Part 2 :

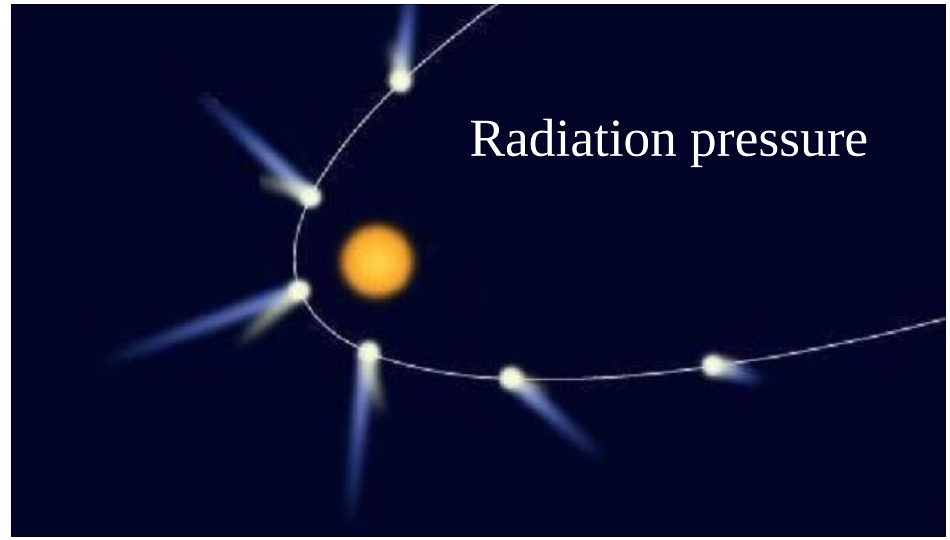
- Light forces : :
  - radiation pressure
  - optical tweezers
  - $E = mc^2$ inertial mass, rest mass

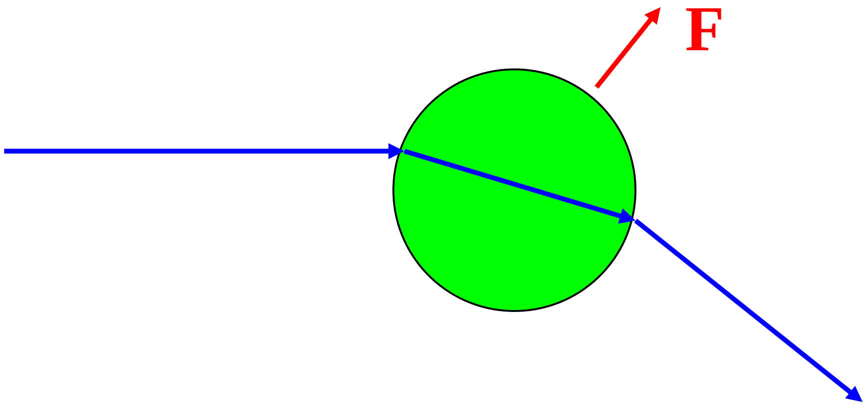
- Generalities of photonic theory :
  - Subtleties with electromagnetic units in SI
  - Time-harmonic formalism and Fourier transforms
  - Transverse and longitudinal fields
  - Light matter interactions in terms of response functions

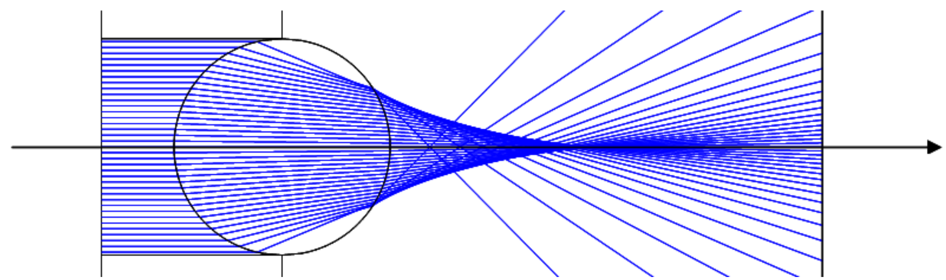
Peter Debye (1909) – radiation pressure

A photon transports
both energy and momentum

$E = cp$



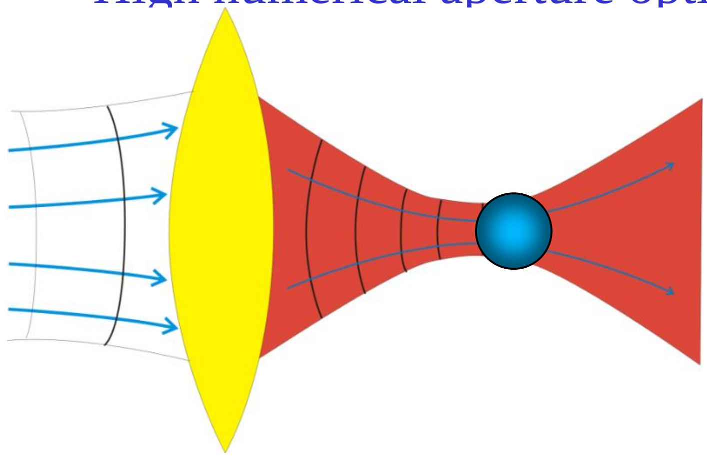


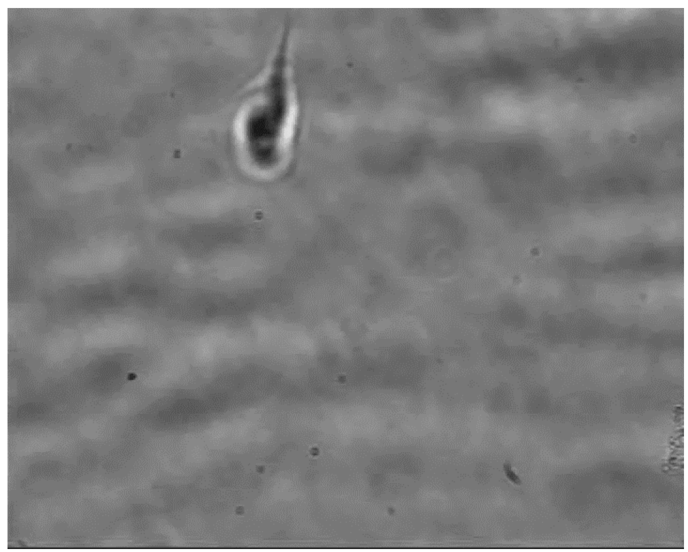


Radiation forces

Optical Tweezers

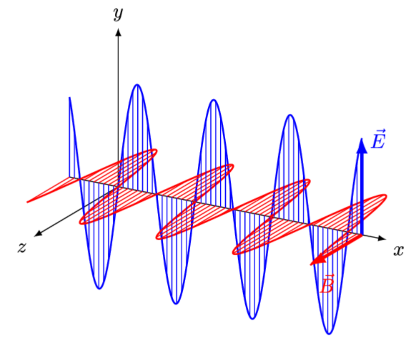
High numerical aperture optics :





Light transports energy and momentum

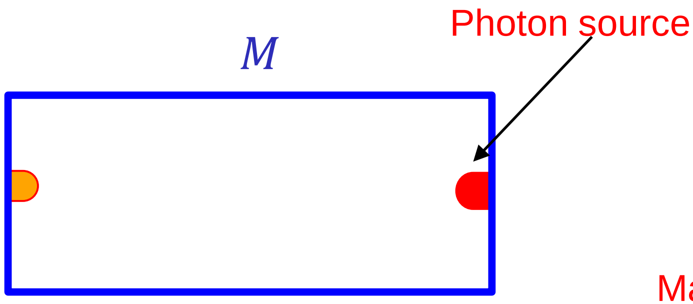
$$\nu\lambda = c$$



Quantum mechanics : $E = h\nu = c\dfrac{h}{\lambda} = cp$

Maxwell's equations : $I = cP \rightarrow E = cp$

A consequence of light carrying momentum



Maxwell equations + Lorentz equation tells us that :

I has the units of power per unit surface $[\mathrm{W.m^{-2}}] = [\mathrm{J.m^{-2}}.s^{-1}] = [\mathrm{N.m^{-1}}.s^{-1}]$

Light pressure, P, has the units $\mathrm{N.m^{-2}}$

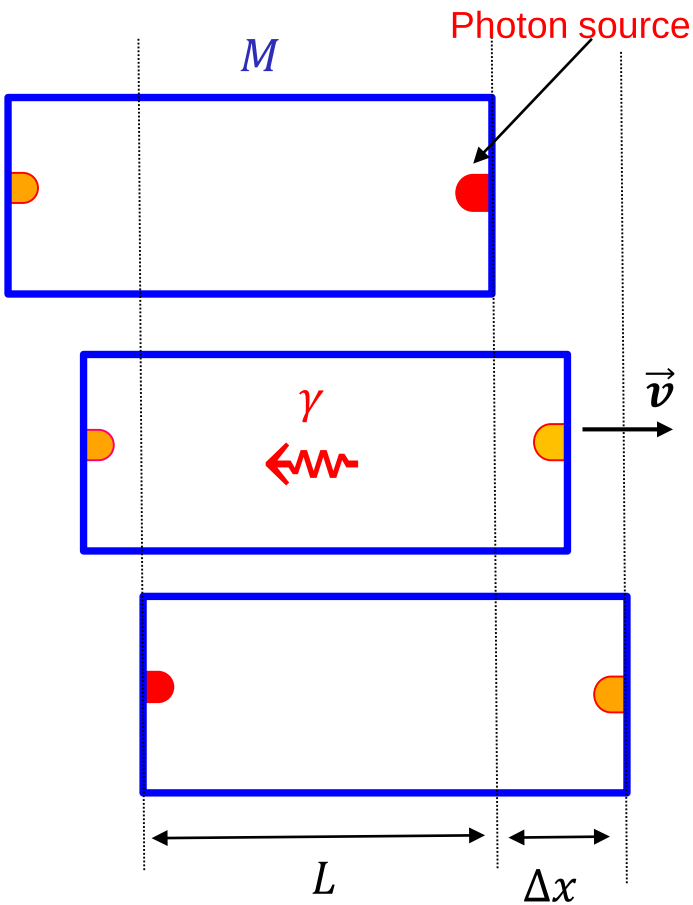
$$\boxed{I = cP \rightarrow E = cp}$$

Quantum mechanics also tells us that :

$$\boxed{E = h\nu = c\frac{h}{\lambda} = cp}$$

The patent clerk argument : step 1

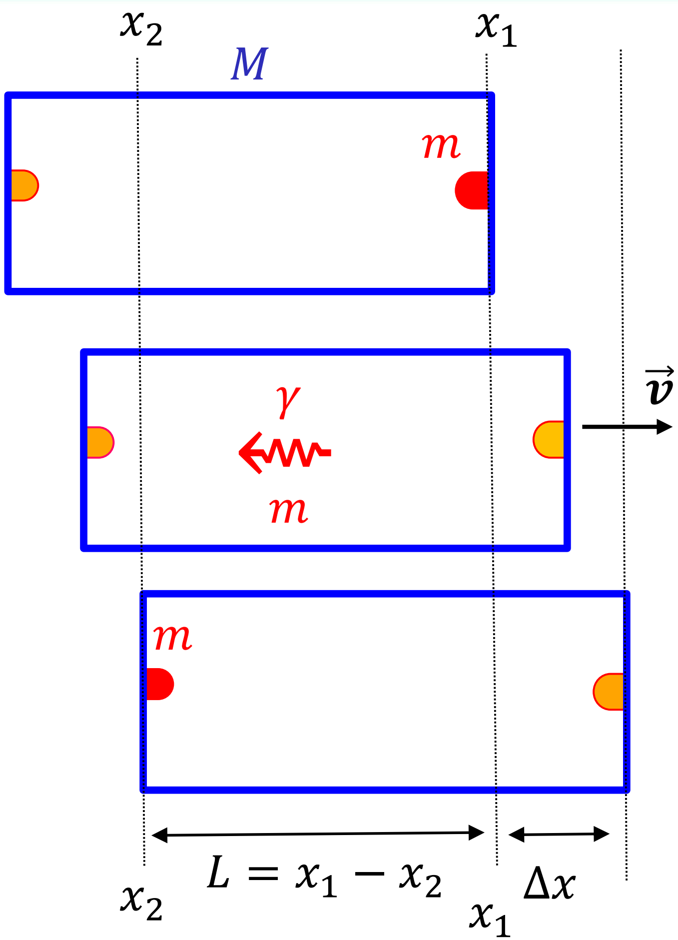
$$E = cp \rightarrow p = \frac{E}{c}$$



$$v = \frac{p}{M} = \frac{E}{Mc}$$

$$\begin{cases} v = \dfrac{\Delta x}{\Delta t} = \dfrac{E}{Mc} \\ \Delta t = \dfrac{L}{c} \end{cases} \Longrightarrow \boxed{\Delta x = L\,\frac{E}{Mc^2}} \quad (1)$$

The patent clerk argument Step 2



Box initially centered at $x = 0$

$$CM = \frac{m x_1}{M + m}$$

$$CM = \frac{M\Delta x + m x_2}{M + m}$$

$$\frac{m x_1}{M + m} = \frac{M\Delta x + m x_2}{M + m} \Rightarrow \boxed{\Delta x = L\frac{m}{M}} \quad (2)$$

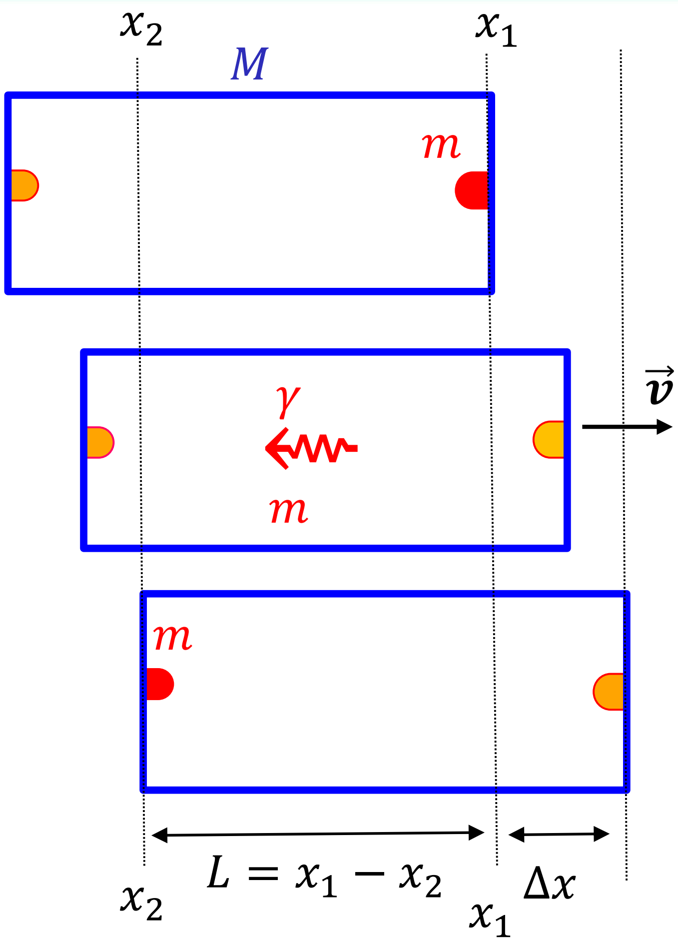
(1)

$$\boxed{\Delta x = L\frac{E}{Mc^2}}$$

$(1) = (2)$

$$L\frac{m}{M} = L\frac{E}{Mc^2}$$

The patent clerk argument Step 2



Box initially centered at $x = 0$

$$CM = \frac{mx_1}{M + m}$$

$$\frac{mx_1}{M + m} = \frac{M\Delta x + mx_2}{M + m}$$

$$CM = \frac{M\Delta x + mx_2}{M + m}$$

(2)

$$\boxed{\Delta x = L\frac{m}{M}}$$

(1)

$$\boxed{\Delta x = L\frac{E}{Mc^2}}$$

$(1) = (2)$

$$L\frac{m}{M} = L\frac{E}{Mc^2}$$

$$\boxed{E = mc^2}$$

Relativistic (inertial) mass

We deduced that photons have a Relativistic (inertial) mass : $E = mc^2$

Recall for photons that : $E = cp \Longrightarrow m = \frac{p}{c}$

Recall for ordinary particles of mass m in Newton's mechanics that $p = mv$

This leads us to define inertial mass as : $m \equiv \frac{p}{v}$

Putting $E = mc^2$ back into the LHS of $m \equiv \frac{p}{v}$, we have : $\frac{E}{c^2} \equiv \frac{p}{v} \Longrightarrow \boxed{E = \frac{pc^2}{v}}$

Recall from Newton's laws that $dE = Fdx = \frac{dp}{dt}dx = \frac{dx}{dt}dp = vdp$ $\quad dE = vdp$

Rest mass m_0

Multiplying both sides of $dE = vdp$ by $E = \frac{pc^2}{v}$, we obtain $EdE = c^2 pdp$

Integrating this relation $\int EdE = c^2 \int pdp$ $\qquad E^2 = c^2p^2 + E_0^2$

Where E_0 is a constant of integration

Given $E = mc^2$, it is natural to define $E_0 = m_0c^2$, with m_0 being henceforth referred to as the rest mass

$$\boxed{E^2 = c^2p^2 + m_0^2c^4}$$

This is the final relativistic relation between the energy and momentum with a rest mass of m_0

Taking rest mass of light to be $m_0 = 0$, we recover our starting point, $E = cp$ for light.
Light has inertial mass but no rest mass !
(A distinction that continues to be a source of confusion !)

Relativistic (inertial) mass : $m(v)\mathrm{v}$

$E^2 = c^2p^2 + m_0^2c^4$ is the relativistic relation between energy, E , momentum, p and rest mass m_0

Note that if we take the particle to be at rest, $p = 0$, then we obtain $E_0 = m_0c^2$

$$E = \frac{pc^2}{v} \Longrightarrow c^2p^2 = \frac{E^2v^2}{c^2} \Longrightarrow E^2 = c^2p^2 + m_0^2c^4 = \frac{E^2v^2}{c^2} + m_0^2c^4$$

$$\Longrightarrow E^2 = \frac{m_0^2c^4}{1 - \frac{v^2}{c^2}} \Longrightarrow E = \frac{m_0c^2}{\sqrt{1 - \frac{v^2}{c^2}}} = m(v)c^2$$

where $m(v)$ is the inertial mass

$$m(v) = \frac{m_0c^2}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma m_0c^2 \qquad \gamma \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad m^2 = \frac{p^2}{c^2} + m_0^2$$

Check that this agrees with non-relativistic mechanics

$$E = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad\qquad x \ll 1 \;\longrightarrow (1-x)^\alpha \cong 1 - \alpha x$$

$$\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \cong 1 + \frac{1}{2}\frac{v^2}{c^2}$$

$$E = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} = m_0 c^2 \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \cong m_0 c^2 \left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) = m_0 c^2 + \frac{1}{2} m_0 v^2$$

We recognize our ordinary mechanics expression for kinetic energy : $\mathrm{K} = \frac{1}{2} m_0 v^2$

Electromagnetic field equations in the vacuum

There are only 2 "Maxwell" equations

Local charge conservation : $\dfrac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$

Electromagnetic equations in International SI units

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}_{\mathrm{SI}}}{\partial t}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B}_{\mathrm{SI}} = \epsilon_0 \mu_0 \frac{\partial \vec{\boldsymbol{E}}}{\partial t} + \mu_0 \vec{\boldsymbol{j}}$$

$$\boldsymbol{F} = q\boldsymbol{E} + q\boldsymbol{v} \times \boldsymbol{B}_{\mathrm{SI}}$$

Uniformed field units

$$\boldsymbol{B} \equiv c\boldsymbol{B}_{\mathrm{SI}} \Longrightarrow$$

Electromagnetic equations for uniformed units

$$\begin{cases} \dfrac{\partial \boldsymbol{B}}{\partial t} = -c\boldsymbol{\nabla} \times \boldsymbol{E} \\[2ex] \dfrac{\partial \boldsymbol{E}}{\partial t} = c\boldsymbol{\nabla} \times \boldsymbol{B} - \dfrac{\boldsymbol{j}}{\epsilon_0} \\[2ex] \boldsymbol{F} = q\boldsymbol{E} + q\dfrac{\boldsymbol{v}}{c} \times \boldsymbol{B} \end{cases}$$

$$\epsilon_0 \frac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \boldsymbol{E}) + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0 \Longrightarrow \boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}$$

Electromagnetic field equations in the vacuum

Uniformed `S.I.' units

$$\frac{\partial \boldsymbol{E}}{\partial t} = c\boldsymbol{\nabla} \times \boldsymbol{B} - \frac{\vec{\boldsymbol{j}}}{\epsilon_0}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = -c\boldsymbol{\nabla} \times \boldsymbol{E}$$

$$\boldsymbol{F} = q\boldsymbol{E} + q\frac{\boldsymbol{v}}{c} \times \boldsymbol{B}$$

Gaussian units $\epsilon_0 \to \frac{1}{4\pi}$

$$\frac{\partial \boldsymbol{E}}{\partial t} = c\boldsymbol{\nabla} \times \boldsymbol{B} - 4\pi\boldsymbol{j}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = -c\boldsymbol{\nabla} \times \boldsymbol{E}$$

$$\boldsymbol{F} = q\boldsymbol{E} + q\frac{\boldsymbol{v}}{c} \times \boldsymbol{B}$$

Electromagnetic field equations in the vacuum

`Quantum' formulation definitions

$$\langle \boldsymbol{r}|\Psi(t)\rangle \equiv \begin{pmatrix} \boldsymbol{e}(\boldsymbol{r},t) \\ \boldsymbol{b}(\boldsymbol{r},t) \end{pmatrix} \qquad \hat{\mathbb{L}} \equiv ic\begin{pmatrix} 0 & \boldsymbol{\nabla}\times \\ \boldsymbol{\nabla}\times & 0 \end{pmatrix} \qquad \hat{\Gamma}_0 \equiv \begin{pmatrix} \overleftrightarrow{\mathbb{I}} & 0 \\ 0 & -\overleftrightarrow{\mathbb{I}} \end{pmatrix} \qquad \langle \boldsymbol{r}|j(t)\rangle \equiv \begin{pmatrix} \boldsymbol{j}(\boldsymbol{r},t)/i\epsilon_0 \\ \boldsymbol{0} \end{pmatrix}$$

$$i\frac{\partial}{\partial t}\begin{pmatrix} \overleftrightarrow{\mathbb{I}} & 0 \\ 0 & -\overleftrightarrow{\mathbb{I}} \end{pmatrix}\begin{pmatrix} \boldsymbol{e}(\boldsymbol{r},t) \\ \boldsymbol{b}(\boldsymbol{r},t) \end{pmatrix} - ic\begin{pmatrix} 0 & \boldsymbol{\nabla}\times \\ \boldsymbol{\nabla}\times & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{e}(\boldsymbol{r},t) \\ \boldsymbol{b}(\boldsymbol{r},t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{j}(\boldsymbol{r},t)/i\epsilon_0 \\ \boldsymbol{0} \end{pmatrix}$$

$$i\frac{\partial}{\partial t}\hat{\Gamma}_0|\Psi\rangle - \hat{\mathbb{L}}|\Psi\rangle = |j\rangle$$

Maxwell equations become a single equation
(currents are an electric "source" of fields)

$$\boldsymbol{F} = q\boldsymbol{e} + q\frac{\boldsymbol{v}}{c}\times\boldsymbol{b}$$

Lorentz force equation determines how fields affect particles

Electromagnetic field equations in the vacuum

$$\boxed{i\frac{\partial}{\partial t}\begin{pmatrix}\overleftrightarrow{\mathbb{I}} & 0\\ 0 & -\overleftrightarrow{\mathbb{I}}\end{pmatrix}\begin{pmatrix}\boldsymbol{e}(\boldsymbol{r},t)\\ \boldsymbol{b}(\boldsymbol{r},t)\end{pmatrix} - ic\begin{pmatrix}0 & \boldsymbol{\nabla}\times\\ \boldsymbol{\nabla}\times & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{e}(\boldsymbol{r},t)\\ \boldsymbol{b}(\boldsymbol{r},t)\end{pmatrix} = \begin{pmatrix}\boldsymbol{j}(\boldsymbol{r},t)/i\epsilon_0\\ \boldsymbol{0}\end{pmatrix}}$$

$$\frac{\partial \boldsymbol{e}}{\partial t} = c\boldsymbol{\nabla}\times\boldsymbol{b} - \frac{\boldsymbol{j}}{\epsilon_0}$$

$$\frac{\partial^2 \boldsymbol{e}}{\partial t^2} = c\boldsymbol{\nabla}\times\frac{\partial \boldsymbol{b}}{\partial t} - \frac{1}{\epsilon_0}\frac{\partial \boldsymbol{j}}{\partial t}$$

$$\frac{\partial \boldsymbol{b}}{\partial t} = -c\boldsymbol{\nabla}\times\boldsymbol{e}$$

$$\begin{cases} c^2\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{e} + \dfrac{\partial^2 \boldsymbol{e}}{\partial t^2} = -\dfrac{1}{\epsilon_0}\dfrac{\partial \boldsymbol{j}}{\partial t} \\[2ex] c^2\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{b} + \dfrac{\partial^2 \boldsymbol{b}}{\partial t^2} = \dfrac{c}{\epsilon_0}\boldsymbol{\nabla}\times\boldsymbol{j} \end{cases}$$

Helmholtz theorem :

a field can be separated into transverse and longitudinal components

$$\boldsymbol{j} = \boldsymbol{j}_\perp + \boldsymbol{j}_\parallel$$

Such that : $\boldsymbol{\nabla} \cdot \boldsymbol{j}_\perp \equiv 0$ ***everywhere !*** $\boldsymbol{j}_\perp$ is called the transverse field

Such that : $\boldsymbol{\nabla} \times \boldsymbol{j}_\parallel \equiv 0$ ***everywhere !*** $\boldsymbol{j}_\parallel$ is called the longitudinal field

Since

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{f}) \equiv 0$$

$$c^2 \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{b} + \frac{\partial^2 \boldsymbol{b}}{\partial t^2} = \frac{c}{\epsilon_0} \boldsymbol{\nabla} \times \boldsymbol{j} \quad \Longrightarrow \quad c^2 \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{b}_\perp + \frac{\partial^2 \boldsymbol{b}_\perp}{\partial t^2} = \frac{c}{\epsilon_0} \boldsymbol{\nabla} \times \boldsymbol{j}_\perp$$

$$\boldsymbol{b}_\parallel = \boldsymbol{0}$$

$$c^2 \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{e} + \frac{\partial^2 \boldsymbol{e}}{\partial t^2} = -\frac{1}{\epsilon_0} \frac{\partial \boldsymbol{j}}{\partial t} \quad \Longrightarrow \quad c^2 \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{e}_\perp + \frac{\partial^2 \boldsymbol{e}_\perp}{\partial t^2} = -\frac{1}{\epsilon_0} \frac{\partial \boldsymbol{j}_\perp}{\partial t}$$

$$\frac{\partial}{\partial t} \left(\frac{\partial \boldsymbol{\nabla} \cdot \boldsymbol{e}_\parallel}{\partial t} + \frac{1}{\epsilon_0} \boldsymbol{\nabla} \cdot \boldsymbol{j}_\parallel \right) = 0$$

Helmholtz theorem :
a field can be separated into transverse and longitudinal components

We can define charge density as $\boldsymbol{\nabla}\cdot\boldsymbol{e}_{\parallel}\equiv\frac{\rho}{\epsilon_0}$ so that $\frac{\partial}{\partial t}\left(\frac{\partial\boldsymbol{\nabla}\cdot\boldsymbol{e}_{\parallel}}{\partial t}+\frac{1}{\epsilon_0}\boldsymbol{\nabla}\cdot\boldsymbol{j}_{\parallel}\right)=0$

is the statement of charge conservation $\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot\boldsymbol{j}_{\parallel}=0$

We can solve for $\boldsymbol{e}_{\parallel}$ by remembering that a longitudinal field can be written as $\boldsymbol{e}_{\parallel}=-\boldsymbol{\nabla}\phi$

$$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\phi\equiv\Delta\phi\equiv-\frac{\rho}{\epsilon_0}\quad\Longrightarrow\quad\phi(\boldsymbol{r},t)\equiv\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\boldsymbol{r}',t)}{|\boldsymbol{r}-\boldsymbol{r}'|}d\boldsymbol{r}'$$

$$\Longrightarrow\quad\boldsymbol{e}_{\parallel}(\boldsymbol{r},t)\equiv\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\boldsymbol{r}',t)}{|\boldsymbol{r}-\boldsymbol{r}'|^2}d\boldsymbol{r}'$$

The problem is that this interaction is “Instantaneous” and “violates” causality

Antenna theory – Quantum field theory solution to the transverse-longitudinal problem

$$\mu_0 = \frac{1}{\epsilon_0 c^2}$$

We introduce a "vector potential" such that : $\boldsymbol{b}_\perp = \boldsymbol{\nabla} \times \boldsymbol{a} = \boldsymbol{\nabla} \times \boldsymbol{a}_\perp$

And a scalar potential : $\boldsymbol{e} = -\frac{\partial \boldsymbol{a}}{\partial t} - \boldsymbol{\nabla}\phi$ $\Rightarrow$ $\boldsymbol{\nabla}^2\phi = -\frac{\rho}{\epsilon_0} - \boldsymbol{\nabla}\cdot\frac{\partial \boldsymbol{a}_\parallel}{\partial t}$

$$\boldsymbol{\nabla}^2\boldsymbol{a} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{a}}{\partial t^2} = \mu_0 \boldsymbol{j} + \boldsymbol{\nabla}\left[\boldsymbol{\nabla}\cdot\boldsymbol{a} + \frac{\partial\phi}{\partial t}\right] \Rightarrow \left\{\begin{array}{l} \boldsymbol{\nabla}^2\boldsymbol{a}_\perp - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{a}_\perp}{\partial t^2} = \frac{1}{\epsilon_0 c^2}\boldsymbol{j}_\perp \\ \boldsymbol{\nabla}^2\boldsymbol{a}_\parallel - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{a}_\parallel}{\partial t^2} = \frac{1}{\epsilon_0 c^2}\boldsymbol{j}_\parallel + \boldsymbol{\nabla}\left[\boldsymbol{\nabla}\cdot\boldsymbol{a}_\parallel + \frac{\partial\phi}{\partial t}\right] \end{array}\right.$$

Antenna theory – Quantum field theory solution to the transverse-longitudinal problem

We introduce a "vector potential" such that : $\boldsymbol{b}_\perp = \nabla \times \boldsymbol{a} = \nabla \times \boldsymbol{a}_\perp$ $\qquad \boldsymbol{e} = -\frac{\partial \boldsymbol{a}}{\partial t} - \nabla \phi$

$$\nabla^2 \boldsymbol{a} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{a}}{\partial t^2} = \mu_0 \boldsymbol{j} + \nabla\left[\nabla \cdot \boldsymbol{a} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}\right]$$

$$\nabla^2 \boldsymbol{a}_\perp - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{a}_\perp}{\partial t^2} = \frac{1}{\epsilon_0 c^2}\boldsymbol{j}_\perp \qquad\qquad \nabla^2 \phi = -\frac{\rho}{\epsilon_0} - \nabla \cdot \frac{\partial \boldsymbol{a}_\parallel}{\partial t}$$

The potentials have introduced a "gauge" degree of freedom

$\boldsymbol{a}_\perp$ is "gauge" independent, while $\boldsymbol{a}_\parallel$ can be chosen to "simplify" the equations

Lorenz gauge : $\nabla \cdot \boldsymbol{a}_\parallel + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0 \quad \Longrightarrow \quad \nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}$

Coulomb gauge : $\boldsymbol{a}_\parallel = \boldsymbol{0} \quad \Longrightarrow \quad \nabla^2 \phi = -\frac{\rho}{\epsilon_0}$

Photonics typically requires dealing with material properties

Electromagnetic field equations in lossless local media

Electromagnetic equations in International SI units

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}_{\mathrm{SI}}}{\partial t}$$

$$\nabla \times \boldsymbol{H}_{\mathrm{SI}} = \frac{\partial \boldsymbol{D}_{\mathrm{SI}}}{\partial t} + \boldsymbol{j}_s$$

$$\boldsymbol{F} = q\boldsymbol{E} + q\boldsymbol{v} \times \boldsymbol{B}_{\mathrm{SI}}$$

Uniformed field units

$$\boldsymbol{D} \equiv \frac{\boldsymbol{D}_{\mathrm{SI}}}{\epsilon_0}$$

$$\boldsymbol{B} \equiv c\boldsymbol{B}_{\mathrm{SI}}$$

$$\boldsymbol{H} \equiv \sqrt{\frac{\mu_0}{\epsilon_0}}\boldsymbol{H}_{\mathrm{SI}} = \frac{\boldsymbol{H}_{\mathrm{SI}}}{\epsilon_0 c}$$

$\Longrightarrow$

Electromagnetic equations for uniformed units

$$\begin{cases} \dfrac{\partial \boldsymbol{B}}{\partial t} = -c\nabla \times \boldsymbol{E} \\[2ex] \dfrac{\partial \boldsymbol{D}}{\partial t} = c\nabla \times \boldsymbol{H} - \dfrac{\boldsymbol{j}_s}{\epsilon_0} \end{cases}$$

$$\boldsymbol{F} = q\boldsymbol{E} + q\frac{\boldsymbol{v}}{c} \times \boldsymbol{B}$$

Local charge conservation

$$\frac{\partial}{\partial t}(\nabla \cdot \overleftrightarrow{\boldsymbol{\varepsilon}} \cdot \boldsymbol{E}) + \nabla \cdot \boldsymbol{j}_s = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{j}_s = 0 \Longrightarrow \nabla \cdot \boldsymbol{D} = \frac{\rho_s}{\epsilon_0}$$

Transverse electromagnetic fields in optics ($\mu = 1$ no sources) :

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{E} - \varepsilon(\boldsymbol{r}) \left(\frac{\omega}{c}\right)^2 \boldsymbol{E} = \boldsymbol{0}$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{F}) \equiv 0 \quad \Longrightarrow \quad \boldsymbol{\nabla} \cdot \varepsilon(\boldsymbol{r})\vec{\boldsymbol{E}}(\boldsymbol{r}) = \boldsymbol{\nabla} \cdot \boldsymbol{D}(\boldsymbol{r}) = 0$$

Source-free fields, $\boldsymbol{D}$ and $\boldsymbol{B}$ are ***transverse*, ** in ***heterogeneous*** media

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \qquad \boldsymbol{\nabla} \times \boldsymbol{D} \neq \boldsymbol{0} \qquad \boldsymbol{\nabla} \times \boldsymbol{B} \neq \boldsymbol{0}$$

The fields $\boldsymbol{E}$ and $\boldsymbol{H}$ are ***transverse*,** $\boldsymbol{\nabla} \cdot \boldsymbol{E} = \boldsymbol{\nabla} \cdot \boldsymbol{H} = \boldsymbol{0}$**,** only in ***source-free*** and ***homogenous*** media

In free space without sources, ***longitudinal*** electric fields ($\boldsymbol{\nabla} \times \boldsymbol{E}_{\parallel} = \boldsymbol{0}$ but $\boldsymbol{\nabla} \cdot \boldsymbol{E}_{\parallel} \neq 0$)

Instantaneous constitutive relations

$$\frac{\partial \boldsymbol{B}}{\partial t} = -c\boldsymbol{\nabla} \times \boldsymbol{E} \qquad \frac{\partial \boldsymbol{D}}{\partial t} = c\boldsymbol{\nabla} \times \boldsymbol{H} - \frac{\boldsymbol{j}_s}{\epsilon_0}$$

$$\boldsymbol{D}(\boldsymbol{r},t) = \int_{-\infty}^{t} \overleftrightarrow{\varepsilon}(\boldsymbol{r},t-t') \cdot \boldsymbol{E}(\boldsymbol{r},t')dt'$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \int_{-\infty}^{t} \overleftrightarrow{\mu}(\boldsymbol{r},t-t') \cdot \boldsymbol{H}(\boldsymbol{r},t')dt'$$

Instantaneous interaction
or quasi-static limit

$$\overleftrightarrow{\varepsilon}(\boldsymbol{r},t-t') = \overleftrightarrow{\varepsilon}(\boldsymbol{r})\delta(t-t')$$

$$\overleftrightarrow{\mu}_m(\boldsymbol{r},t-t') = \overleftrightarrow{\mu}(\boldsymbol{r})\delta(t-t')$$

Real symmetric
matrices

$$\overleftrightarrow{\mu}(\vec{\boldsymbol{r}}) \cdot \frac{\partial \boldsymbol{H}(\vec{\boldsymbol{r}},t)}{\partial t} = -c\boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{r},t)$$

$$\overleftrightarrow{\varepsilon}(\vec{\boldsymbol{r}}) \cdot \frac{\partial \boldsymbol{E}(\vec{\boldsymbol{r}},t)}{\partial t} = c\boldsymbol{\nabla} \times \boldsymbol{H}(\boldsymbol{r},t) - \frac{\vec{\boldsymbol{j}}_s}{\epsilon_0}$$

Time-harmonic vs Fourier Transform

$$\begin{pmatrix} \boldsymbol{e}(\boldsymbol{r},t) \\ \boldsymbol{b}(\boldsymbol{r},t) \end{pmatrix} = \mathrm{Re}\left\{ \begin{pmatrix} \boldsymbol{e}(\boldsymbol{r}) \\ \boldsymbol{b}(\boldsymbol{r}) \end{pmatrix} e^{-i\omega t} \right\} \qquad j_\omega(t)\rangle = \mathrm{Re}\left\{ \begin{pmatrix} \boldsymbol{j}(\boldsymbol{r})/i\epsilon_0 \\ \mathbf{0} \end{pmatrix} e^{-i\omega t} \right\}$$

Time-harmonic

$$\begin{pmatrix} \boldsymbol{e}(\boldsymbol{r},t) \\ \boldsymbol{b}(\boldsymbol{r},t) \end{pmatrix} = \int_{-\infty}^{\infty} d\omega \begin{pmatrix} \boldsymbol{e}(\boldsymbol{r},\omega) \\ \boldsymbol{b}(\boldsymbol{r},\omega) \end{pmatrix} e^{-i\omega t} \quad \Longrightarrow \quad \begin{aligned} \boldsymbol{e}(\boldsymbol{r},-\omega^*) &= \boldsymbol{e}^*(\boldsymbol{r},\omega) \\ \boldsymbol{b}(\boldsymbol{r},-\omega^*) &= \boldsymbol{b}^*(\boldsymbol{r},\omega) \\ \boldsymbol{j}(\boldsymbol{r},-\omega^*) &= \boldsymbol{j}^*(\boldsymbol{r},\omega) \end{aligned}$$

Fourier transform

Maxwell equations in the frequency domain

$$\boldsymbol{D}(\boldsymbol{r},t) = \int_{-\infty}^{t} \overleftrightarrow{\varepsilon}(\boldsymbol{r},t-t')\cdot\boldsymbol{E}(\boldsymbol{r},t')dt'$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \int_{-\infty}^{t} \overleftrightarrow{\mu}(\boldsymbol{r},t-t')\cdot\boldsymbol{H}(\boldsymbol{r},t')dt'$$

$\longrightarrow$

$$\boldsymbol{D}(\boldsymbol{r},\omega) = \overleftrightarrow{\varepsilon}(\boldsymbol{r},\omega)\cdot\boldsymbol{E}(\boldsymbol{r},\omega)$$

$$\boldsymbol{B}(\boldsymbol{r},\omega) = \overleftrightarrow{\mu}(\vec{\mathbf{r}},\omega)\cdot\boldsymbol{H}(\vec{\mathbf{r}},\omega)$$

$$\frac{\partial\boldsymbol{B}}{\partial t} = -c\boldsymbol{\nabla}\times\boldsymbol{E}$$

$$\frac{\partial\boldsymbol{D}}{\partial t} = c\boldsymbol{\nabla}\times\boldsymbol{H} - \frac{\boldsymbol{j}_s}{\epsilon_0}$$

$\longrightarrow$

$$i\omega\boldsymbol{B} = c\boldsymbol{\nabla}\times\boldsymbol{E}$$

$$-i\omega\boldsymbol{D} = c\boldsymbol{\nabla}\times\boldsymbol{H} - \frac{\boldsymbol{j}_s}{\epsilon_0}$$

Time-harmonic vs Fourier Transform

$$\underbrace{\psi_\omega(\boldsymbol{r},t) = \mathrm{Re}\left\{\psi(\boldsymbol{r},\omega)e^{-i\omega t}\right\}}_{\text{Time-harmonic formalism}}$$

$$\underbrace{\psi(\boldsymbol{r},t) = \int_{-\infty}^{\infty} d\omega\, \psi(\boldsymbol{r},\omega)e^{-i\omega t}}_{\text{Temporal Fourier transform}} \Longrightarrow \psi(\boldsymbol{r},-\omega^*) = \psi^*(\boldsymbol{r},\omega)$$

Space-time evolution of `classical' electromagnetic fields

In-medium Maxwell equation

$$i\frac{\partial}{\partial t}\int_{-\infty}^{t} dt' \begin{pmatrix} \overleftrightarrow{\varepsilon}(\boldsymbol{r}, t-t') & 0 \\ 0 & -\overleftrightarrow{\mu}(\boldsymbol{r}, t-t') \end{pmatrix} \begin{pmatrix} \boldsymbol{e}(\boldsymbol{r}, t') \\ \boldsymbol{h}(\boldsymbol{r}, t') \end{pmatrix} = ic \begin{pmatrix} 0 & \boldsymbol{\nabla}\times \\ \boldsymbol{\nabla}\times & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{e}(\boldsymbol{r}, t) \\ \boldsymbol{h}(\boldsymbol{r}, t) \end{pmatrix} + \begin{pmatrix} \boldsymbol{j}(\boldsymbol{r}, t)/i\epsilon_0 \\ \boldsymbol{0} \end{pmatrix}$$

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`Quantum' formulation definitions

$$\langle \boldsymbol{r}|\Psi(t)\rangle \equiv \begin{pmatrix} \boldsymbol{e}(\boldsymbol{r}, t) \\ \boldsymbol{h}(\boldsymbol{r}, t) \end{pmatrix} \qquad \hat{\mathbb{L}} \equiv ic \begin{pmatrix} 0 & \boldsymbol{\nabla}\times \\ \boldsymbol{\nabla}\times & 0 \end{pmatrix} \qquad \langle \boldsymbol{r}|j(t)\rangle \equiv \begin{pmatrix} \boldsymbol{j}(\boldsymbol{r}, t)/i\epsilon_0 \\ \boldsymbol{0} \end{pmatrix}$$

$$\hat{\Gamma}(t-t') \equiv \begin{pmatrix} \overleftrightarrow{\varepsilon}(\boldsymbol{r}, t-t') & 0 \\ 0 & -\overleftrightarrow{\mu}(\boldsymbol{r}, t-t') \end{pmatrix}$$

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Time dependent Maxwell equations

$$i\frac{\partial}{\partial t}\int_{-\infty}^{t} dt' \hat{\Gamma}(t-t')|\Psi(t')\rangle = ic\hat{\mathbb{L}}|\Psi(t)\rangle + |j(t)\rangle$$

Frequency Domain Maxwell equations

$$|\Psi(\boldsymbol{r},t)\rangle = \int_{-\infty}^{\infty} d\omega \begin{pmatrix} \boldsymbol{e}(\boldsymbol{r},\omega) \\ \boldsymbol{h}(\boldsymbol{r},\omega) \end{pmatrix} e^{-i\omega t} = \int_{-\infty}^{\infty} d\omega\, |\Psi(\omega)\rangle e^{-i\omega t} \Longrightarrow$$

$$\boldsymbol{e}(\boldsymbol{r},-\omega^*) = \boldsymbol{e}^*(\boldsymbol{r},\omega)$$

$$\boldsymbol{h}(\boldsymbol{r},-\omega^*) = \boldsymbol{h}^*(\boldsymbol{r},\omega)$$

$$\boldsymbol{j}(\boldsymbol{r},-\omega^*) = \boldsymbol{j}^*(\boldsymbol{r},\omega)$$

$$\boxed{\omega \underbrace{\begin{pmatrix} \overleftrightarrow{\varepsilon}(\boldsymbol{r},\omega) & 0 \\ 0 & -\overleftrightarrow{\mu}(\boldsymbol{r},\omega) \end{pmatrix}}_{\hat{\Gamma}_\omega} \begin{pmatrix} \boldsymbol{e}(\boldsymbol{r},\omega) \\ \boldsymbol{h}(\boldsymbol{r},\omega) \end{pmatrix} - ic \begin{pmatrix} 0 & \boldsymbol{\nabla}\times \\ \boldsymbol{\nabla}\times & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{e}(\boldsymbol{r},\omega) \\ \boldsymbol{h}(\boldsymbol{r},\omega) \end{pmatrix} = \begin{pmatrix} \boldsymbol{j}(\boldsymbol{r},\omega)/i\epsilon_0 \\ \boldsymbol{0} \end{pmatrix}}$$

Time independent Maxwell equation !

$$\boxed{\omega\hat{\Gamma}_\omega|\Psi\rangle - \hat{\mathbb{L}}\big|\Psi\rangle = |j\rangle}$$

Inhomogeneous media in the frequency domain

$$\boldsymbol{D}(\boldsymbol{r},\omega) = \varepsilon(\boldsymbol{r},\omega)\boldsymbol{E}(\boldsymbol{r},\omega)$$

$$\boldsymbol{B}(\boldsymbol{r},\omega) = \mu(\boldsymbol{r},\omega)\boldsymbol{H}(\boldsymbol{r},\omega)$$

$$i\omega\boldsymbol{B} = c\boldsymbol{\nabla}\times\boldsymbol{E}$$

$$c\boldsymbol{\nabla}\times\boldsymbol{H} = -i\omega\boldsymbol{D} + \frac{\boldsymbol{j}_s}{\epsilon_0}$$

$$\Longrightarrow$$

$$i\omega\mu(\boldsymbol{r},\omega)\boldsymbol{H}(\boldsymbol{r},\omega) = c\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r},\omega)$$

$$c\boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{r},\omega) = -i\omega\varepsilon(\boldsymbol{r},\omega)\boldsymbol{E}(\boldsymbol{r},\omega) + \frac{\boldsymbol{j}_s(\boldsymbol{r},\omega)}{\epsilon_0}$$

$$\boxed{\boldsymbol{\nabla}\times\frac{1}{\mu(\boldsymbol{r},\omega)}\boldsymbol{\nabla}\times\boldsymbol{E} - \frac{\omega^2}{c^2}\varepsilon(\boldsymbol{r},\omega)\boldsymbol{E}(\boldsymbol{r},\omega) = \frac{i\omega\boldsymbol{j}_s}{\epsilon_0 c^2}}$$

$$\boldsymbol{\nabla}\times\frac{1}{\varepsilon(\boldsymbol{r},\omega)}\boldsymbol{\nabla}\times\boldsymbol{H} - \frac{\omega^2}{c^2}\mu(\boldsymbol{r},\omega)\boldsymbol{H}(\boldsymbol{r},\omega) = \boldsymbol{\nabla}\times\frac{1}{\varepsilon(\boldsymbol{r},\omega)}\frac{\boldsymbol{j}_s}{\epsilon_0 c}$$

Resonant States (Quasi-Normal Modes)

Complex eigenstates of the homogeneous Maxwell equation

$$[\omega\Gamma(\omega) - \mathbb{L}]\left|\Psi_q^{(\pm)}\right\rangle = 0 \Longrightarrow ic\begin{pmatrix} 0 & \nabla\times \\ \nabla\times & 0 \end{pmatrix}\begin{pmatrix} \vec{\boldsymbol{e}}_q^{(\pm)} \\ \vec{\boldsymbol{h}}_q^{(\pm)} \end{pmatrix} = \omega_q^{(\pm)}\begin{pmatrix} \varepsilon\left(\boldsymbol{r}, \omega_q^{(\pm)}\right) & 0 \\ 0 & -\mu\left(\boldsymbol{r}, \omega_q^{(\pm)}\right) \end{pmatrix}\begin{pmatrix} \vec{\boldsymbol{e}}_q^{(\pm)} \\ \vec{\boldsymbol{h}}_q^{(\pm)} \end{pmatrix}$$

$$\Big\Downarrow$$

$$\hat{\mathbb{L}}\left|\Psi_q^{(\pm)}\right\rangle = \omega_q^{(\pm)}\Gamma\left(\omega_q^{(\pm)}\right)\left|\Psi_q^{(\pm)}\right\rangle$$

Electric dipole RS frequencies satisfying outgoing boundary conditions (blue dots) in the complex plane while red dots are eigenvalues satisfying incoming boundary conditions