } \mathbf{r}_{<} = \mathbf{r}'\}$$

$$|\mathbf{r}| < |\mathbf{r}'| \Rightarrow \{\mathbf{r}_{>} = \mathbf{r}' \text{ and } \mathbf{r}_{<} = \mathbf{r}\}$$

Often, the distribution $-\frac{\mathbf{u}_r \mathbf{u}_r}{\kappa^2} \delta^{(3)}(\mathbf{r} - \mathbf{r}')$ can be safely ignored, but sometimes must be included in certain applications:

The distribution $-\frac{\mathbf{u}_r \mathbf{u}_r}{\kappa^2} \delta^{(3)}(\mathbf{r} - \mathbf{r}')$ compensates for strongly divergent fields that are only defined outside an infinitesimal exclusion volume around the origin
(3D principal volume : cf. Jackson, Chew, Tsang and Kong, ...)

Closed form for the Green's function : $\kappa^2 = \epsilon_b \mu_b \left(\frac{\omega}{c}\right)^2$

Letting $\mathbf{r}' \rightarrow \mathbf{0}$ in $\vec{g}(\mathbf{r}, \mathbf{r}')$ and remarking that only $N_{1,m}(\kappa \mathbf{0})$ is non-zero in this limit so the Green's function then simplifies to:

$$\begin{aligned}\vec{g}(\mathbf{r}, \mathbf{0}) &\rightarrow \frac{i}{\kappa^2} \sum_{m=-1,0,1} (-)^m \left\{ N_{1,m}^{(+)}(\kappa \vec{r}) N_{1,-m}(\kappa \mathbf{0}) \right\} - \frac{\mathbf{u}_r \mathbf{u}_r}{\kappa^2} \delta^{(3)}(\mathbf{r}) \\ &= \frac{e^{i\kappa r}}{4\pi r^3 \epsilon_b \left(\frac{\omega}{c}\right)^2} \left\{ (1 - i\kappa r) [3\mathbf{u}_r \mathbf{u}_r - \vec{\mathbb{I}}] + \kappa^2 r^2 [\vec{\mathbb{I}} - \mathbf{u}_r \mathbf{u}_r] \right\} - \frac{\vec{\mathbb{I}}}{3\kappa^2} \delta^{(3)}(\mathbf{r})\end{aligned}$$

Electromagnetic Green's function :

$$\vec{g}(\mathbf{r}_1, \mathbf{r}_2) = \frac{e^{i\kappa r_{12}}}{4\pi r_{12}^3 \epsilon_b \left(\frac{\omega}{c}\right)^2} \left\{ \kappa^2 r_{12}^2 [\vec{\mathbb{I}} - \mathbf{u}_{12} \mathbf{u}_{12}] + (1 - i\kappa r_{12}) [3\mathbf{u}_{12} \mathbf{u}_{12} - \vec{\mathbb{I}}] \right\} - \frac{\vec{\mathbb{I}}}{3\epsilon_b \left(\frac{\omega}{c}\right)^2} \delta^{(3)}(\mathbf{r}_1 - \mathbf{r}_2)$$

Example: Electric dipole emission:

A ‘point’ electric dipole, $\mathbf{p}$ at the position, $\mathbf{r}_2$, oscillating at angular frequency of ω i.e. characterized by a current : $\mathbf{j}(\mathbf{r}', t) = -i\omega\mathbf{p}e^{-i\omega t}\delta^{(3)}(\mathbf{r}' - \mathbf{r}_2)$.

The electric field measured at a position $\mathbf{r}_1$ is :

$$\mathbf{E}(\mathbf{r}_1) = \frac{i\omega}{\epsilon_0 c^2} \int \vec{\mathbf{g}}(\mathbf{r}_1, \mathbf{r}') \cdot \mathbf{j}(\mathbf{r}') d\mathbf{r}' \rightarrow \frac{\omega^2}{\epsilon_0 c^2} \vec{\mathbf{g}}(\mathbf{r}_1, \mathbf{r}_2) \cdot \mathbf{j}(\mathbf{r}_2)$$

The electric field by a point electric dipole at the position $\vec{\mathbf{r}}_2$ is thus :

$$\mathbf{E}(\mathbf{r}_1) = \frac{e^{ikr_{12}}}{4\pi r_{12}^3 \epsilon_b \epsilon_0} \{ \kappa^2 r_{12}^2 [\mathbf{p} - \mathbf{u}_{12}(\mathbf{u}_{12} \cdot \mathbf{p})] + (1 - i\kappa r_{12}) [3\mathbf{u}_{12}(\mathbf{u}_{12} \cdot \mathbf{p}) - \mathbf{p}] \} - \frac{\mathbf{p}}{3\epsilon_b \epsilon_0} \delta^{(3)}(\mathbf{r}_{12})$$

“far” field

“near” field

“intermediate” field

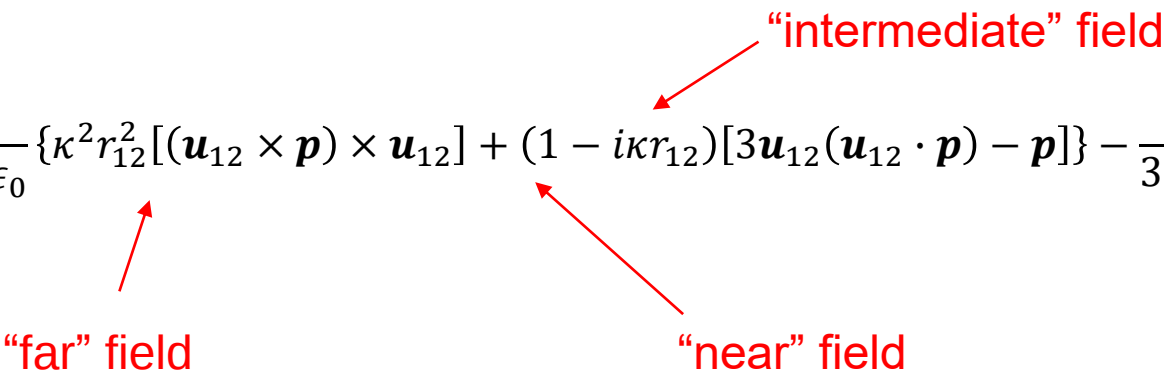
$$\begin{aligned} \mathbf{r}_{12} &\equiv \mathbf{r}_1 - \mathbf{r}_2 \\ r_{12} &\equiv |\mathbf{r}_1 - \mathbf{r}_2| \\ \vec{\mathbf{u}}_{12} &\equiv \frac{\mathbf{r}_1 - \mathbf{r}_2}{r_{12}} \end{aligned}$$

Electric dipole emission:

$$\kappa^2 = \varepsilon_b \mu_b \left(\frac{\omega}{c} \right)^2$$

Alternative expression:

$$\mathbf{E}(\mathbf{r}_1) = \frac{e^{i\kappa r_{12}}}{4\pi r_{12}^3 \varepsilon_b \epsilon_0} \{ \kappa^2 r_{12}^2 [(\mathbf{u}_{12} \times \mathbf{p}) \times \mathbf{u}_{12}] + (1 - i\kappa r_{12}) [3\mathbf{u}_{12}(\mathbf{u}_{12} \cdot \mathbf{p}) - \mathbf{p}] \} - \frac{\mathbf{p}}{3\varepsilon_b \epsilon_0} \delta^{(3)}(\mathbf{r}_{12})$$



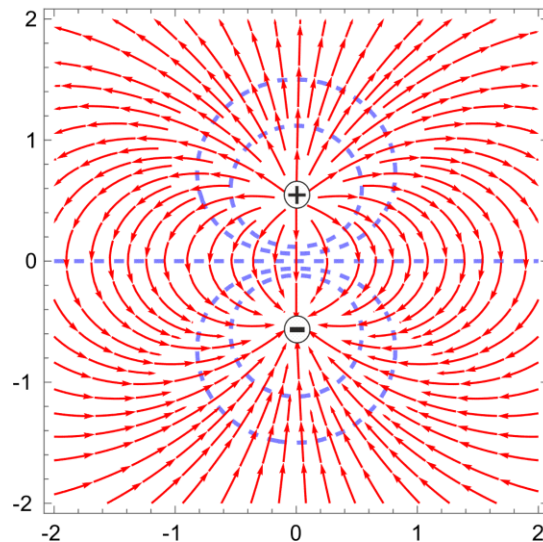
“far” field “near” field “intermediate” field

Such expressions for electric dipole emission are very important but notoriously difficult to derive and have a rich physical interpretation !

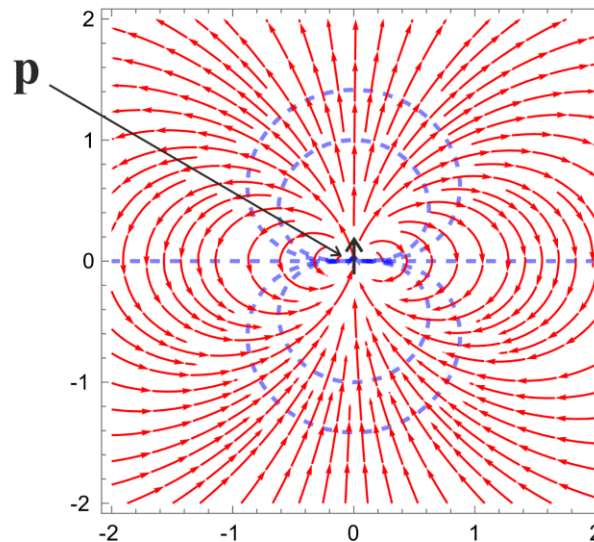
Origin of the delta function contribution: $\kappa^2 = \epsilon_b \mu_b \left(\frac{\omega}{c}\right)^2$

$$\mathbf{E}(\mathbf{r}_1) = \frac{e^{i\kappa r_{12}}}{4\pi r_{12}^3 \epsilon_b \epsilon_0} \{ \kappa^2 r_{12}^2 [(\mathbf{u}_{12} \times \mathbf{p}) \times \mathbf{u}_{12}] + (1 - i\kappa r_{12}) [3\mathbf{u}_{12}(\mathbf{u}_{12} \cdot \mathbf{p}) - \mathbf{p}] \} - \frac{\mathbf{p}}{3\epsilon_b \epsilon_0} \delta^{(3)}(\mathbf{r}_{12})$$

Field of a real two charge dipole



point dipole model



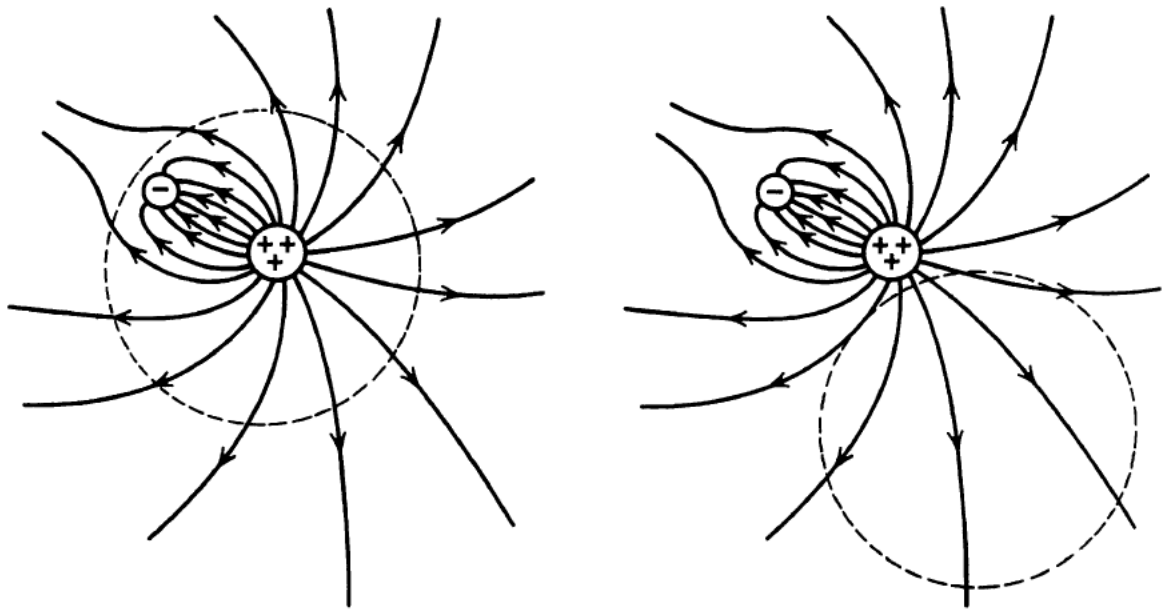
Distribution theory:

The 3D **delta function** accounts for the strong fields between charges in the “point-like” limit

Other references :

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \left[\frac{3\mathbf{n}(\mathbf{p} \cdot \mathbf{n}) - \mathbf{p}}{|\mathbf{x} - \mathbf{x}_0|^3} - \frac{4\pi}{3} \mathbf{p} \delta(\mathbf{x} - \mathbf{x}_0) \right]$$

Jackson (3rd edition: pp 148-150) explains that the delta function contribution to the dipole field corrects for the fact that in the electrostatic limit, the average field inside a spherical (charge-free) region is equal to the field at the center of the sphere, but if the sphere surrounds an electric dipole then the average field is the value at the center of the sphere **plus** $\frac{\mathbf{p}}{3\epsilon_b\epsilon_0}$. The symmetry of the field around a “point” dipole would naively suggest that the volume average in a sphere centered on the dipole vanishes (but this reasoning ignores the strong fields between the positive and negative charges). The 3D delta function $\delta^{(3)}(\mathbf{r}_{12})$ is consequently a necessary addition to the dipole field.



Electro-magnetic field produced by an oscillating electric dipole:

$$\mathbf{E}(\mathbf{r}) = \left[\frac{e^{ikr}}{4\pi r^3 \epsilon_b \epsilon_0} \{k^2 r^2 [\vec{\mathbb{I}} - \mathbf{u}_r \mathbf{u}_r] + (1 - ikr)[3\mathbf{u}_r \mathbf{u}_r - \vec{\mathbb{I}}]\} - \frac{\vec{\mathbb{I}}}{3\epsilon_b \epsilon_0} \delta^{(3)}(\mathbf{r}) \right] \cdot \mathbf{p}$$

$$E_r(r, \theta) = \frac{||\vec{\mathbf{p}}|| \cos \theta \exp(i\kappa r)}{4\pi \epsilon_b \epsilon_0} \frac{1}{r} \kappa^2 \left[\frac{2}{\kappa^2 r^2} - \frac{2i}{\kappa r} \right]$$

$$E_\theta(r, \theta) = \frac{||\vec{\mathbf{p}}|| \sin \theta \exp(i\kappa r)}{4\pi \epsilon_b \epsilon_0} \frac{1}{r} \kappa^2 \left[\frac{1}{\kappa^2 r^2} - \frac{i}{\kappa r} - 1 \right]$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

$$i\omega \mathbf{B} = \nabla \times \mathbf{E}$$

$$\mathbf{H}_{\text{SI}} = \frac{c}{i\omega \mu_b \mu_0} \nabla \times \mathbf{E}$$

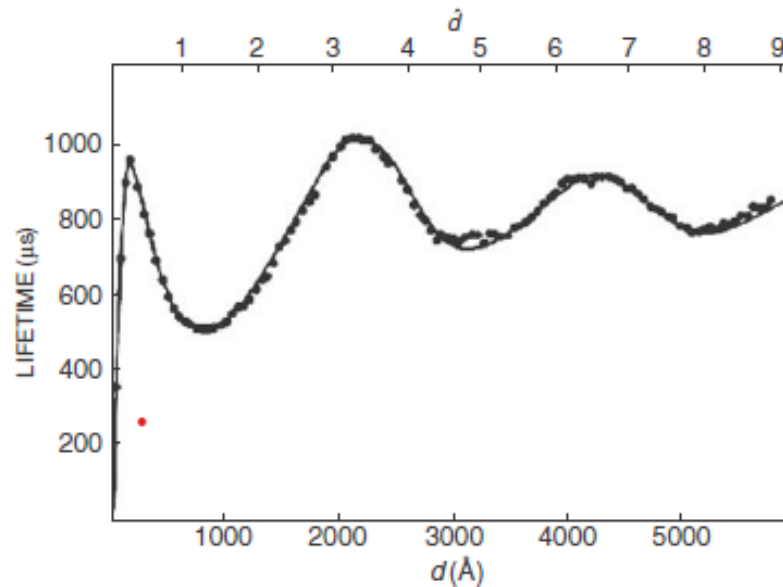
$$H_\phi = \frac{||\vec{\mathbf{p}}|| \sin \theta \exp(i\kappa r)}{4\pi \epsilon_b \epsilon_0} \frac{1}{r} \kappa^2 \left[-\frac{i}{\kappa r} - 1 \right] \sqrt{\frac{\epsilon_0 \epsilon_b}{\mu_0 \mu_b}} ;$$

Environmental effects on Light-matter emission

“Purcell effect”

$$F_P = \frac{\Gamma}{\Gamma_0} = \frac{3}{4\pi^2} \left(\frac{\lambda_{\text{free}}}{n} \right)^3 \frac{Q}{V} = \frac{3}{4\pi^2} \left(\frac{\lambda_{\text{free}}}{n} \right)^3 \rho_P(\mathbf{r}, \omega)$$

Experimental verification of the “Purcell effect”



Comparison of classical theory (curve) and experimental data (points) of molecular lifetimes in inhomogeneous environments. In the experiment, a layer of Eu^{3+} ions is held by fatty-acid spacers of variable thickness close to a silver surface (data after Drexhage [24]). The calculated curve is due to Chance *et al.* [25].

Classical radiation by a point-source (monopolar 'scalar' light source)

$$\bigcirc \bigodot s(O, t) = s_c \operatorname{Re}[e^{-i\omega_0 t}]$$

$$g(\mathbf{r}_1, \mathbf{r}_2) = \frac{e^{ik_b|\mathbf{r}_1 - \mathbf{r}_2|}}{4\pi|\mathbf{r}_1 - \mathbf{r}_2|} \equiv \frac{e^{ik_b r_{12}}}{4\pi r_{12}}$$

$$\phi(\vec{\mathbf{r}}, t) \propto g(\mathbf{r}, O) s(t) \quad \frac{d\mathbf{P}_s}{dS} \propto \phi \nabla \phi \rightarrow c_b k_b \mathbf{u}_r \phi^2$$

$$P_{r,0} \equiv \langle P_r \rangle_0 \propto \frac{1}{T} \int_0^T dt \oint d\mathbf{S} \cdot \frac{d\mathbf{P}_s}{dS} \rightarrow \frac{c_b}{2} \oint d\mathbf{S} \cdot k_b \vec{\mathbf{u}}_r |\phi|^2 \rightarrow \frac{c_b k_b |s_c|^2}{4\pi}$$

$$P_{e,0} \equiv \langle P_e \rangle_0 \propto \frac{1}{T} \int_0^T dt \phi(\vec{\mathbf{0}}, t) s(\vec{\mathbf{0}}, t) \rightarrow \frac{c_b}{2} \operatorname{Im}[g(\vec{\mathbf{0}}, \vec{\mathbf{0}})] |s_c|^2 \rightarrow \frac{c_b \operatorname{Re}(k_b) |s_c|^2}{4\pi}$$

In a lossless host medium : $P_{r,0} = P_{e,0} \equiv P_0$

Modifying the decay rate by the environment (`weak' coupling model - scalar theory)



$$s(O, t) = \text{Re}[s_c e^{-i\omega_0 t}]$$

$$G(\mathbf{r}_1, \mathbf{r}_2) = g(\mathbf{r}_1, \mathbf{r}_2) + G_s(\mathbf{r}_1, \mathbf{r}_2) \quad g(\mathbf{r}_1, \mathbf{r}_2) = \frac{e^{ikr_{12}}}{4\pi r_{12}}$$

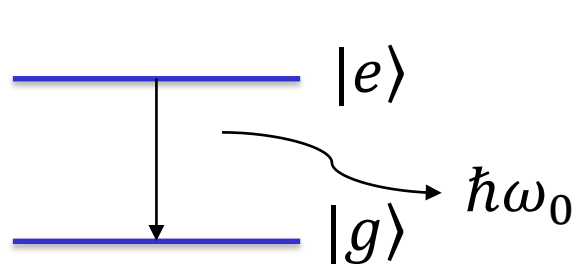
$$P_{e,0} \equiv \langle P_e \rangle_0 \propto \frac{c_b}{2} \text{Im}[g(\mathbf{0}, \mathbf{0})] |s_c|^2 \rightarrow \frac{c_b \text{Re}(k_b) |s_c|^2}{4\pi} = P_0$$

$$P_e^* \equiv \langle P_e \rangle \propto \frac{c_b}{2} \text{Im}[g(\mathbf{0}, \mathbf{0}) + G_s(\mathbf{0}, \mathbf{0})] |s_c|^2$$

$$\frac{P_e^*}{P_{e,0}} = 1 + \frac{2\pi}{\text{Re}(k_b)} \text{Im}[G_s(\mathbf{0}, \mathbf{0})]$$

Unlike $g(\vec{\mathbf{0}}, \vec{\mathbf{0}})$, – the scattering feedback $G_s(\vec{\mathbf{0}}, \vec{\mathbf{0}})$ is regular (no divergence)

Modeling quantum emission as a classical “point-like” antenna far from resonance



$$\mathbf{j}_s(\vec{r}, t) = \text{Re}[-i\omega_0 \mathbf{p} e^{-i\omega_0 t}]$$

$$\vec{E}(\vec{r}, t) = \frac{i\omega_0^2}{\epsilon_0 c^2} \vec{g}(\mathbf{r}, \mathbf{0}) \cdot \mathbf{j}_s(t)$$

$$\langle P_e \rangle = \frac{1}{T} \int_0^T d\mathbf{x} \vec{E}(\mathbf{r}, t) \cdot \mathbf{j}_s(\mathbf{r}, t)$$

$$= \frac{\omega_0^3}{2\epsilon_0 c^2} |\mathbf{p}|^2 \text{Im}[\hat{\mathbf{n}}^* \cdot \vec{g}(\mathbf{0}, \mathbf{0}; \omega) \cdot \hat{\mathbf{n}}] = \gamma_{0,\text{cl}}$$

$$\tau \propto \frac{1}{\gamma_{0,\text{cl}}}$$

Radiation by “point-like” dipole in free space (Weak coupling - high impedance mismatch)

$$\mathbf{0} \uparrow \hat{\mathbf{n}} \quad \vec{\mathbf{j}}_s(\mathbf{r}, t) = \frac{d}{dt} \mathbf{p}_e(\mathbf{0}, t) = \text{Re}[-i\omega_0 p_e \hat{\mathbf{n}} e^{-i\omega_0 t}]$$

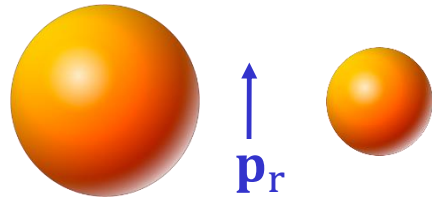
$$\vec{\mathbf{g}}_0(\mathbf{r}) = \frac{c^2 e^{ikr}}{4\pi\omega^2 \epsilon_0 \epsilon_b r^3} \text{P.V.} \{ (-1 + ik_b r) (\vec{\mathbf{I}} - 3\mathbf{u}_r \mathbf{u}_r) + k_b^2 r^2 (\vec{\mathbf{I}} - \mathbf{u}_r \mathbf{u}_r) \} - \frac{c^2 \delta(\mathbf{r})}{3\omega^2 \epsilon_0 \epsilon_b} \vec{\mathbf{I}}$$

$$\mathbf{E}(\vec{\mathbf{r}}, t) = \frac{\omega_0^3}{2\epsilon_0 c^2} \vec{\mathbf{g}}(\mathbf{r}, \mathbf{0}) \cdot \mathbf{p}_e(t)$$

$$\begin{aligned} P_{e,0} \equiv \langle P_e \rangle_0 &= \frac{1}{T} \int_0^T dt \mathbf{E}(\mathbf{0}, t) \cdot \vec{\mathbf{j}}_s(\vec{\mathbf{0}}, t) = \frac{\omega_0^3}{2\epsilon_0 c^2} |\vec{\mathbf{p}}_e|^2 \text{Im}[\hat{\mathbf{n}}^* \cdot \vec{\mathbf{g}}(\vec{\mathbf{0}}, \mathbf{0}; \omega) \cdot \hat{\mathbf{n}}] \\ &= \frac{\omega_0^3 \text{Re}(k_b)}{12\pi\epsilon_0 c^2} |\vec{\mathbf{p}}_e|^2 \equiv P_0 \quad \tau \propto \frac{1}{P_{e,0}} \end{aligned}$$

“Nano” antenna

Sub-wavelength “nano” elements will modify the :
Decay rate, emission frequency, and directivity of quantum emissions

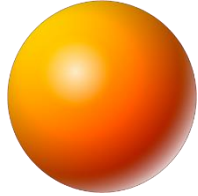


$$E(\mathbf{r}) = \frac{\omega_0^3}{\epsilon_0 c^2} \vec{G}(\mathbf{r}, \mathbf{0}) \cdot \mathbf{p}$$

$$\langle P_e \rangle = \frac{\omega_0^3}{2\epsilon_0 c^2} \text{Im}[\mathbf{p}^* \cdot \vec{G}(\mathbf{0}, \mathbf{0}; \omega) \cdot \mathbf{p}] \propto \gamma^* \propto \text{Re}[Z_{\vec{0}}]$$

Decay rate : $\tau \propto \frac{1}{\gamma^*}$

Modifying the emission rate with a nano-environment



$$\mathbf{j}_s(\mathbf{r}, t) = \frac{d}{dt} \mathbf{p}_e(\mathbf{0}, t) = \text{Re}[-i\omega_0 p_e \hat{\mathbf{n}} e^{-i\omega_0 t}]$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{k_b^2}{\epsilon_0 \epsilon_b c^2} \vec{\mathbf{G}}(\mathbf{r}, \mathbf{0}) \cdot \mathbf{p}_e(t)$$

$$\vec{\mathbf{G}}(\mathbf{r}_1, \mathbf{r}_2) = \vec{\mathbf{g}}(\mathbf{r}_1, \mathbf{r}_2) + \vec{\mathbf{G}}_s(\mathbf{r}_1, \mathbf{r}_2)$$

$$P_e^* \equiv \langle P_e \rangle = \frac{1}{T} \int_0^T dt \vec{\mathbf{E}}(\mathbf{0}, t) \cdot \mathbf{j}_s(\mathbf{0}, t) = \frac{\omega_0^3}{2\epsilon_0 c^2} |\mathbf{p}_e|^2 \text{Im} \left[\hat{\mathbf{n}}^* \cdot \left(\vec{\mathbf{g}}(\mathbf{0}, \mathbf{0}) + \vec{\mathbf{G}}_s(\mathbf{0}, \mathbf{0}) \right) \cdot \hat{\mathbf{n}} \right]$$

$$P_{e,0} \equiv \langle P_e \rangle_0 = \frac{\omega_0^3 \text{Re}(k_b)}{12\pi \epsilon_0 c^2} |\vec{\mathbf{p}}_e|^2$$

$$\frac{P_e^*}{P_{e,0}} = 1 + 6\pi \frac{\text{Im}[\hat{\mathbf{n}}^* \cdot \vec{\mathbf{G}}_s(\mathbf{0}, \mathbf{0}) \cdot \hat{\mathbf{n}}]}{\text{Re}(k_b)}$$

$$\tau \propto \frac{1}{P_e^*}$$

Unlike $\vec{\mathbf{g}}(\vec{\mathbf{0}}, \vec{\mathbf{0}})$, the scattering feedback, $\vec{\mathbf{G}}_s(\vec{\mathbf{0}}, \vec{\mathbf{0}})$ is regular (no divergence)

Radiative emission – “point-like” antenna model (high impedance mismatch)



$$\mathbf{E}(\mathbf{r}, t) = \frac{k_b^2}{\epsilon_0 \epsilon_b c^2} \vec{g}(\mathbf{r}, \mathbf{0}) \cdot \mathbf{p}_e(t)$$

$$\lim_{r \rightarrow \infty} \mathbf{H}(\mathbf{r}, t) = \frac{\epsilon_0 c^2 k_b}{\omega_0} \mathbf{u}_r \times \mathbf{E}(\mathbf{r}, t)$$

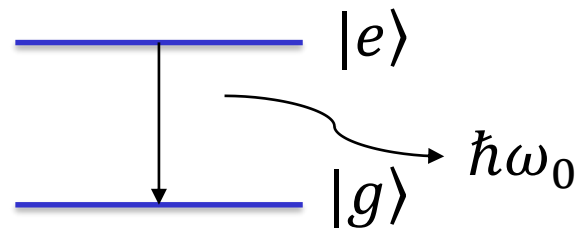
$$P_{r,0} \equiv \langle P_r \rangle_0 = \lim_{r \rightarrow \infty} \frac{1}{T} \int_0^T dt \oint \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t) \cdot d\mathbf{S} \xrightarrow{\text{f.f. free-space}} \frac{\omega_0 k_b^3}{12\pi \epsilon_0 \epsilon_b} |\mathbf{p}_e|^2$$

$$\frac{P_r^*}{P_{r,0}} = \lim_{r \rightarrow \infty} \frac{1}{P_{r,0}} \oint \text{Re}[\mathbf{E}^*(\mathbf{r}, \omega_0) \times \mathbf{H}(\mathbf{r}, \omega_0)] \cdot d\mathbf{S}$$

lossless host medium : $P_{r,0} = P_{e,0} \equiv P_0$

Lossy local environment : $P_r^* < P_e^*$

Quantum emission



$$\tau = \frac{1}{\gamma_{r,q}}$$

“Classical” result

$$P_0 = \frac{\omega_0^4}{12\pi\epsilon_0 c_b^3} |\mathbf{p}|^2$$

“Quantum” – Fermi golden-rule result

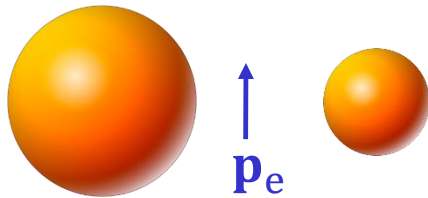
$$\begin{aligned} \gamma_{r,q} &= \frac{4}{\hbar\omega_0} P_0 = \frac{\omega_0^3}{3\pi\epsilon_0 \hbar c_b^3} |\langle g|\mathbf{p}|e\rangle|^2 \\ &= \frac{\pi\omega_0}{3\epsilon_0 \hbar} \frac{\omega_0^2}{\pi^2 c_b^3} |\langle g|\mathbf{p}|e\rangle|^2 \end{aligned}$$

Properties of quantum emission modified by “nano” antennas

Weak coupling approximation $\gamma^* \ll \omega_0$

Sub-wavelength “nano” elements will modify the :

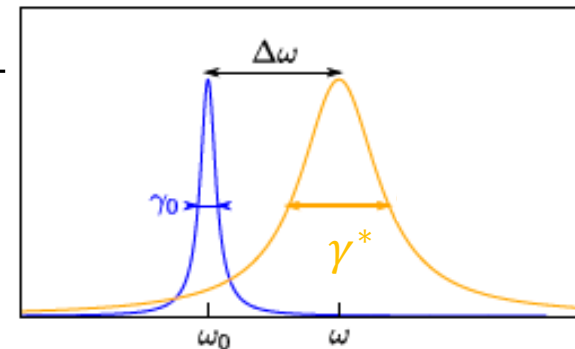
Decay rate, emission frequency, and directivity of quantum emissions



Quantum efficiency : $\eta \equiv \frac{\gamma_r}{\gamma_r + \gamma_{n.r.}} \equiv \frac{\gamma_r}{\gamma_{0,t}}$

lifetime modification : $\frac{\tau_0}{\tau^*} = \frac{\gamma^*}{\gamma_{0,t}} = \frac{P_e^*}{P_{e,0}} = 1 + \eta 6\pi \frac{\text{Im}[\hat{\mathbf{n}}^* \cdot \vec{\mathbf{G}}_s(\mathbf{0}, \mathbf{0}) \cdot \hat{\mathbf{n}}]}{\text{Re}(k_b)}$

“Lamb” shift : $\frac{\Delta\omega}{\gamma_{0,t}} = \eta 3\pi \frac{\text{Re}[\hat{\mathbf{n}}^* \cdot \vec{\mathbf{G}}_s(\mathbf{0}, \mathbf{0}) \cdot \hat{\mathbf{n}}]}{\text{Re}(k_b)}$



Radiation modification : $\frac{\tau_0}{\tau_r^*} = \frac{\gamma_r^*}{\gamma_{0,t}} = 1 + \lim_{r \rightarrow \infty} \frac{\eta}{P_0} \oint \text{Re}[\mathbf{E}^*(\mathbf{r}, \omega_0) \times \mathbf{H}(\mathbf{r}, \omega_0)] \cdot d\mathbf{S}$