Introduction to nanophotonics

Part 1 :

(Light properties, Green Functions, Density of states/Black-body)

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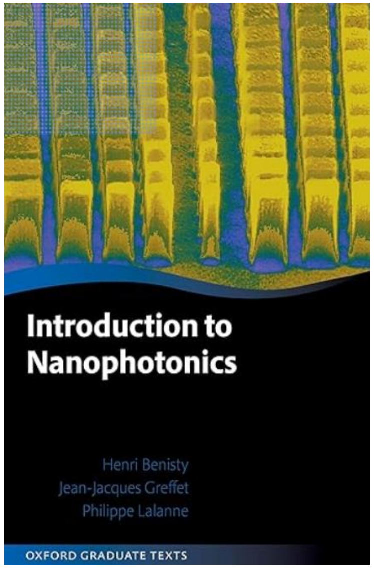
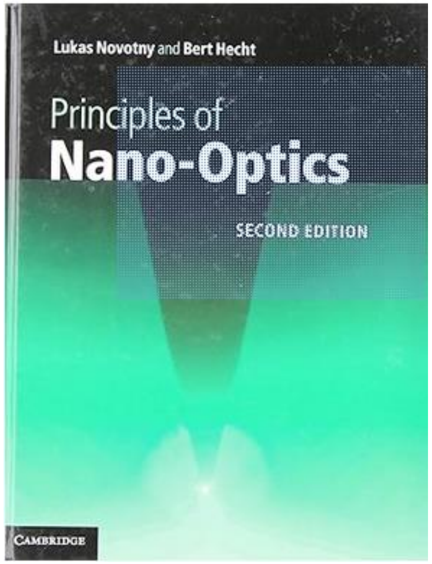
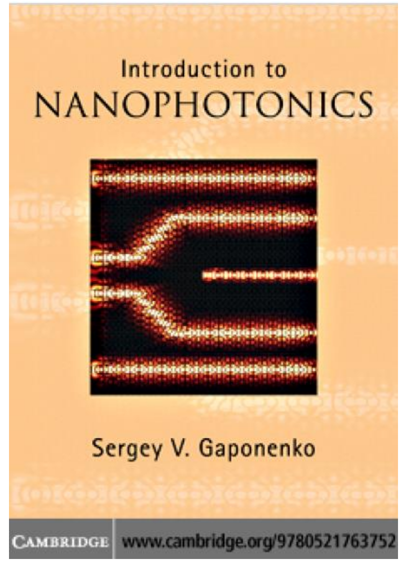
Part 1:



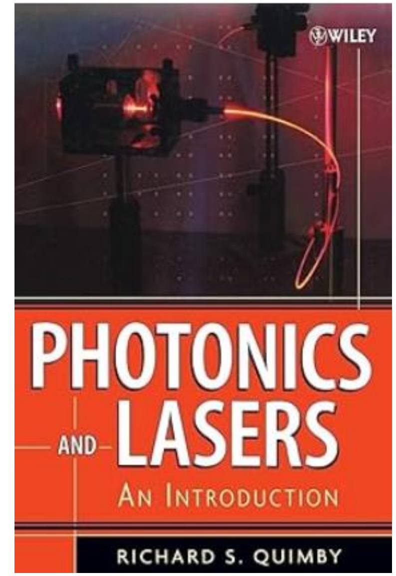
2022



2012



2010



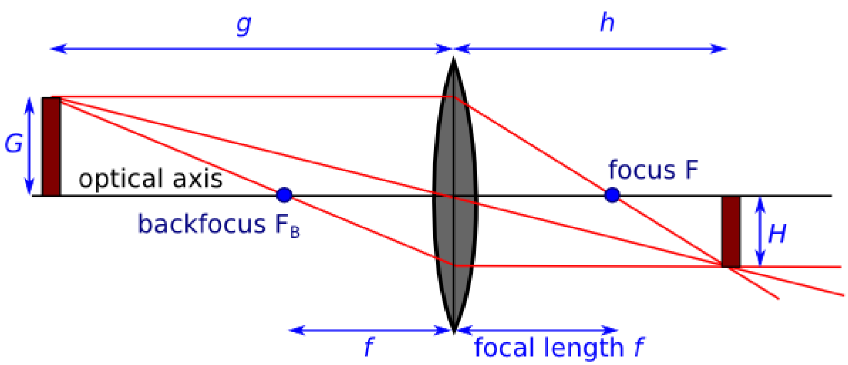
2006

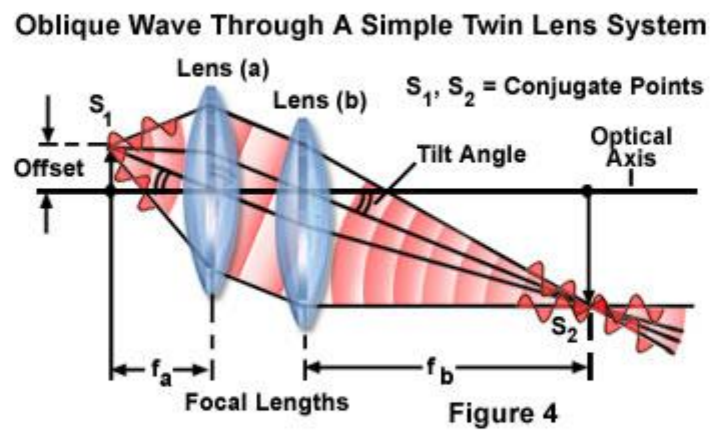
Part 1:

- Properties of light and light propagation
- Green’s functions
- Density of states and black-body radiation

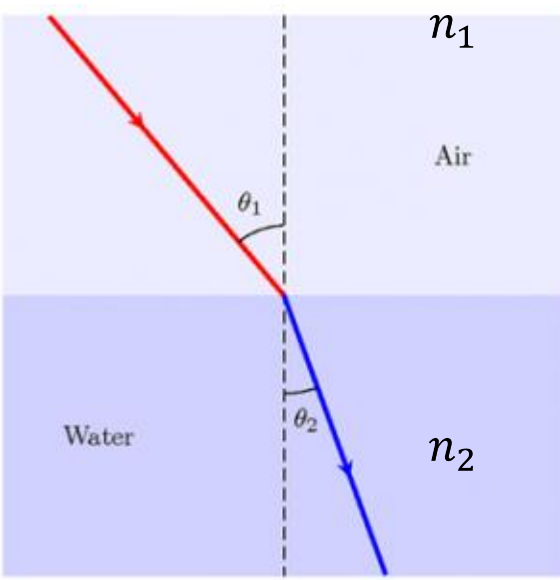
Geometric optics approximation - Newton:

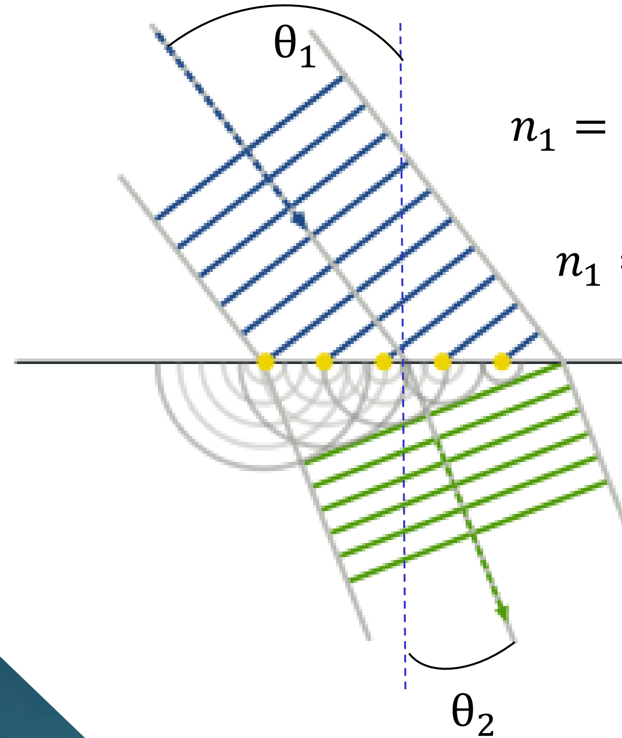
$$\lambda \to 0$$





Both Newton's 'particle model' and Huygens's 'wave model' can explain refraction !

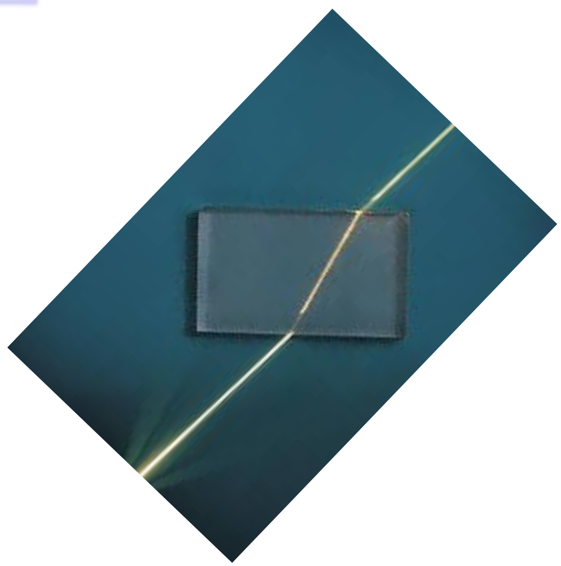




$n_1 =$

$$n_1 = \frac{v_1}{c}$$

$$n_2 = \frac{v_2}{c}$$



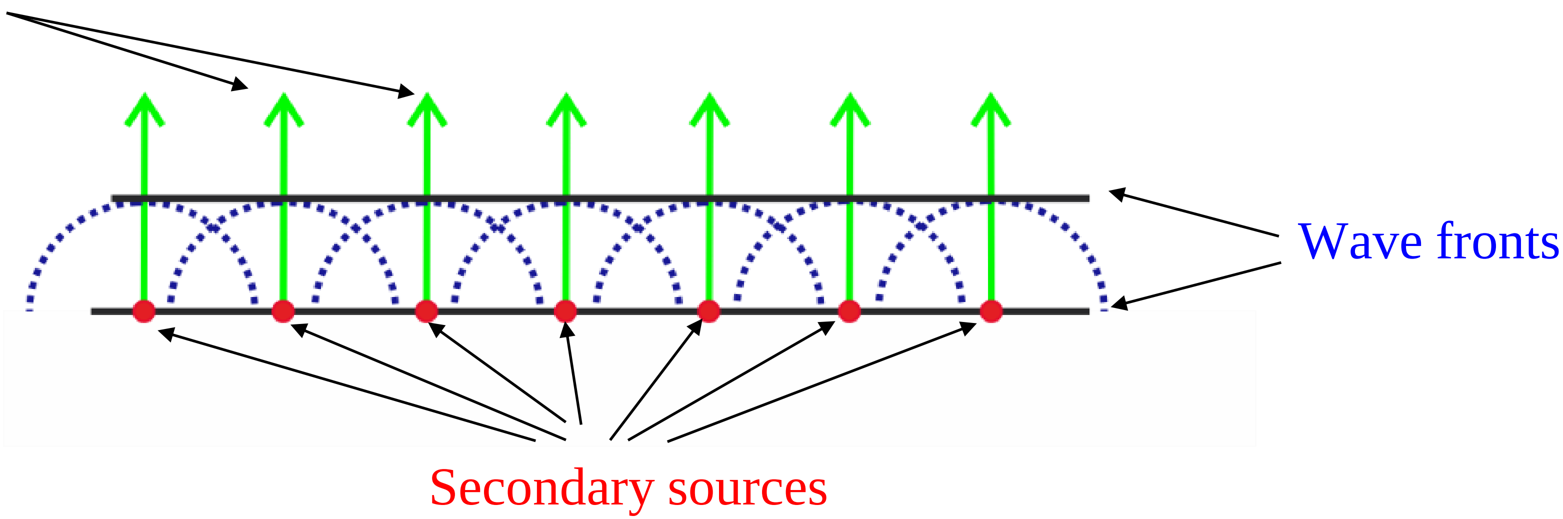
Ibn Sahl (983)

Snell (1621) Descartes (1637)

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Light as a wave - Huygens principle

The "propagation" of a wave is can be understood
as the "interference" of secondary sources in the wave fronts

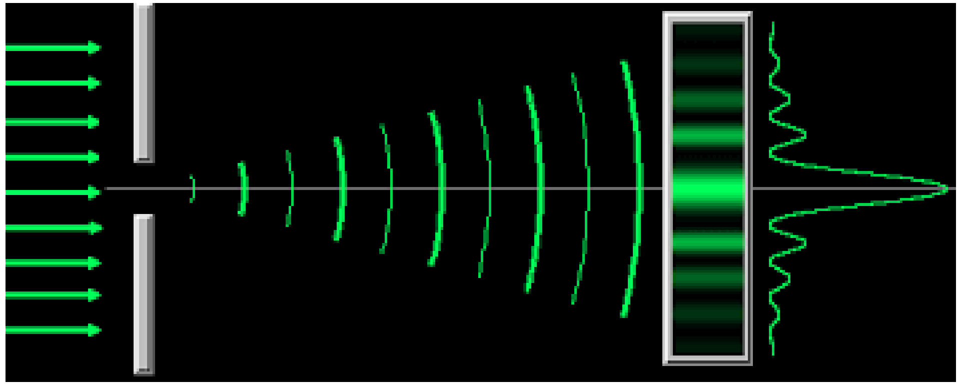


Why do the "secondary" waves only propagate in the forward direction ?
Meta-surfaces give us an answer !

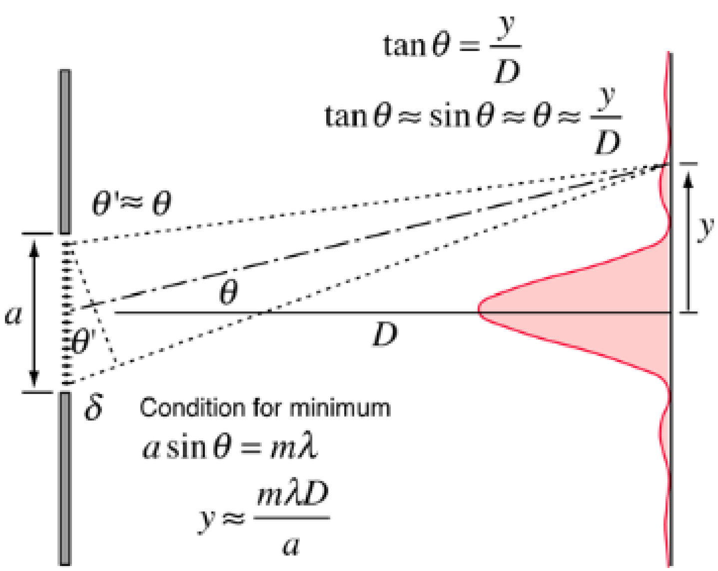
Need a wave theory of light needs to describe diffraction
(Grimaldi ~1665)

Screen

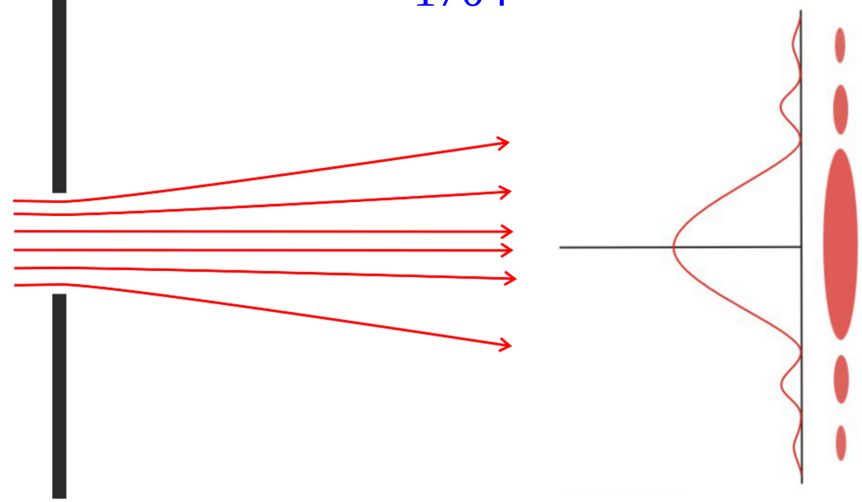
$\lambda \sim a$



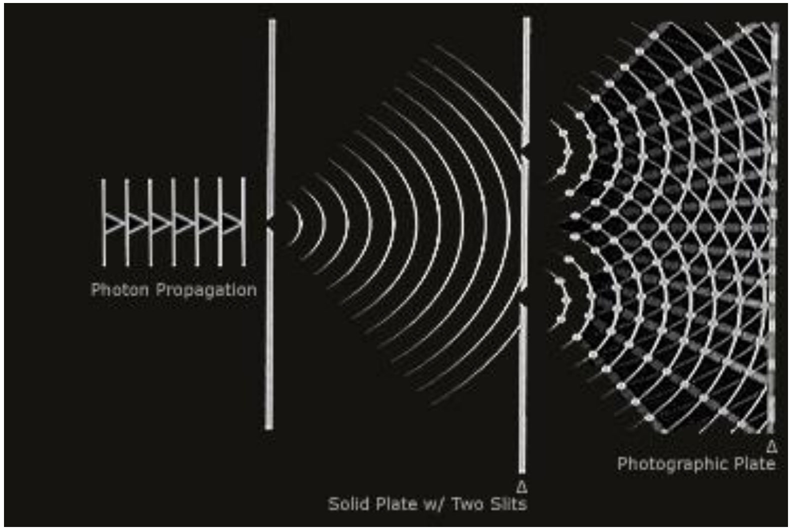
Textbook wave diffraction theory

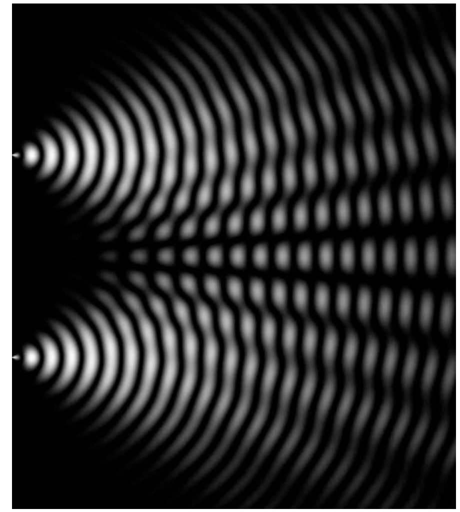


Newton's particle theory of diffraction ~1704



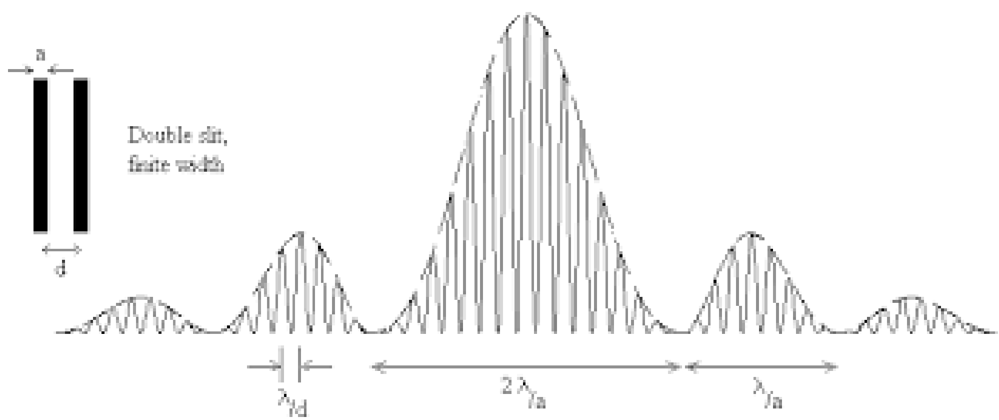
Young's double slit experiment demonstrated the Interference of light waves (1801)

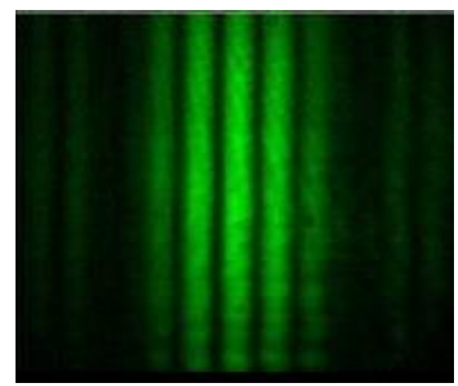




Wave interference

Double slit interference "screen view"





Rayleigh-Sommerfeld diffraction theory ~1900

Fresnel (1819), Green's theorem (1828), Kirchhoff (1883)

'Exact' Rayleigh-Sommerfeld formula
(acoustics, scalar light, quantum mechanics ?)

$$E(x,y,z) = \frac{1}{i\lambda}\iint_{\Sigma} E(X,Y,0)\left(1+\frac{i}{kr}\right)\frac{\exp(ikr)}{r}F(\theta)\,dX\,dY$$

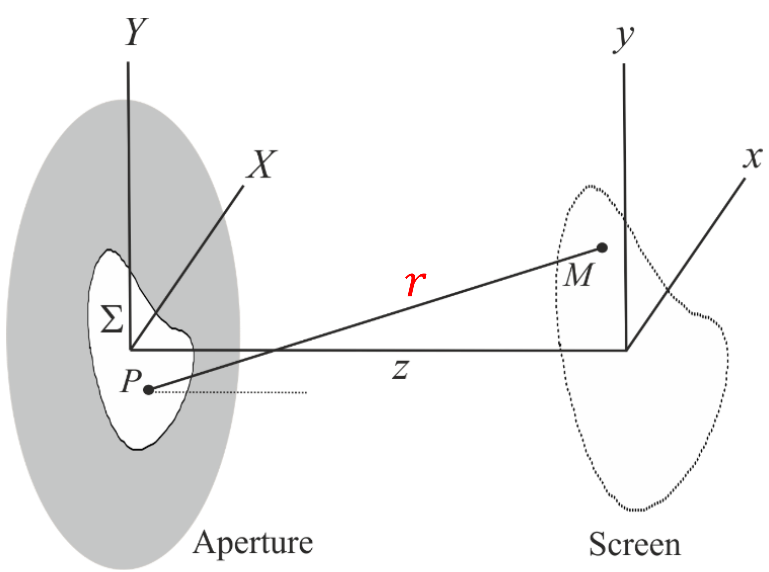
$$r = PM = \sqrt{z^2+(x-X)^2+(y-Y)^2}$$

'Obliquity' factor

$$F(\theta) = \cos\theta = \frac{z}{r}$$

wave number

$$k = \frac{2\pi}{\lambda}$$



Green's theorem (1828)

$$\boxed{\iiint_V (U\Delta g - g\Delta U)d\boldsymbol{r} = \oiint_S \left(U\frac{\partial g}{\partial n} - g\frac{\partial U}{\partial n}\right)ds}$$

Where $\frac{\partial}{\partial n}$ signifies a partial derivative in the outward normal direction at each point on S

$$\Delta U(\boldsymbol{r}) + k^2 U(\boldsymbol{r}) = 0$$

Green functions :

$$\Delta g(\boldsymbol{r}_1, \boldsymbol{r}_0) + k^2 g(\boldsymbol{r}_1, \boldsymbol{r}_0) = -\delta^{(3)}(\boldsymbol{r}_1 - \boldsymbol{r}_0)$$

$$g\,(\boldsymbol{r}_1, \boldsymbol{r}_0) = g(|\boldsymbol{r}_1 - \boldsymbol{r}_0|) = g(r) = \frac{e^{ikr}}{4\pi r} \qquad r \equiv |\boldsymbol{r}_1 - \boldsymbol{r}_0|$$

$$\frac{\partial g(r)}{\partial n} = \cos(\hat{\boldsymbol{n}}, r)\left(ik - \frac{1}{r}\right)\frac{e^{ikr}}{4\pi r}$$

Interlude on the scalar Green's function

Homogeneous wave equation : $\Delta U(\boldsymbol{r}) + k^2 U(\boldsymbol{r}) = 0$

Inhomogeneous wave equation : $\Delta\psi(\boldsymbol{r}) + k^2\psi(\boldsymbol{r}) = -j(\boldsymbol{r})$

Green functions :

$$(\Delta_1 + k^2) g(\boldsymbol{r}_1, \boldsymbol{r}_0) = -\delta^{(3)}(\boldsymbol{r}_1, \boldsymbol{r}_0)$$

$$g(\boldsymbol{r}_1, \boldsymbol{r}_0) = (\Delta_1 + k^2)^{-1}$$

$$\psi(\boldsymbol{r}_1) = \int d\boldsymbol{r}_0 g(\boldsymbol{r}_1, \boldsymbol{r}_0) j(\boldsymbol{r}_0)$$

$$\boxed{(\Delta_1 + k^2)\psi(\boldsymbol{r}_1) = \int d\boldsymbol{r}_0 (\Delta_1 + k^2) g(\boldsymbol{r}_1, \boldsymbol{r}_0) j(\boldsymbol{r}_0) = -\int d\boldsymbol{r}_0 \delta^{(3)}(\boldsymbol{r}_1 - \boldsymbol{r}_0) j(\boldsymbol{r}_0) = -j(\boldsymbol{r}_1)}$$

Interlude on the scalar Green's function

How do we know/find that the solution to $(\Delta_1 + k^2) g(\boldsymbol{r}_1, \boldsymbol{r}_0) = -\delta^{(3)}(\boldsymbol{r}_1 - \boldsymbol{r}_0)$

Verify that the scalar Green's function is :$g(\boldsymbol{r}_1, \boldsymbol{r}_0) = g(|\boldsymbol{r}_1 - \boldsymbol{r}_0|) = G(r) = \dfrac{e^{ikr}}{4\pi r}$

$$\Delta\psi(r,\theta,\phi) = \frac{1}{r}\frac{\partial^2(r\psi)}{\partial r^2} + \frac{1}{r^2\sin^2\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial^2\psi}{\partial\phi^2}\right]$$

$\boldsymbol{r}_0 \to \boldsymbol{0}$

$\boldsymbol{r} \equiv \boldsymbol{r}_1 - \boldsymbol{r}_0$

$$g(r,\theta,\phi) = \frac{e^{ikr}}{4\pi r}$$

$$\Longrightarrow \quad \Delta g(\boldsymbol{r}) = \frac{1}{r}\frac{\partial^2\left(r\dfrac{e^{ikr}}{4\pi r}\right)}{\partial r^2} = \frac{1}{4\pi r}\frac{\partial^2\left(e^{ikr}\right)}{\partial r^2} = -\frac{k^2 e^{ikr}}{4\pi r} = -k^2 g(\boldsymbol{r})$$

$$\Longrightarrow \quad (\Delta + k^2)\frac{e^{ikr}}{4\pi r} = 0 \qquad r \neq 0$$

Interlude on the scalar Green's function

$$(\Delta + k^2) g(\boldsymbol{r}) = -\delta^{(3)}(\boldsymbol{r}) \qquad\qquad g(r) = \frac{e^{ikr}}{4\pi r}$$

Integrate this equation over an infinitely small volume around $\boldsymbol{r} = \boldsymbol{0}$

$$\int_{V\to 0} d\boldsymbol{r} (\Delta + k^2) g(\boldsymbol{r}) = -\int_V d\boldsymbol{r}\, \delta^{(3)}(\boldsymbol{r}) = -1$$

$$\Rightarrow \quad \int_{V\to 0} d\boldsymbol{r} \Delta g(\boldsymbol{r}) = -1 \qquad\qquad \Delta g(\boldsymbol{r}) = \nabla \cdot \nabla g(\boldsymbol{r})$$

$$\Rightarrow \quad \int_{V\to 0} d\boldsymbol{r} \nabla \cdot \frac{\boldsymbol{r}}{4\pi r^3} = 1 \ {\color{red}?} \qquad\qquad r \to 0 \qquad\qquad \nabla g(\boldsymbol{r}) \to -\frac{\boldsymbol{r}}{4\pi r^3}$$

Yes ! $\displaystyle\int_V d\boldsymbol{r} \nabla \cdot \boldsymbol{A}(\boldsymbol{r}) = \int_S \boldsymbol{A}(\boldsymbol{r}) \cdot d\boldsymbol{S} \quad \Rightarrow \quad \int_{V\to 0} d\boldsymbol{r} \nabla \cdot \frac{\boldsymbol{r}}{4\pi r^3} = \int_{S\to 0} \frac{\boldsymbol{u}_r}{4\pi r^2} \cdot \boldsymbol{u}_r r^2 d\Omega = 1$

Scalar Green's function in time domain

$$\Delta_1 g + k^2 g = -\delta^{(3)}(\boldsymbol{r}_1 - \boldsymbol{r}_0)$$

$$g(r) = \frac{e^{ikr}}{4\pi r}$$

$$\boldsymbol{r} \equiv \boldsymbol{r}_1 - \boldsymbol{r}_0$$
$$r \equiv |\boldsymbol{r}_1 - \boldsymbol{r}_0|$$

$$k \equiv \frac{\omega}{c}$$

$$\Delta_1 g - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} g = -\delta^{(3)}(\boldsymbol{r}_1 - \boldsymbol{r}_0)\delta(t - t_0)$$

$$\tau \equiv t - t_0$$

'Causal' time domain solution

$$g(\boldsymbol{r}, \tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\frac{\omega}{c}r}}{4\pi r} e^{-i\omega\tau} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\left(\frac{r}{c}-\tau\right)\omega}}{4\pi r} = \frac{\delta\left(\tau - \frac{r}{c}\right)}{4\pi r}$$

$$\psi(\boldsymbol{r}, t) = \int_{-\infty}^{t} dt' \int_{-\infty}^{\infty} d\boldsymbol{r}\, g(\boldsymbol{r}, \boldsymbol{r}'; t - t') j(\boldsymbol{r}', t')$$

Different Green's functions from different boundary conditions !

$$\Delta_1 g + k^2 g = -\delta^{(3)}(\boldsymbol{r}_1 - \boldsymbol{r}_0)$$

$$\boldsymbol{r} \equiv \boldsymbol{r}_1 - \boldsymbol{r}_0$$
$$r \equiv |\boldsymbol{r}_1 - \boldsymbol{r}_0|$$

$$\Delta_1 g - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} g = -\delta^{(3)}(\boldsymbol{r}_1 - \boldsymbol{r}_0)\delta(t - t_0)$$

$$k \equiv \frac{\omega}{c}$$

$$\tau \equiv t - t_0$$

Retarded Green's function : $g_R(r) = \dfrac{e^{ikr}}{4\pi r}$ $\quad g_R(\boldsymbol{r},\tau) = \dfrac{\delta\left(\tau - \frac{r}{c}\right)}{4\pi r}$

Advanced Green's function : $g_A(r) = \dfrac{e^{-ikr}}{4\pi r}$ $\quad g_A(\boldsymbol{r},\tau) = \dfrac{\delta\left(\tau + \frac{r}{c}\right)}{4\pi r}$

Solid state Green's function : $g_{\text{s.t.}}(r) = \dfrac{1}{2}\dfrac{e^{ikr}}{4\pi r} + \dfrac{1}{2}\dfrac{e^{-ikr}}{4\pi r} = \dfrac{\cos kr}{4\pi r}$

$$g_{\text{s.t.}}(\boldsymbol{r},\tau) = \left[\frac{1}{2}\frac{\Theta\left(\tau - \frac{r}{c}\right)}{4\pi r} + \frac{1}{2}\frac{\Theta\left(\tau + \frac{r}{c}\right)}{4\pi r}\right]\delta\left(|\tau| - \frac{r}{c}\right)$$

Attn : boundary conditions !

Approximations to Sommerfeld diffraction formula

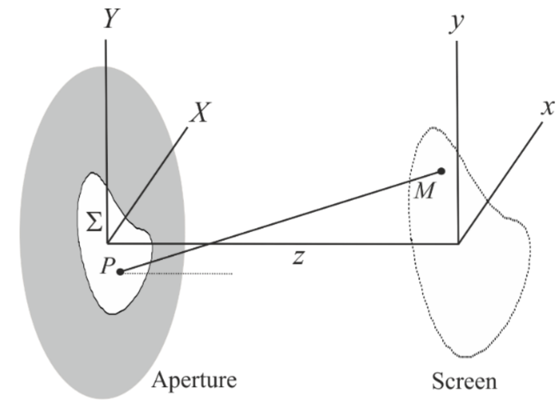
$$F(\theta) = \cos\theta = \frac{z}{r}$$

$$k = \frac{2\pi}{\lambda}$$

$$E(x,y,z) = \frac{1}{i\lambda}\iint_{\Sigma} E(X,Y,0)\left(1+\frac{i}{kr}\right)\frac{\exp(ikr)}{r}F(\theta)dXdY$$

$$\cong \frac{1}{i\lambda}\iint_{\Sigma} E(X,Y,0)\frac{\exp(ikr)}{r}F(\theta)dXdY$$

$$r = PM = \sqrt{z^2+(x-X)^2+(y-Y)^2} \cong z\left(1+\frac{[(x-X)^2+(y-Y)^2]}{2z^2}\right)$$

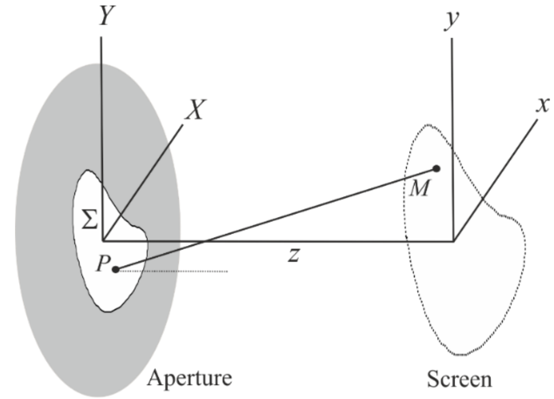


Fresnel-Kirchhoff intermediate-field ‘approximation’ :

$$E(x,y,z) \cong \frac{e^{ikz}}{i\lambda z}\iint_{\Sigma} E(X,Y,0)\exp\left\{\frac{i\pi}{\lambda z}[(x-X)^2+(y-Y)^2]\right\}dXdY$$

Fourier Optics

$$E(x,y,z) \cong \frac{e^{ikz}}{i\lambda z} \iint_{\Sigma} E(X,Y,0) \exp\left\{\frac{ik}{2z}\left[(x-X)^2 + (y-Y)^2\right]\right\} dXdY$$



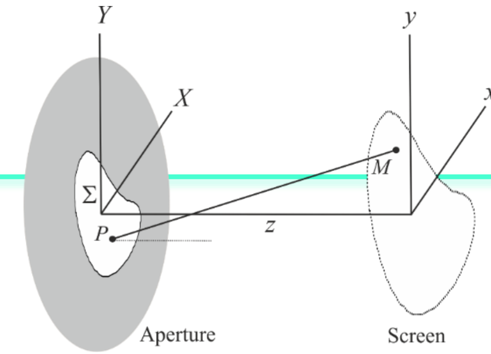
$$E(x,y,z) \cong \frac{e^{ikz}}{i\lambda z} \exp\left[\frac{ik}{2z}(x^2+y^2)\right] \iint_{\Sigma} E(X,Y,0) e^{\frac{ik}{2z}(X^2+Y^2)} e^{\frac{ik}{z}(xX+yY)} dXdY$$

Fresnel-Kirchhoff diffraction :

$$E(x,y,z) \cong C \iint_{\Sigma} E(X,Y) e^{\frac{ik}{2z}(X^2+Y^2)} e^{-\frac{ik}{z}(xX+yY)} dXdY \propto \mathcal{F}\left\{E(X,Y) e^{\frac{ik}{2z}(X^2+Y^2)}\right\}$$

('Parabolic' wavelets)

Fourier Optics in the ‘far field’



Fresnel-Kirchhoff diffraction :

$$E(x,y,z) \cong C \iint_{\Sigma} E(X,Y) e^{\frac{ik}{2z}(X^2+Y^2)} e^{-\frac{ik}{z}(xX+yY)} dX dY \propto \mathcal{F}\left\{E(X,Y) e^{\frac{ik}{2z}(X^2+Y^2)}\right\}$$

Fraunhoffer diffraction : $e^{\frac{ik}{2z}(X^2+Y^2)} \Longrightarrow 1 \qquad z \gg X, Y$

$$E(x,y,z) \cong C \iint_{\Sigma} E(X,Y) dX dY \propto \mathcal{F}\{E(X,Y)\}$$

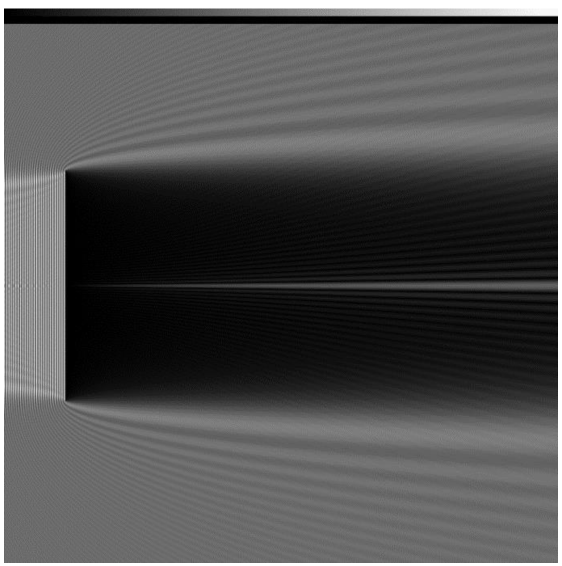
$$\mathcal{F}\{E(X,Y)\} = \iint_{\Sigma} E(X,Y) e^{-\frac{ik}{z}(xX+yY)} dX dY = \iint_{\Sigma} E(X,Y) e^{-ik(X\sin\theta_X + Y\sin\theta_Y)} dX dY$$

$$\sin\theta_X \equiv \frac{X}{z} \qquad \sin\theta_Y \equiv \frac{Y}{z}$$

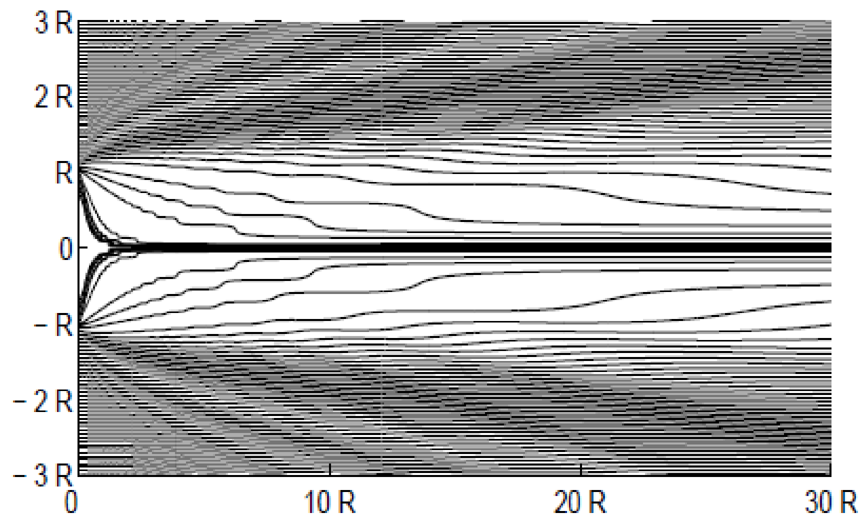
Newton's Particle theory woefully fails to explain the Poisson-Fresnel-Arago "spot"

$$E(x,y,z) = \frac{e^{ikz}}{i\lambda z} \iint_{\Sigma} E(X,Y,0) \exp\left\{\frac{i\pi}{\lambda z}\left[(x-X)^2 + (y-Y)^2\right]\right\} dX dY$$

Side views of the 'Poisson' spot 'simulations'

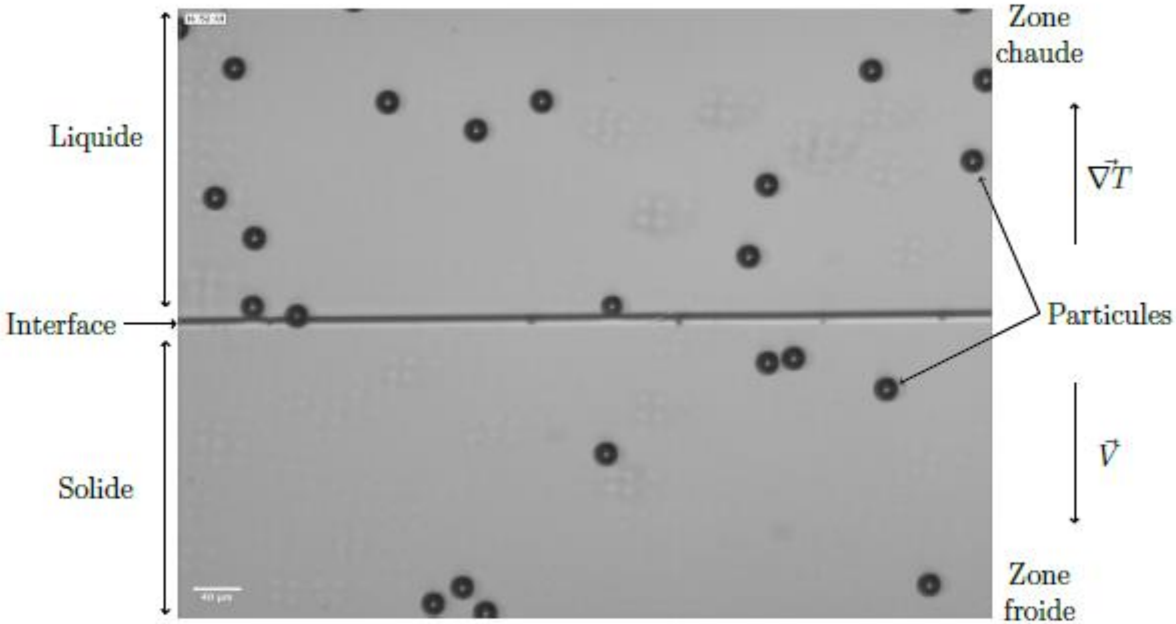


$|E|^2$



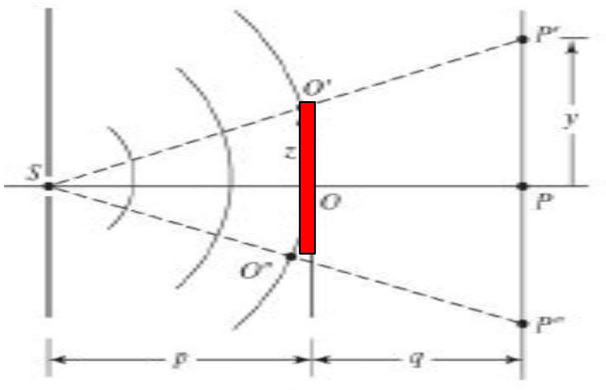
$\vec{S}$

Arago-Poisson spot is common in microscopy of small round objects



What actually impressed the French academy ?

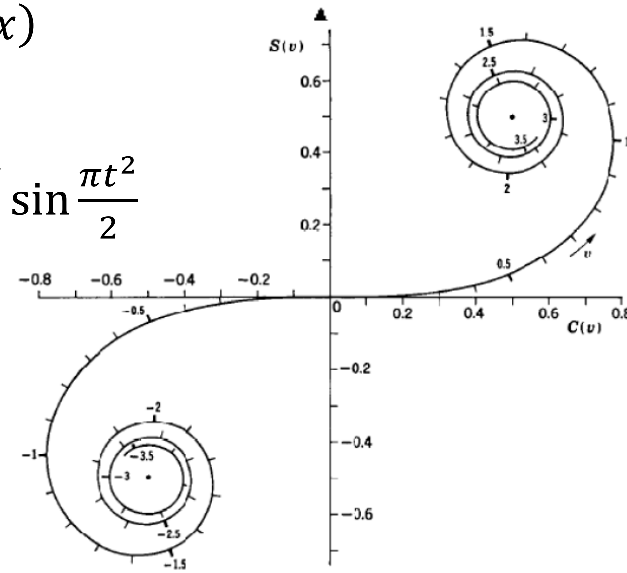
$$E(x,y,z) = K(z) \iint_{\Sigma} E(X,Y,0) \exp\left\{\frac{i\pi}{\lambda z}\left[(x-X)^2 + (y-Y)^2\right]\right\} dX dY$$



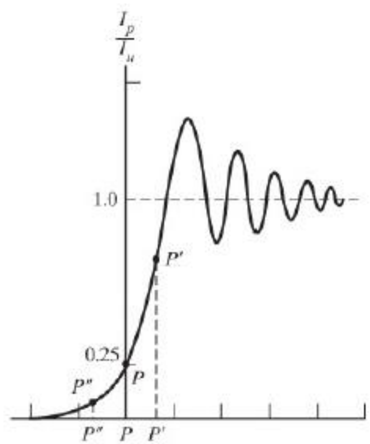
(a)

$$I(x) = \int_0^x e^{\frac{i\pi t^2}{2}} dt = C(x) + iS(x)$$

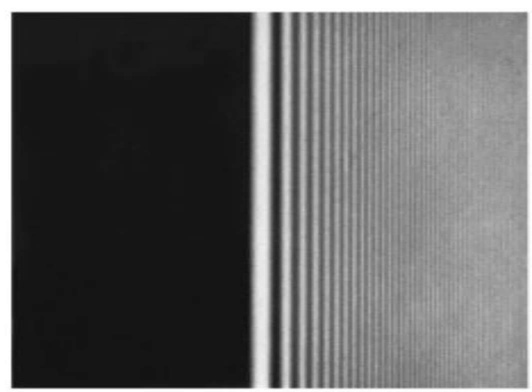
$$C(x) = \int_0^x \cos\frac{\pi t^2}{2} dt \qquad S(x) = \int_0^x \sin\frac{\pi t^2}{2}$$



- Precise measurements
- Rigorous mathematics
- Parameter free theory

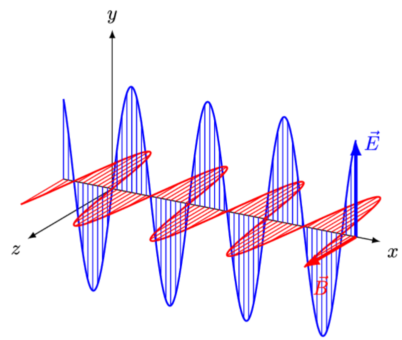


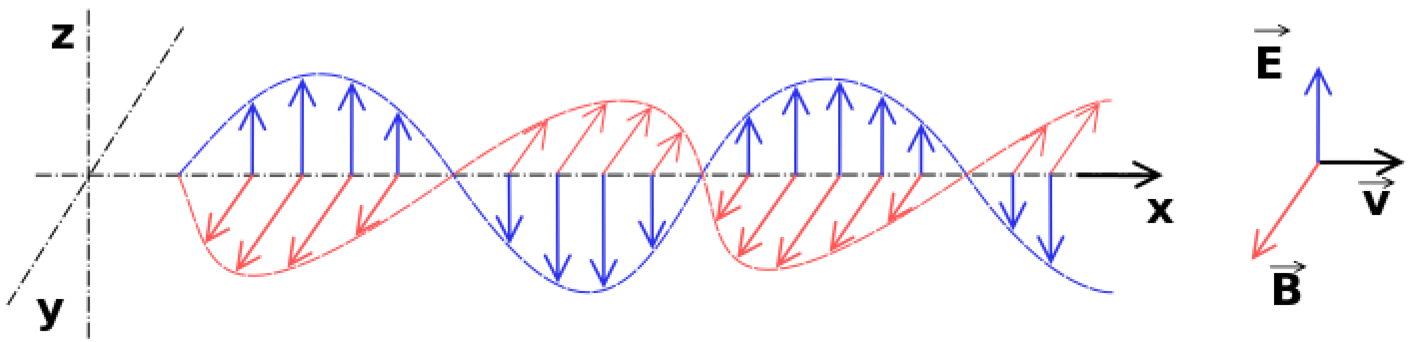
Vertical screen position



$|\boldsymbol{E}|^2/|\boldsymbol{E}_0|^2$

Since Maxwell's 1865 theory
light is an electromagnetic wave





Explains most of the physics of the Fresnel Institute !
("photonics")

It took ~100 years before one began to seriously explain light phenomena in terms of Maxwell equations (why is that ?)

Maxwell's equations of electromagnetism in free space

Maxwell equations in free space :

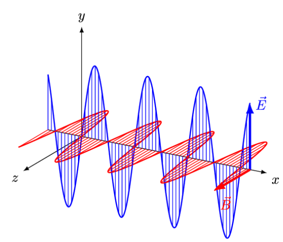
$$\boxed{\begin{array}{ll} \nabla \cdot \boldsymbol{E} = 0 & \nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t} \\ \nabla \cdot \boldsymbol{B} = 0 & \nabla \times \boldsymbol{B} = \dfrac{\partial \boldsymbol{E}}{c^2 \partial t} \end{array}}$$

$$\epsilon_0 \mu_0 = \frac{1}{c^2}$$

$$\left.\begin{array}{l} \nabla \times \nabla \times \boldsymbol{E}(\boldsymbol{r}, t) = -\dfrac{\partial^2 \boldsymbol{E}(\boldsymbol{r}, t)}{c^2 \partial t^2} \\ \\ \nabla \times \nabla \times \boldsymbol{A} \equiv \nabla(\nabla \cdot \boldsymbol{A}) - \Delta \boldsymbol{A} \end{array}\right\}$$

$$\Delta \boldsymbol{E}(\boldsymbol{r}, t) - \frac{\partial^2 \boldsymbol{E}(\boldsymbol{r}, t)}{c^2 \partial t^2} = 0$$

$$\Delta \boldsymbol{B}(\boldsymbol{r}, t) - \frac{\partial^2 \boldsymbol{B}(\boldsymbol{r}, t)}{c^2 \partial t^2} = 0$$



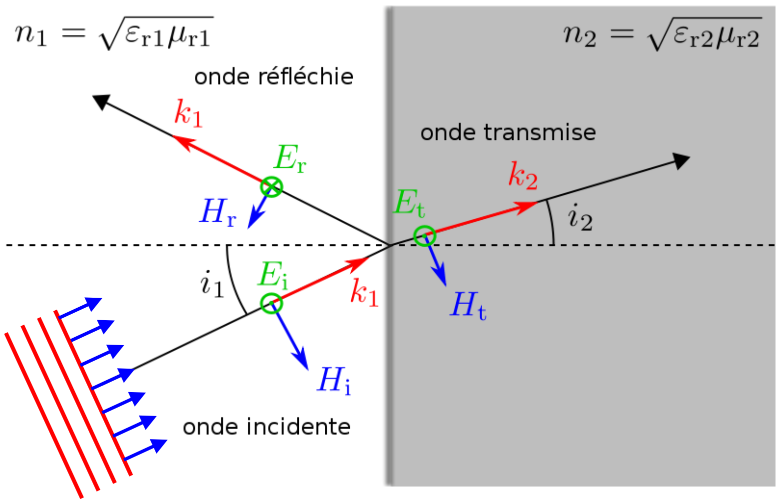
$$\boldsymbol{E}(\boldsymbol{r}, t) = \mathcal{E} \vec{\boldsymbol{e}}_{\boldsymbol{p}} e^{i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)}$$

$$\omega \equiv 2\pi\nu$$

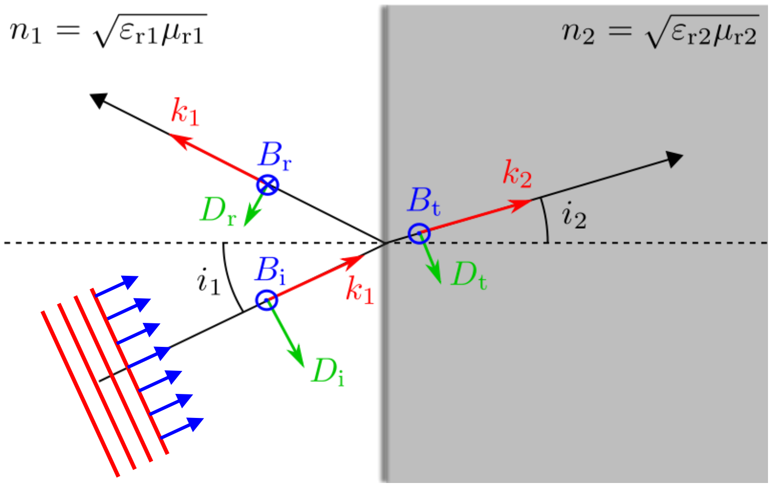
$$k \equiv |\boldsymbol{k}| \equiv \frac{2\pi}{\lambda} = \frac{\omega}{c}$$

Wave equations remain indispensable to describe light propagation like Fresnel coefficients

s-polarization



p-polarization



Fresnel coefficients

plane waves, planar interface,

$$r_s = r_\perp = \frac{n_1\cos\theta_i - n_2\cos\theta_t}{n_1\cos\theta_i + n_2\cos\theta_t}$$

$$t_s = t_\perp = \frac{2n_1\cos\theta_i}{n_1\cos\theta_i + n_2\cos\theta_t}$$

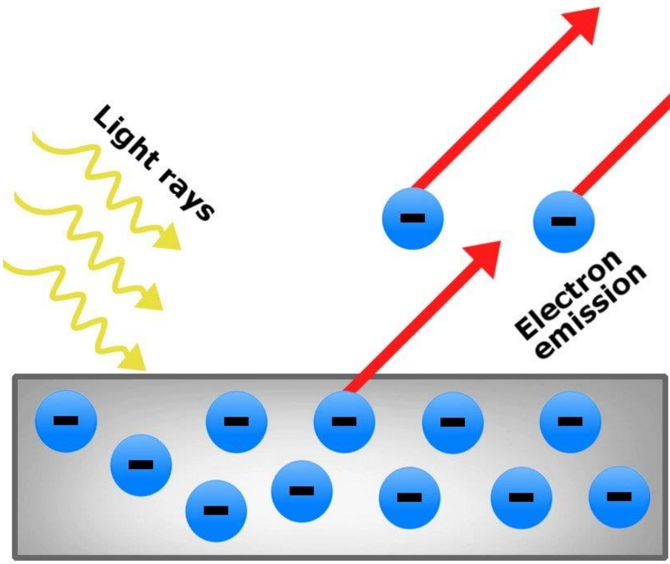
$$r_p = r_\parallel = \frac{n_2\cos\theta_i - n_1\cos\theta_t}{n_1\cos\theta_t + n_2\cos\theta_i}$$

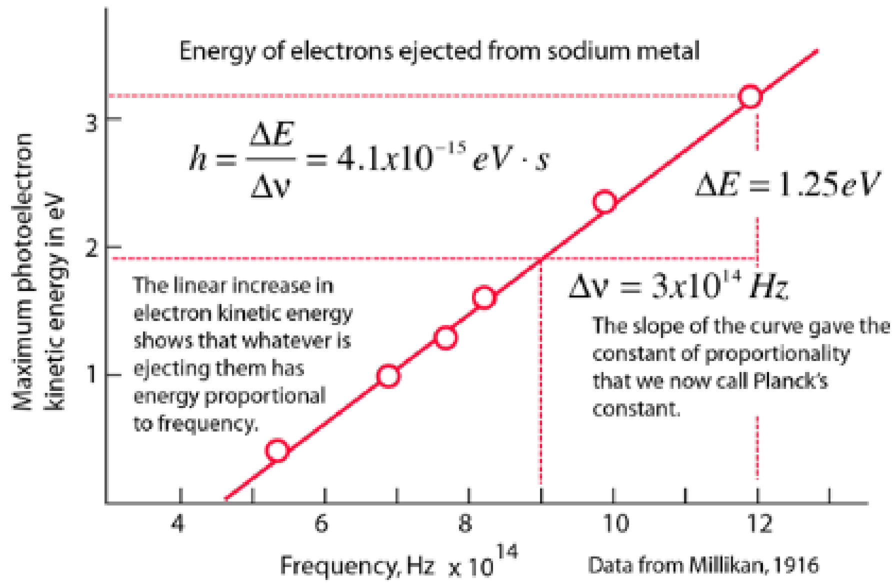
$$t_p = t_\parallel = \frac{2n_1\cos\theta_i}{n_1\cos\theta_t + n_2\cos\theta_i}$$

$$n(\omega) = \sqrt{\varepsilon_r(\omega)\mu_r(\omega)}$$

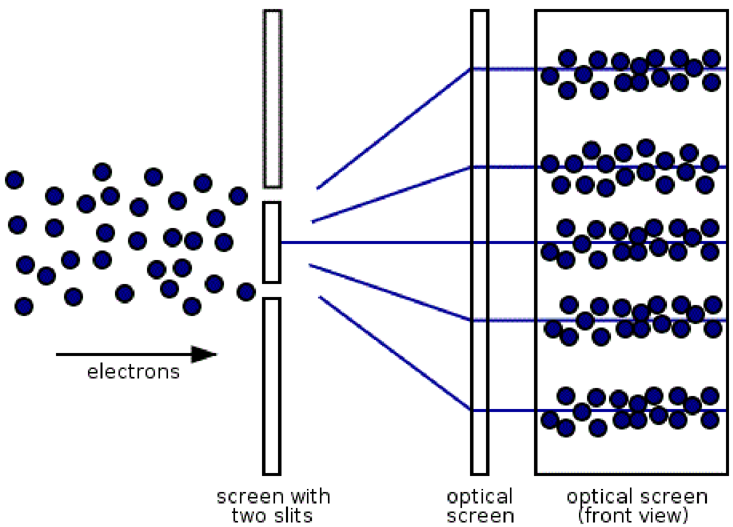
Light became a particle ? “Lichtquanta”

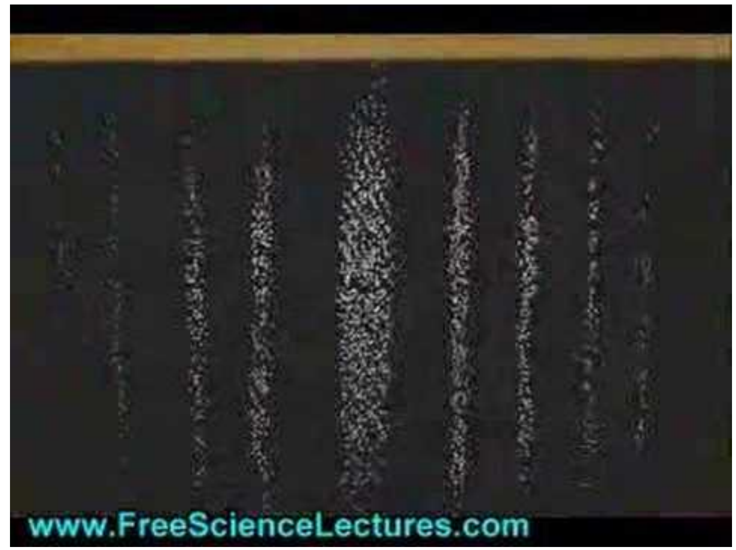
Photo-electric Effect (Einstein 1905)





Quantum mechanics ?
Particles or waves



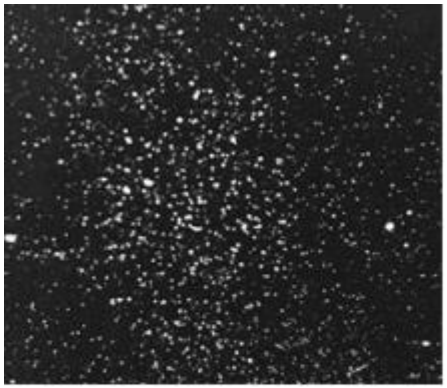


Quantum mechanics played a key role in the technological developments of the 20th century

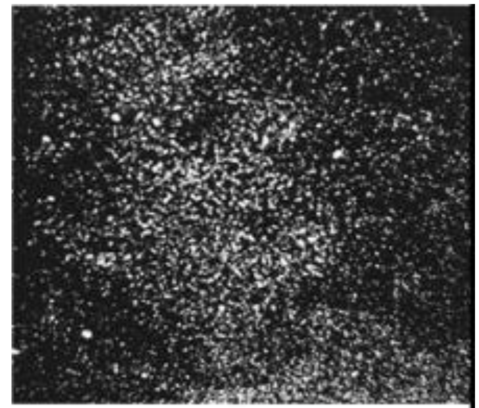
But do photons truly exist ?

(The semi-classical picture of light can explain blackbody radiation, photo-electric effect, stimulated emission (lasers), ultra-fast photography, …)

3×10^3 photons



1.2×10^4 photons



9.3×10^4 photons

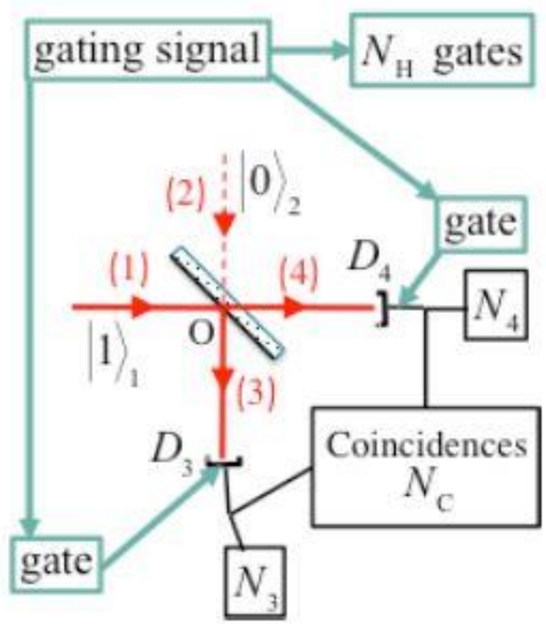
2.8×10^7 photons

Since the 1980's we know that light obeys quantum mechanical laws!

1-photon sources

(Wave particle duality, entanglement, Bell inequalities, quantum cryptography,…)

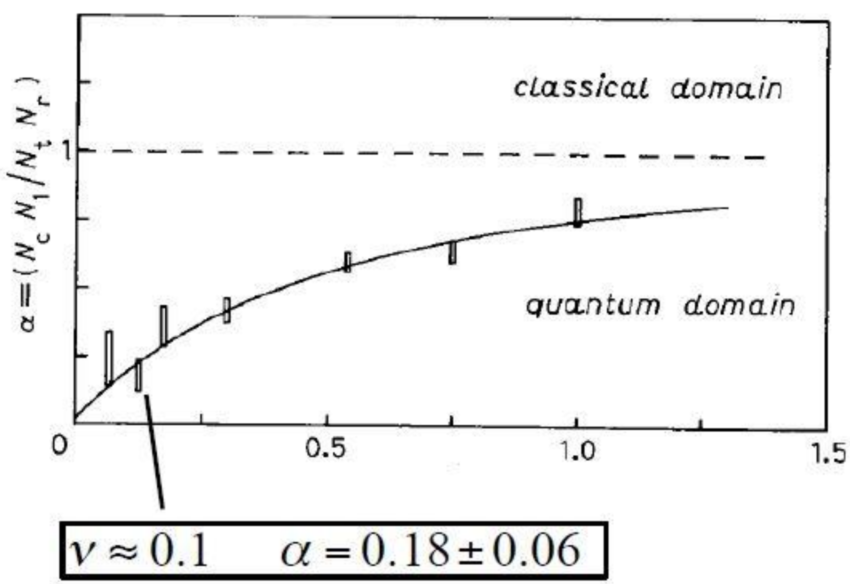
True 'Quantum Optics' often requires 1-photon (2-photon) sources, and low temperatures !



P. Grangier G. Roger and A. Aspect

EUROPHYSICS LETTERS

Europhys. Lett., **1** (4), pp. 173-179 (1986)



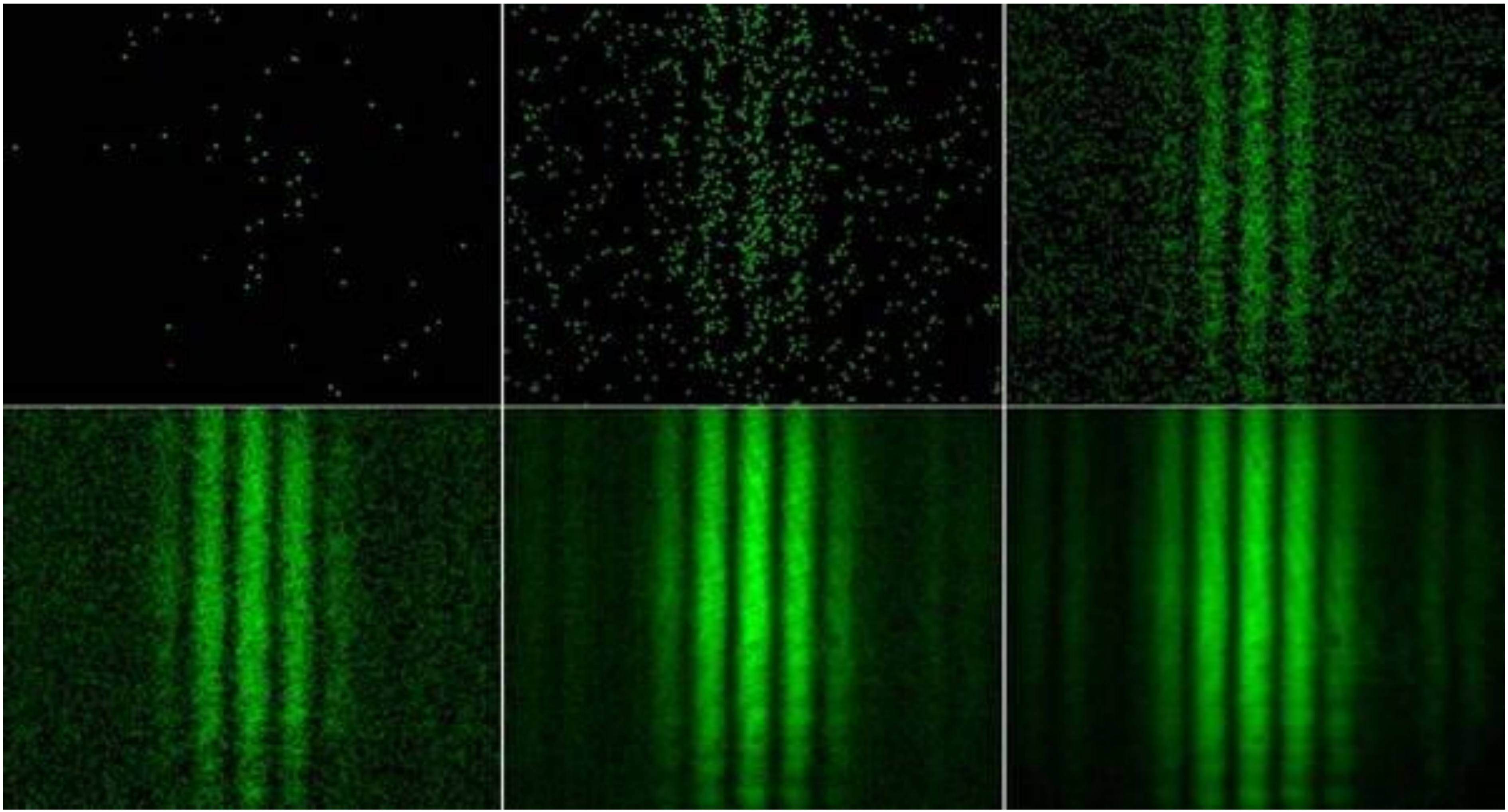
Is this the ***second*** quantum revolution?

French version : https://www.youtube.com/watch?v=_kGqkxQo-Tw

English version : https://www.youtube.com/watch?v=RSXpeDgqUO4

Alain Aspect : https://www.coursera.org/learn/quantum-optics-single-photon

Interference patterns exist even with true 1-photon wave-packets

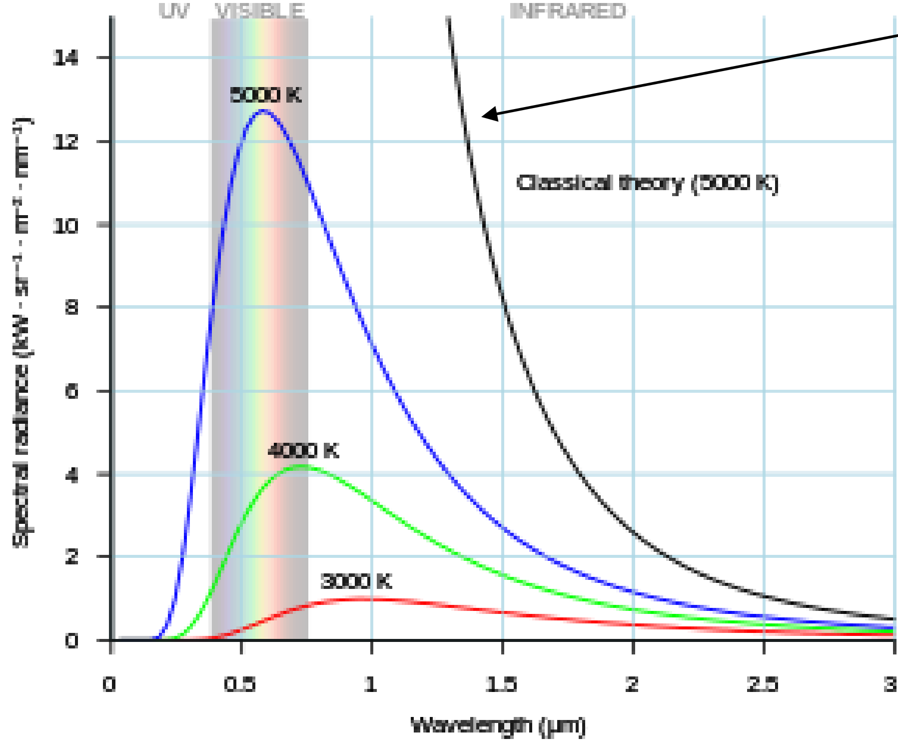


Quantum theory got started with light!

Black body radiation Planck (1900)

Thermal radiation states, density of states, and Planck's black-body radiation formula

Ultra-violet "catastrophe"



Useful properties of density matrices:

$\mathrm{Tr}\{\hat{\rho}\} = 1$

Expectation values : $\langle \hat{A} \rangle = \mathrm{tr}\{\hat{\rho}\hat{A}\} = \mathrm{tr}\{\rho A\}$

Time evolution (von Neumann equation) : $i\hbar \frac{\partial}{\partial t}\hat{\rho} = [\hat{H}, \hat{\rho}]$

Pure states: $\mathrm{tr}\{\rho^2\} = 1$ Ex: $\rho = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ $\rho^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

Mixed states $\mathrm{tr}\{\rho^2\} < 1$ Ex: $\rho = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ $\rho^2 = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$

Thermal Radiation : $E_{n_\ell} = n_\ell \hbar \omega_\ell$

Thermal Radiation States of a Single Field Mode (incoherent superposition!)

Density matrix of a mode ℓ :

$$\hat{\rho}_\ell = \sum_{n_\ell=0}^{\infty} p_{n_\ell} |\psi_{n_\ell}\rangle\langle\psi_{n_\ell}| = \sum_{n_\ell=0}^{\infty} p_{n_\ell} |n_\ell\rangle\langle n_\ell|$$

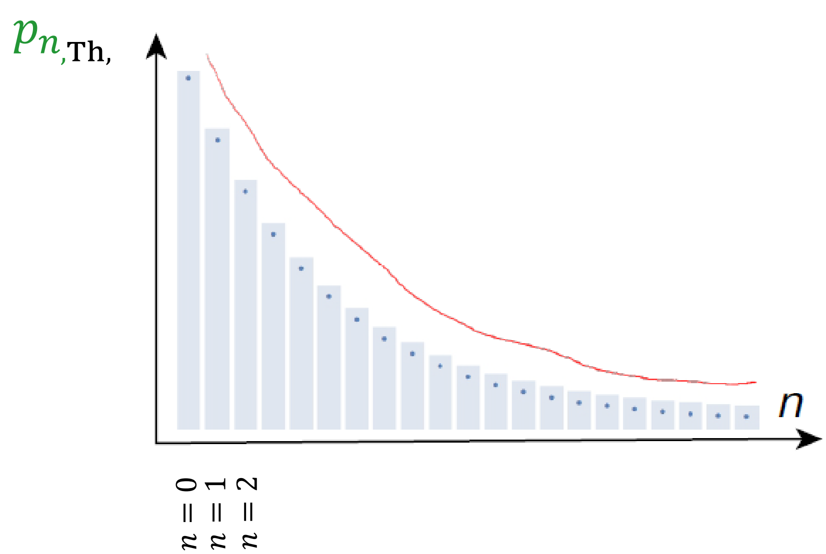
Boltzmann probability distribution

$$p_{n_\ell,\mathrm{Th}} = \frac{e^{-\frac{E_{n_\ell}}{k_B T}}}{Z_\ell} = \frac{e^{-\frac{n_\ell \hbar\omega_\ell}{k_B T}}}{1 - e^{-\frac{\hbar\omega_\ell}{k_B T}}}$$

Z_ℓ is chosen such that:

$$\sum_{n=0}^{\infty} p_{n_\ell} = \sum_{n_\ell=0}^{\infty} \frac{e^{-\frac{n_\ell \hbar\omega_\ell}{k_B T}}}{Z_\ell} = \frac{1}{Z_\ell} \sum_{n_\ell=0}^{\infty} x^n = 1 \qquad x \equiv e^{-\frac{\hbar\omega}{k_B T}}$$

$$\Rightarrow \quad Z_\ell = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = \frac{1}{1 - e^{-\frac{\hbar\omega}{k_B T}}}$$



Average thermal mode occupation

Thermal radiation state of a single electromagnetic mode

Attn! : we drop the mode index ℓ

$$\bar{n}_\omega \equiv \langle \hat{n} \rangle = \mathrm{Tr}\left(\hat{n}\hat{\rho}_{n_\ell,\mathrm{Th}}\right) = \sum_{n=0}^{\infty} n p_n = \frac{1}{Z}\sum_{n=0}^{\infty} n e^{-\frac{E_n}{k_B T}} = \frac{1}{Z}\sum_{n=0}^{\infty} n e^{-\frac{n\hbar\omega}{k_B T}}$$

$$E_n = n\hbar\omega$$

$$= \frac{1}{Z}\sum_{n=0}^{\infty} n \left(e^{-\frac{\hbar\omega}{k_B T}}\right)^n = \frac{1}{Z}\sum_{n=0}^{\infty} n x^n$$

$$x \equiv e^{-\frac{\hbar\omega}{k_B T}}$$

$$Z = \sum_{n=0}^{\infty} e^{-\frac{n\hbar\omega}{k_B T}} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$= \frac{1}{Z}\sum_{n=0}^{\infty} x \frac{d}{dx} x^n = \frac{1}{Z} x \frac{d}{dx} \sum_{n=0}^{\infty} x^n = \frac{1}{Z} x \frac{d}{dx} \frac{1}{1-x}$$

$$= \frac{1}{Z}\frac{x}{(1-x)^2} = \frac{x}{1-x} = \frac{1}{\frac{1}{x} - 1} = \frac{1}{e^{\frac{\hbar\omega_\ell}{k_B T}} - 1}$$

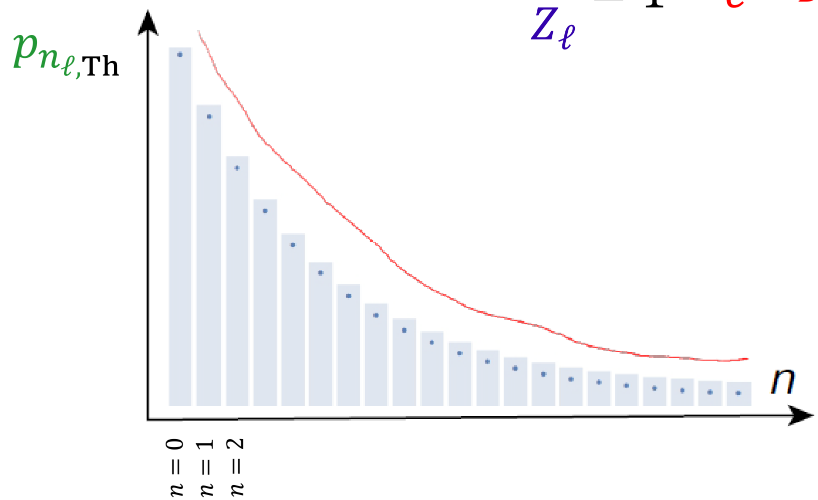
$$\boxed{\bar{n}_{\omega_\ell} = \frac{1}{e^{\frac{\hbar\omega_\ell}{k_B T}} - 1}}$$

Photon probability occupation as a function of $\bar{n}$

$$\bar{n}_\omega = \frac{1}{e^{\hbar\omega/k_BT} - 1} \Rightarrow e^{\hbar\omega/k_BT} = 1 + \frac{1}{\bar{n}_\omega} = \frac{\bar{n}_\omega + 1}{\bar{n}_\omega} \Rightarrow e^{-\frac{\hbar\omega}{k_BT}} = \frac{\bar{n}_\omega}{\bar{n}_\omega + 1}$$

$$\frac{1}{Z_\ell} = 1 - e^{-\frac{\hbar\omega}{k_BT}} \quad \frac{1}{\bar{n}_\omega + 1}$$

$$p_{\mathrm{Th}}(n) = \frac{e^{-\frac{E_{n_\ell}}{k_BT}}}{Z_\ell} = \frac{1}{\bar{n}_\omega + 1}\left(\frac{\bar{n}_\omega}{\bar{n}_\omega + 1}\right)^n$$



$$\boxed{\hat{\rho}_\ell = \sum_{n_\ell=0}^{\infty} p_{n_\ell} |\psi_{n_\ell}\rangle\langle\psi_{n_\ell}| = \frac{1}{\bar{n}_\ell + 1}\sum_{n_\ell=0}^{\infty}\left(\frac{\bar{n}_\ell}{\bar{n}_\ell + 1}\right)^n |n_\ell\rangle\langle n_\ell|}$$

Bose-Einstein distribution (for photons)

$$\bar{n}_\omega = \frac{1}{e^{\hbar\omega/k_B T} - 1}$$

$\bar{n}_\omega$: average photon number in mode

$$\bar{E}_\omega = \bar{n}_\omega \hbar\omega$$

$\bar{E}_\omega$: Average photon energy

$\hbar\omega$: Single photon energy

Planck radiation formula

$$u(\omega)d\omega = \bar{n}_\omega \hbar\omega \frac{dN}{d\omega} d\omega \frac{1}{V}$$

$u(\omega)d\omega$: Energy density in the frequency interval $\{\omega, \omega + d\omega\}$

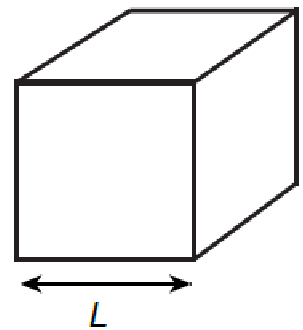
$\frac{dN}{d\omega} d\omega$: # of states in the frequency interval $\{\omega, \omega + d\omega\}$

V volume of the `box'

Density of states in a box : $\frac{dN}{d\omega}$

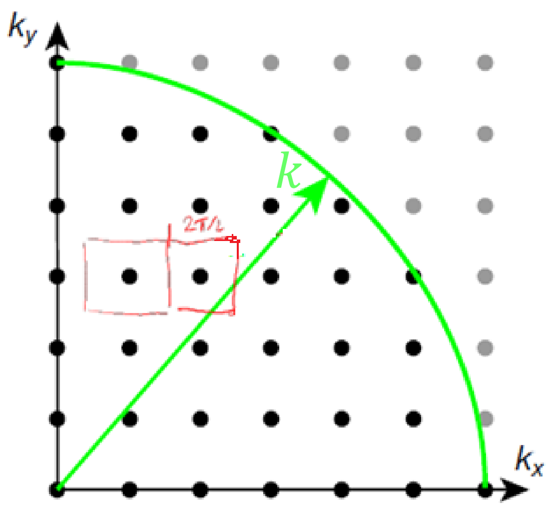
Discretized radiation modes

$k_x = \frac{2\pi}{L} n_x \qquad k_y = \frac{2\pi}{L} n_y \qquad k_z = \frac{2\pi}{L} n_z$



$V = L^3$

$n_i \in -\infty, \dots, -1, 0, 1, 2, \dots \infty$



$$N(k) = 2\,\frac{\frac{4\pi}{3}k^3}{\left(\frac{2\pi}{L}\right)^3} = 2\,\frac{V}{6\pi^2}k^3$$

2 polarization states per $\boldsymbol{k}$-vector

$$dN = \frac{V}{\pi^2}k^2\,dk$$

$$dN = \frac{V}{\pi^2 c^3}\omega^2\,d\omega$$

$k = \frac{\omega}{c} \qquad dk = \frac{d\omega}{c}$

$$dN = \frac{V}{\pi^2 c^3}\omega^2\,d\omega$$

$$\boxed{\frac{dN}{d\omega} = \frac{V}{\pi^2 c^3}\omega^2}$$

Planck radiation formula :

Density of states : $\frac{dN}{d\omega} = \frac{V\omega^2}{\pi^2 c^3}$

$$u(\omega)d\omega = \bar{n}_\omega \hbar\omega \frac{dN}{d\omega}\frac{1}{V}d\omega$$

Density of states per unit volume : $\frac{dn}{d\omega} \equiv \frac{1}{V}\frac{dN}{d\omega} = \frac{\omega^2}{\pi^2 c^3}$

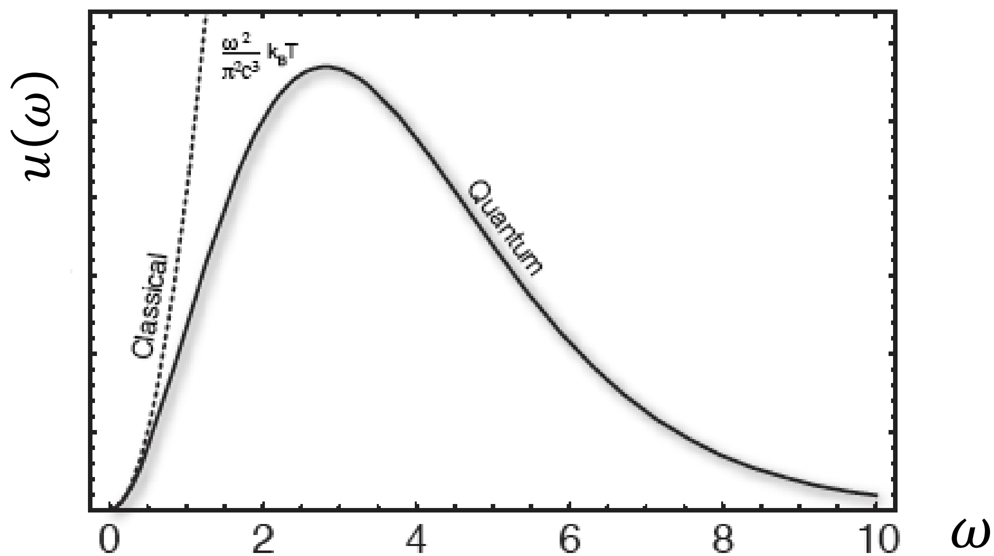
$$u(\omega) = \frac{1}{e^{\hbar\omega/k_B T} - 1}\hbar\omega\frac{\omega^2}{\pi^2 c^3}$$

$$\bar{n}_\omega = \frac{1}{e^{\hbar\omega/k_B T} - 1}$$

$$\boxed{u(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3}\frac{1}{e^{\hbar\omega/k_B T} - 1}}$$

Planck radiation formula vs classical prediction :

$$u(\omega) = \frac{1}{e^{\hbar\omega/k_BT} - 1}\hbar\omega\frac{\omega^2}{\pi^2c^3} = \frac{\hbar\omega^3}{\pi^2c^3}\frac{1}{e^{\hbar\omega/k_BT} - 1}$$



$\hbar\omega/k_BT \to 0$

Classical prediction for the density of states $u(\omega) \cong \frac{\omega^2}{\pi^2c^3}k_BT$

Photon fluctuations in black-body radiation :

$$\langle \hat{n}^2 \rangle = \mathrm{Tr}\left(\hat{n}^2 \hat{\rho}_{n_\ell,\mathrm{Th}}\right) = \sum_{n=0}^{\infty} n^2 p_n = \frac{1}{Z}\sum_{n=0}^{\infty} n^2 e^{-\frac{n\hbar\omega}{k_B T}} = \frac{1}{Z}\sum_{n=0}^{\infty} n^2 x^n$$

$$= \frac{1}{Z}\sum_{n=0}^{\infty} n^2 x^n = \frac{1}{Z} x \frac{d}{dx} x \frac{d}{dx} \sum_{n=0}^{\infty} x^n = \frac{1}{Z} x \frac{d}{dx} x \frac{d}{dx} \frac{1}{1-x}$$

$$= \frac{1}{Z} x \frac{d}{dx} x(1-x)^{-2} = \frac{1}{Z}\left[x(1-x)^{-2} + 2x^2(1-x)^{-3}\right]$$

$$= x(1-x)^{-1} + 2x^2(1-x)^{-2}$$

$$= \frac{1}{\frac{1}{x}-1} + \frac{2}{\left(\frac{1}{x}-1\right)^2} = \langle \hat{n} \rangle + 2\langle \hat{n} \rangle^2$$

$$x \equiv e^{-\frac{\hbar\omega}{k_B T}}$$

$$\boxed{\langle \hat{n}^2 \rangle = \langle \hat{n} \rangle + 2\langle \hat{n} \rangle^2 = \bar{n}_\omega + 2\bar{n}_\omega^2}$$

Photon fluctuations in black-body radiation :

$$(\Delta n)^2 = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 = \bar{n}_\omega + \bar{n}_\omega^2$$

Particle like fluctuations   `Wave like' fluctuations

Wave-particle duality (Einstein 1909)