Part 2 :

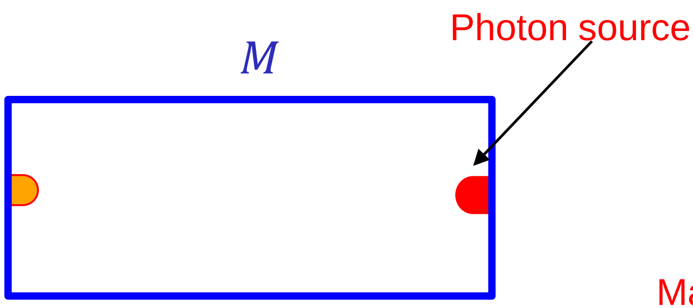
- Light forces : $E = mc^2$ and optical tweezers

- Generalities of photonic theory :
  - Subtleties with electromagnetic units in SI
  - Time-harmonic formalism and Fourier transforms
  - Transverse and longitudinal fields
  - Light matter interactions in terms of response functions

- Scattering theory :
  - Basic definitions
  - Applications and basics of Mie theory
  - Electric polarizability theory for small scatterers.

- Multipole theory :
  - Multipole basis functions
  - Expansions of Green's functions

A consequence of light carrying momentum



Maxwell equations tell us that :

I has the units of power per unit surface $[\mathrm{W.m^{-2}}] = [\mathrm{J.m^{-2}}.s^{-1}] = [\mathrm{N.m^{-1}}.s^{-1}]$

Light pressure, P, has the units $\mathrm{N.m^{-2}}$

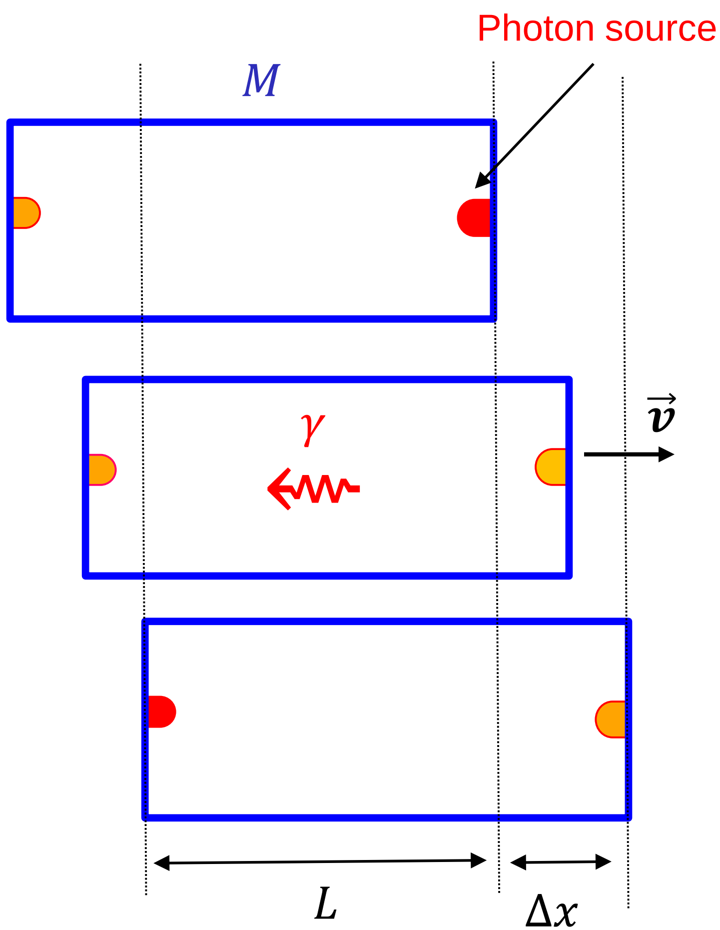
$$I = cP \rightarrow E = cp$$

Quantum mechanics also tells us that :

$$E = h\nu = c\frac{h}{\lambda} = cp$$

The patent clerk argument : step 1

"Gedanken" experiment



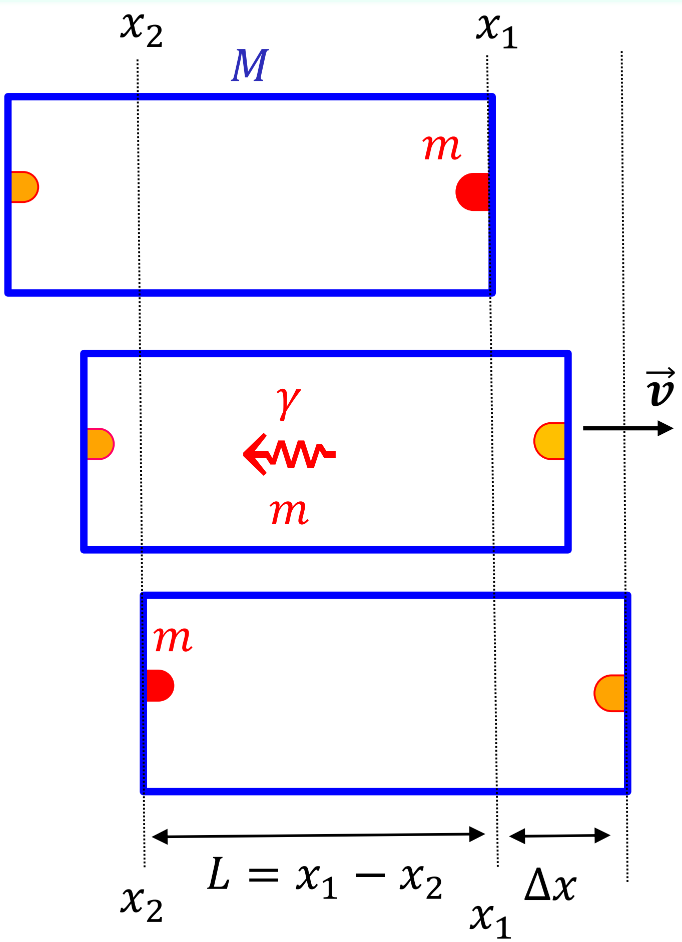
Quantum mechanics : $E = h\nu = c\dfrac{h}{\lambda} = cp$

Maxwell's equations : $I = cP \rightarrow \boxed{E = cp} \rightarrow p = \dfrac{E}{c}$

$$v = \frac{p}{M} = \frac{E}{Mc}$$

$$\begin{cases} v = \dfrac{\Delta x}{\Delta t} = \dfrac{E}{Mc} \\ \Delta t = \dfrac{L}{c} \end{cases} \Rightarrow \boxed{\Delta x = L\frac{E}{Mc^2}} \quad (1)$$

The patent clerk argument Step 2



Box initially centered at $x = 0$

$$CM = \frac{m x_1}{M + m}$$

CM is the center of mass of the box + m

$$\frac{m x_1}{M + m} = \frac{M \Delta x + m x_2}{M + m}$$

(2)

$$\Delta x = L \frac{m}{M}$$

$$L = x_1 - x_2$$

$$CM = \frac{M \Delta x + m x_2}{M + m}$$

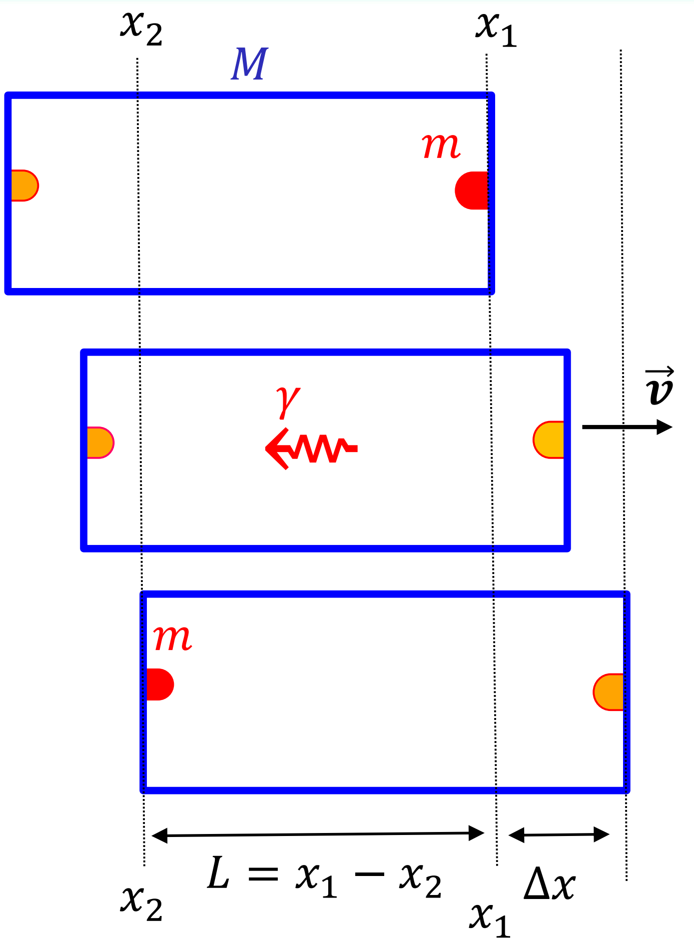
(1)

$$\Delta x = L \frac{E}{M c^2}$$

$(1) = (2)$

$$L \frac{E}{M c^2} = L \frac{m}{M}$$

The patent clerk argument Step 2



Box initially centered at $x = 0$

$$CM = \frac{mx_1}{M + m}$$

$$CM = \frac{M\Delta x + mx_2}{M + m}$$

$$\frac{mx_1}{M + m} = \frac{M\Delta x + mx_2}{M + m} \Rightarrow \Delta x = L\frac{m}{M} \quad (2)$$

(1)

$$\Delta x = L\frac{E}{Mc^2}$$

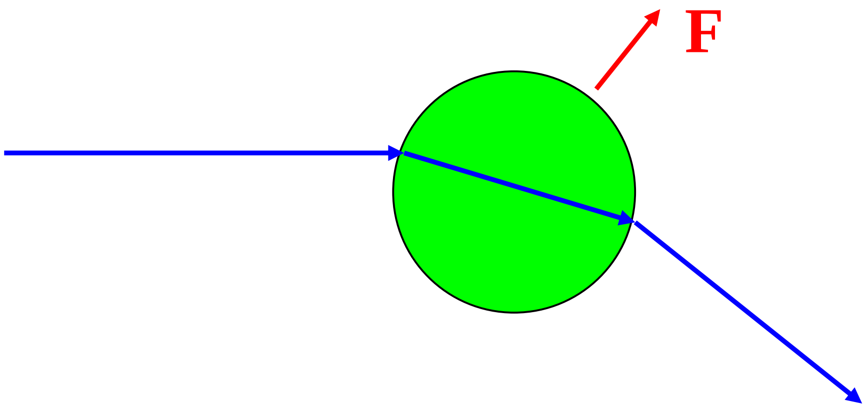
(1) = (2)

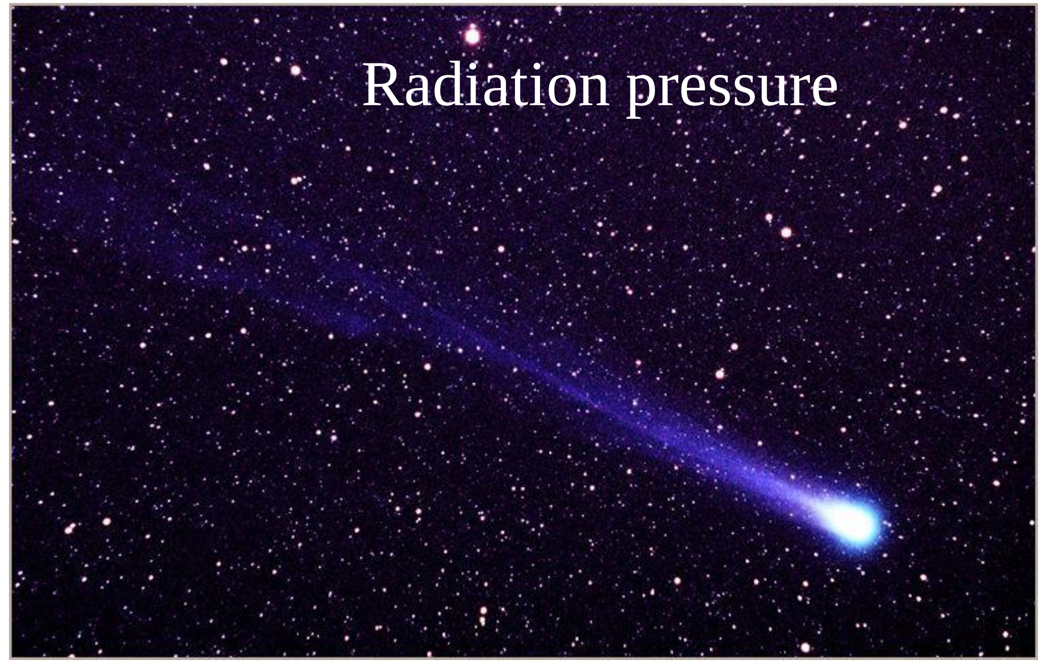
$$L\frac{E}{Mc^2} = L\frac{m}{M} \Rightarrow E = mc^2$$

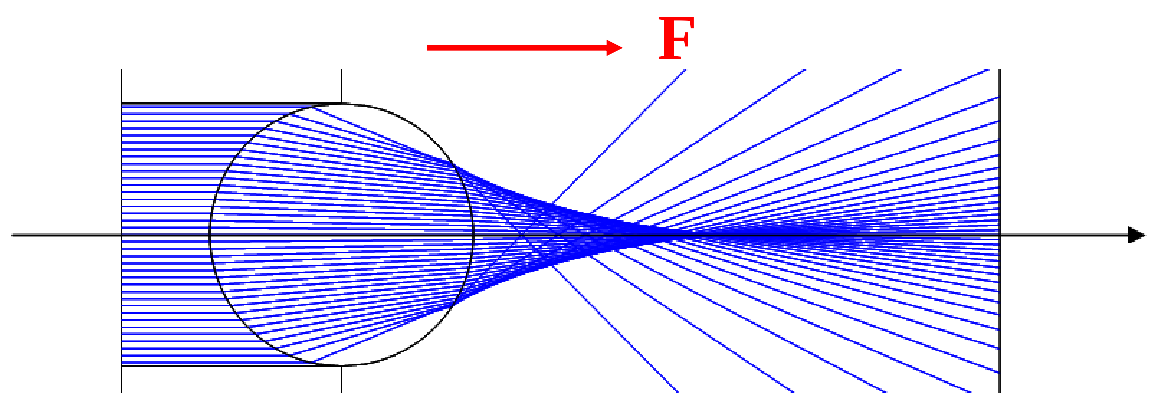
Peter Debye (1909) – radiation pressure

A photon transports
both energy and momentum

$$E = c\,p$$



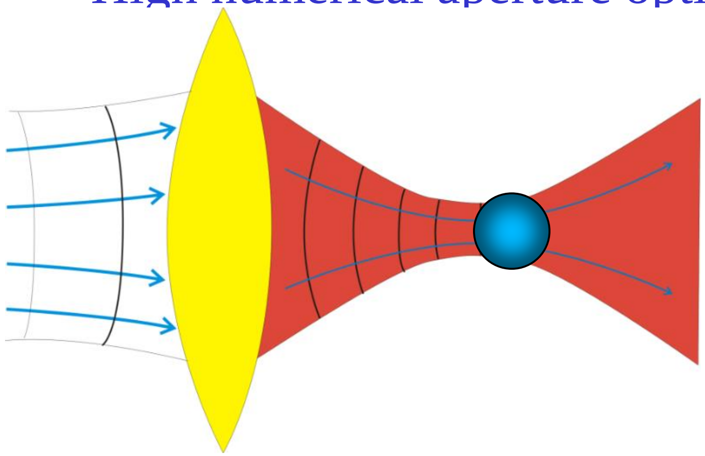


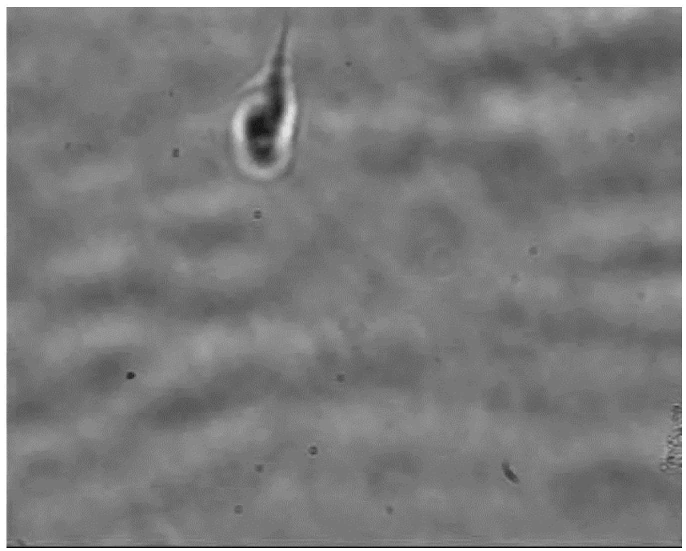


Radiation forces

Optical Tweezers

High numerical aperture optics :





Electromagnetic field equations in the vacuum

There are only 2 “Maxwell” equations

Local charge conservation : $$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{j} = 0$$

Electromagnetic equations in International SI units

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}_{\mathrm{SI}}}{\partial t}$$

$$\nabla \times \boldsymbol{B}_{\mathrm{SI}} = \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{j}$$

$$\boldsymbol{F} = q\boldsymbol{E} + q\boldsymbol{v} \times \boldsymbol{B}_{\mathrm{SI}}$$

Uniformed field units

$$\boldsymbol{B} \equiv c\boldsymbol{B}_{\mathrm{SI}}$$

Electromagnetic equations for uniformed units

$$\begin{cases} \dfrac{\partial \boldsymbol{B}}{\partial t} = -c\nabla \times \boldsymbol{E} \\ \dfrac{\partial \boldsymbol{E}}{\partial t} = c\nabla \times \boldsymbol{B} - \dfrac{\boldsymbol{j}}{\epsilon_0} \\ \boldsymbol{F} = q\boldsymbol{E} + q\dfrac{\boldsymbol{v}}{c} \times \boldsymbol{B} \end{cases}$$

$$\epsilon_0 \frac{\partial}{\partial t}(\nabla \cdot \boldsymbol{E}) + \nabla \cdot \boldsymbol{j} = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{j} = 0 \Longrightarrow \nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}$$

Electromagnetic field equations in the vacuum

Uniformed `S.I.' units

$$\frac{\partial \boldsymbol{E}}{\partial t} = c\boldsymbol{\nabla} \times \boldsymbol{B} - \frac{\vec{\boldsymbol{j}}}{\epsilon_0}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = -c\boldsymbol{\nabla} \times \boldsymbol{E}$$

$$\boldsymbol{F} = q\boldsymbol{E} + q\frac{\boldsymbol{v}}{c} \times \boldsymbol{B}$$

Gaussian units

$$\frac{\partial \boldsymbol{E}}{\partial t} = c\boldsymbol{\nabla} \times \boldsymbol{B} - 4\pi\boldsymbol{j}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = -c\boldsymbol{\nabla} \times \boldsymbol{E}$$

$$\boldsymbol{F} = q\boldsymbol{E} + q\frac{\boldsymbol{v}}{c} \times \boldsymbol{B}$$

$$\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = c\boldsymbol{\nabla} \times \frac{\partial \boldsymbol{B}}{\partial t} - \frac{1}{\epsilon_0}\frac{\partial \boldsymbol{j}}{\partial t}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = -c\boldsymbol{\nabla} \times \boldsymbol{E}$$

$$\Rightarrow \begin{cases} c^2\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{E} + \dfrac{\partial^2 \boldsymbol{E}}{\partial t^2} = \dfrac{1}{\epsilon_0}\dfrac{\partial \boldsymbol{j}}{\partial t} \\[2ex] c^2\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{B} + \dfrac{\partial^2 \boldsymbol{B}}{\partial t^2} = \dfrac{c}{\epsilon_0}\boldsymbol{\nabla} \times \boldsymbol{j} \end{cases}$$

Electromagnetic field equations in lossless local media

Electromagnetic equations in
International SI units

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}_{\mathrm{SI}}}{\partial t}$$

$$\nabla \times \boldsymbol{H}_{\mathrm{SI}} = \frac{\partial \boldsymbol{D}_{\mathrm{SI}}}{\partial t} + \boldsymbol{j}_s$$

$$\boldsymbol{F} = q\boldsymbol{E} + q\boldsymbol{v} \times \boldsymbol{B}_{\mathrm{SI}}$$

Uniformed field units

$$\boldsymbol{D} \equiv \frac{\boldsymbol{D}_{\mathrm{SI}}}{\epsilon_0}$$

$$\boldsymbol{B} \equiv c\boldsymbol{B}_{\mathrm{SI}}$$

$$\boldsymbol{H} \equiv \sqrt{\frac{\mu_0}{\epsilon_0}}\boldsymbol{H}_{\mathrm{SI}} = \frac{\boldsymbol{H}_{\mathrm{SI}}}{\epsilon_0 c}$$

Electromagnetic equations for
uniformed units

$$\begin{cases} \dfrac{\partial \boldsymbol{B}}{\partial t} = -c\nabla \times \boldsymbol{E} \\ \dfrac{\partial \boldsymbol{D}}{\partial t} = c\nabla \times \boldsymbol{H} - \dfrac{\boldsymbol{j}_s}{\epsilon_0} \end{cases}$$

$$\boldsymbol{F} = q\boldsymbol{E} + q\frac{\boldsymbol{v}}{c} \times \boldsymbol{B}$$

Local charge conservation

$$\frac{\partial}{\partial t}(\nabla \cdot \overleftrightarrow{\boldsymbol{\varepsilon}} \cdot \boldsymbol{E}) + \nabla \cdot \boldsymbol{j}_s = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{j}_s = 0 \Longrightarrow \nabla \cdot \boldsymbol{D} = \frac{\rho_s}{\epsilon_0}$$

Instantaneous constitutive relations

$$\frac{\partial \boldsymbol{B}}{\partial t} = -c\boldsymbol{\nabla} \times \boldsymbol{E} \qquad \frac{\partial \boldsymbol{D}}{\partial t} = c\boldsymbol{\nabla} \times \boldsymbol{H} - \frac{\boldsymbol{j}_s}{\epsilon_0}$$

$$\boldsymbol{D}(\boldsymbol{r}, t) = \int_{-\infty}^{t} \overleftrightarrow{\varepsilon}(\boldsymbol{r}, t - t') \cdot \boldsymbol{E}(\boldsymbol{r}, t') dt'$$

$$\boldsymbol{B}(\boldsymbol{r}, t) = \int_{-\infty}^{t} \overleftrightarrow{\mu}(\boldsymbol{r}, t - t') \cdot \boldsymbol{H}(\boldsymbol{r}, t') dt'$$

Instantaneous interaction
or quasi-static limit

$$\overleftrightarrow{\varepsilon}(\boldsymbol{r}, t - t') = \overleftrightarrow{\varepsilon}(\boldsymbol{r})\delta(t - t')$$

$$\overleftrightarrow{\mu}_m(\boldsymbol{r}, t - t') = \overleftrightarrow{\mu}(\boldsymbol{r})\delta(t - t')$$

Real symmetric
matrices

$$\overleftrightarrow{\mu}(\vec{\boldsymbol{r}}) \cdot \frac{\partial \boldsymbol{H}(\vec{\boldsymbol{r}}, t)}{\partial t} = -c\boldsymbol{\nabla} \times \boldsymbol{E}(\boldsymbol{r}, t)$$

$$\overleftrightarrow{\varepsilon}(\vec{\boldsymbol{r}}) \cdot \frac{\partial \boldsymbol{E}(\vec{\boldsymbol{r}}, t)}{\partial t} = c\boldsymbol{\nabla} \times \boldsymbol{H}(\boldsymbol{r}, t) - \frac{\vec{\boldsymbol{j}}_s}{\epsilon_0}$$

Time-harmonic vs Fourier Transform

$$\begin{pmatrix} \boldsymbol{E}(\boldsymbol{r},t) \\ \boldsymbol{B}(\boldsymbol{r},t) \end{pmatrix} = \mathrm{Re}\left\{ \begin{pmatrix} \boldsymbol{E}(\boldsymbol{r}) \\ \boldsymbol{B}(\boldsymbol{r}) \end{pmatrix} e^{-i\omega t} \right\}$$

$$j_\omega(t)\rangle = \mathrm{Re}\left\{ \begin{pmatrix} \boldsymbol{j}(\boldsymbol{r})/i\epsilon_0 \\ \boldsymbol{0} \end{pmatrix} e^{-i\omega t} \right\}$$

Time-harmonic

$$\begin{pmatrix} \boldsymbol{E}(\boldsymbol{r},t) \\ \boldsymbol{B}(\boldsymbol{r},t) \end{pmatrix} = \int_{-\infty}^{\infty} d\omega \begin{pmatrix} \boldsymbol{E}(\boldsymbol{r},\omega) \\ \boldsymbol{B}(\boldsymbol{r},\omega) \end{pmatrix} e^{-i\omega t}$$

$\Rightarrow$

$$\boldsymbol{E}(\boldsymbol{r},-\omega^*) = \boldsymbol{E}^*(\boldsymbol{r},\omega)$$

$$\boldsymbol{B}(\boldsymbol{r},-\omega^*) = \boldsymbol{B}^*(\boldsymbol{r},\omega)$$

$$\boldsymbol{j}(\boldsymbol{r},-\omega^*) = \boldsymbol{j}^*(\boldsymbol{r},\omega)$$

Fourier transform

Maxwell equations in the frequency domain

$$\boldsymbol{D}(\boldsymbol{r},\omega) = \int_{-\infty}^{t} \overleftrightarrow{\varepsilon}(\boldsymbol{r}, t - t') \cdot \boldsymbol{E}(\boldsymbol{r}, t')dt'$$

$$\boldsymbol{B}(\boldsymbol{r},\omega) = \int_{-\infty}^{t} \overleftrightarrow{\mu}(\boldsymbol{r}, t - t') \cdot \boldsymbol{H}(\boldsymbol{r}, t')dt'$$

$\Longrightarrow$

$$\boldsymbol{D}(\boldsymbol{r},\omega) = \overleftrightarrow{\varepsilon}(\boldsymbol{r},\omega) \cdot \boldsymbol{E}(\boldsymbol{r},\omega)$$

$$\boldsymbol{B}(\boldsymbol{r},\omega) = \overleftrightarrow{\mu}(\vec{\mathbf{r}},\omega) \cdot \boldsymbol{H}(\vec{\mathbf{r}},\omega)$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = -c\boldsymbol{\nabla} \times \boldsymbol{E}$$

$$\frac{\partial \boldsymbol{D}}{\partial t} = c\boldsymbol{\nabla} \times \boldsymbol{H} - \frac{\boldsymbol{j}_s}{\epsilon_0}$$

$\Longrightarrow$

$$i\omega\boldsymbol{B} = c\boldsymbol{\nabla} \times \boldsymbol{E}$$

$$-i\omega\boldsymbol{D} = c\boldsymbol{\nabla} \times \boldsymbol{H} - \frac{\boldsymbol{j}_s}{\epsilon_0}$$

Inhomogeneous media in the frequency domain

$$\boldsymbol{D}(\boldsymbol{r},\omega) = \varepsilon(\boldsymbol{r},\omega)\boldsymbol{E}(\boldsymbol{r},\omega)$$

$$\boldsymbol{B}(\boldsymbol{r},\omega) = \mu(\boldsymbol{r},\omega)\boldsymbol{H}(\boldsymbol{r},\omega)$$

$$i\omega\boldsymbol{B} = c\boldsymbol{\nabla}\times\boldsymbol{E}$$

$$c\boldsymbol{\nabla}\times\boldsymbol{H} = -i\omega\boldsymbol{D} + \frac{\boldsymbol{j}_s}{\epsilon_0}$$

⇒

$$i\omega\mu(\boldsymbol{r},\omega)\boldsymbol{H}(\boldsymbol{r},\omega) = c\boldsymbol{\nabla}\times\boldsymbol{E}(\boldsymbol{r},\omega)$$

$$c\boldsymbol{\nabla}\times\boldsymbol{H}(\boldsymbol{r},\omega) = -i\omega\varepsilon(\boldsymbol{r},\omega)\boldsymbol{E}(\boldsymbol{r},\omega) + \frac{\boldsymbol{j}_s(\boldsymbol{r},\omega)}{\epsilon_0}$$

$$\boldsymbol{\nabla}\times\frac{1}{\mu(\boldsymbol{r},\omega)}\boldsymbol{\nabla}\times\boldsymbol{E} - \frac{\omega^2}{c^2}\varepsilon(\boldsymbol{r},\omega)\boldsymbol{E}(\boldsymbol{r},\omega) = \frac{i\omega\boldsymbol{j}_s}{\epsilon_0 c^2}$$

$$\boldsymbol{\nabla}\times\frac{1}{\varepsilon(\boldsymbol{r},\omega)}\boldsymbol{\nabla}\times\boldsymbol{H} - \frac{\omega^2}{c^2}\mu(\boldsymbol{r},\omega)\boldsymbol{H}(\boldsymbol{r},\omega) = \boldsymbol{\nabla}\times\frac{1}{\varepsilon(\boldsymbol{r},\omega)}\frac{\boldsymbol{j}_s}{\epsilon_0 c}$$

Transverse electromagnetic fields in optics ($\mu = 1$ no sources) :

$$\nabla \times \nabla \times \boldsymbol{E} - \varepsilon(\boldsymbol{r})\left(\frac{\omega}{c}\right)^2 \boldsymbol{E} = \boldsymbol{0}$$

$$\nabla \cdot (\nabla \times \boldsymbol{F}) \equiv \vec{\boldsymbol{0}} \quad \Longrightarrow \quad \nabla \cdot \varepsilon(\boldsymbol{r})\vec{\boldsymbol{E}}(\boldsymbol{r}) = \nabla \cdot \boldsymbol{D}(\boldsymbol{r}) = 0$$

Source-free fields, $\boldsymbol{D}$ and $\boldsymbol{B}$ are ***transverse***, in ***heterogeneous*** media

$$\nabla \cdot \boldsymbol{D} = \nabla \cdot \boldsymbol{B} = 0 \qquad \nabla \times \boldsymbol{D} \neq \boldsymbol{0} \qquad \nabla \times \boldsymbol{B} \neq \boldsymbol{0}$$

The fields $\boldsymbol{E}$ and $\boldsymbol{H}$ are ***transverse***, $\nabla \cdot \boldsymbol{E} = \nabla \cdot \boldsymbol{H} = \boldsymbol{0}$, only in source-free ***homogenous*** media

In free space without sources, ***longitudinal*** electric fields ($\nabla \times \boldsymbol{E}_{\parallel} = \boldsymbol{0}$ but $\nabla \cdot \boldsymbol{E}_{\parallel} \neq 0$) are null

Electromagnetic fields in a homogenous space are solutions of the 3D Helmholtz equation

Mathematical identity : $\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{E} \equiv \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{E}) - \Delta \boldsymbol{E}$

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{E} - \varepsilon(\boldsymbol{r}) \left(\frac{\omega}{c}\right)^2 \boldsymbol{E} = \boldsymbol{0}$$

$$\Downarrow$$

$$\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{E}) - \Delta \boldsymbol{E} - \varepsilon(\boldsymbol{r}) \left(\frac{\omega}{c}\right)^2 \boldsymbol{E} = \boldsymbol{0}$$

For a source-free homogeneous medium, $\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0$, and we find that $\boldsymbol{E}$ satisfies a vector 3D Helmholtz equation :

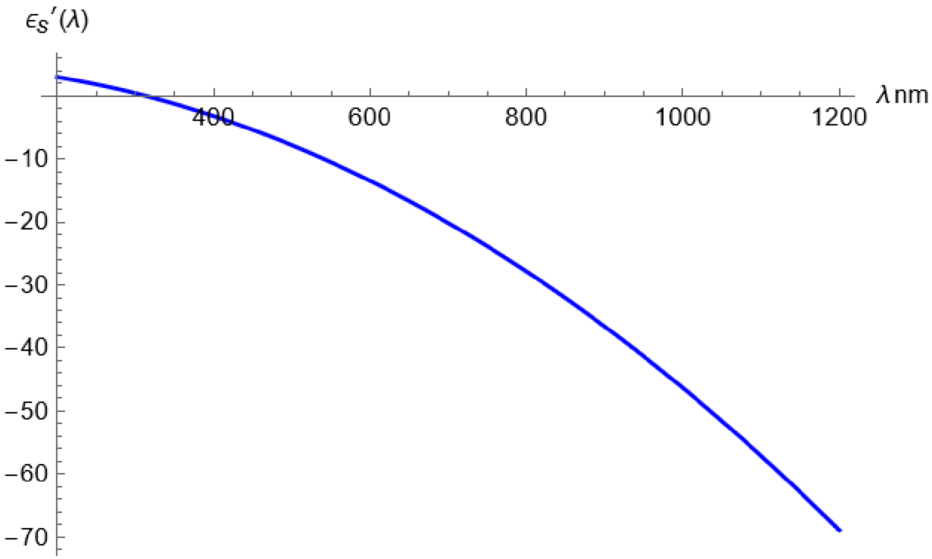
$$\Delta \boldsymbol{E} + \varepsilon \left(\frac{\omega}{c}\right)^2 \boldsymbol{E} = \boldsymbol{0}$$

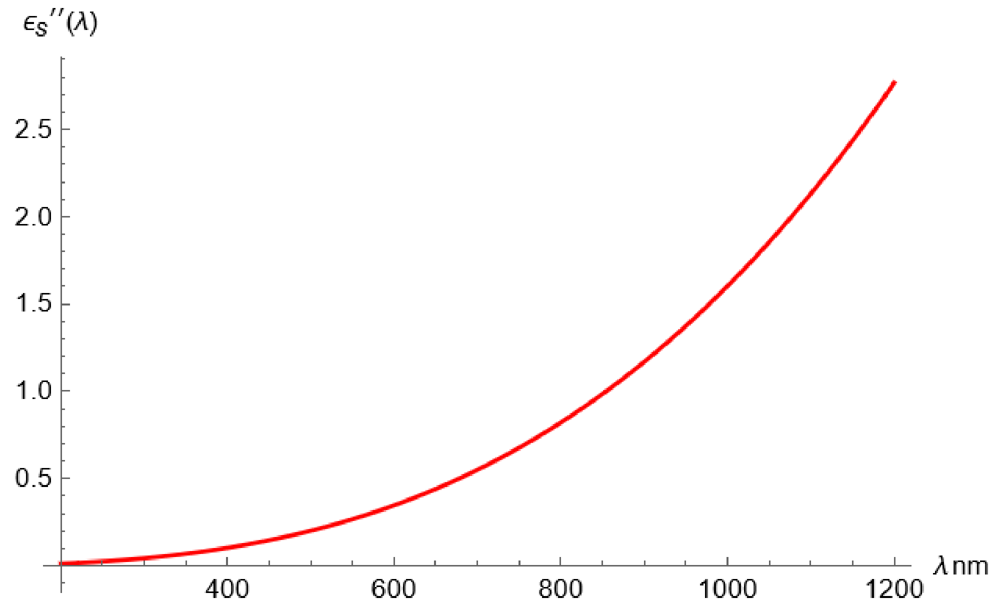
Drude-Lorentz model of material media

Frequency *dispersion* of permittivity

$$\varepsilon(\omega) = \varepsilon_{\mathrm{DC}} - \frac{\omega_{\mathrm{pl}}^2}{\omega(\omega + i\Gamma)} - \sum_{\alpha=1}^{N} \frac{\omega_{\mathrm{pT},\alpha}^2}{\omega^2 + 2i\gamma_\alpha\omega - \omega_{T,\alpha}^2}$$

$$\varepsilon(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega)$$



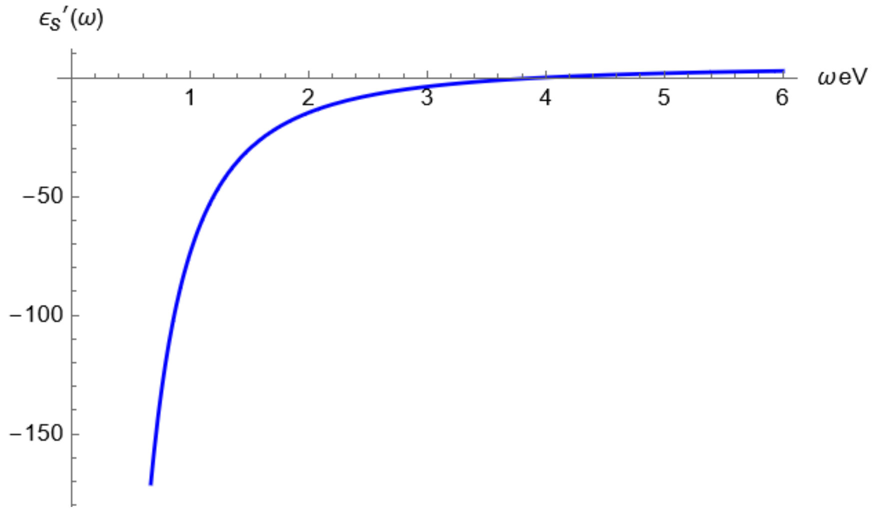


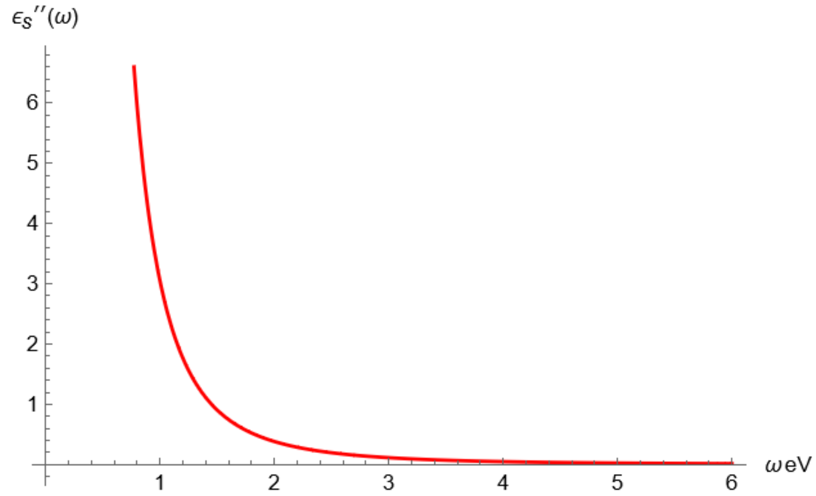
Drude model for the permittivity of silver (Ag)

Drude-Lorentz model of material media

$$\varepsilon(\omega) = \varepsilon_{\mathrm{DC}} - \frac{\omega_{\mathrm{pl}}^2}{\omega(\omega + i\Gamma)} - \sum_{\alpha=1}^{N} \frac{\omega_{\mathrm{pT},\alpha}^2}{\omega^2 + 2i\gamma_\alpha\omega - \omega_{T,\alpha}^2}$$

$$\varepsilon(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega)$$





Drude model for the permittivity of silver (Ag)

Electric constitutive relations (adimensioned units)

$$\boldsymbol{D}(\boldsymbol{r},t) = \int_{-\infty}^{t} \overleftrightarrow{\varepsilon}(\boldsymbol{r},t-t')\cdot\boldsymbol{E}(\boldsymbol{r},t')dt'$$

Homogeneous media

$$\boldsymbol{D}(\boldsymbol{r},t) = \int_{-\infty}^{t} \overleftrightarrow{\varepsilon}(t-t')\cdot\boldsymbol{E}(\boldsymbol{r},t')dt'$$

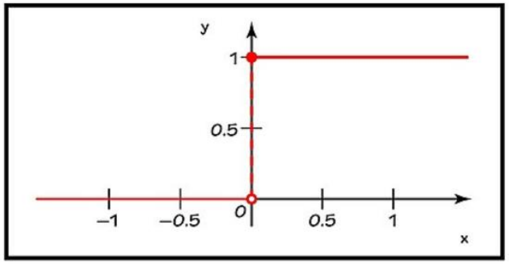
Isotropic media

$$\boldsymbol{D}(\boldsymbol{r},t) = \int_{-\infty}^{t} \varepsilon(t-t')\,\boldsymbol{E}(\boldsymbol{r},t')dt'$$

Convolution integral
(with Heaviside $\Theta(t)$)

$$\boldsymbol{D}(\boldsymbol{r},t) = \int_{-\infty}^{\infty} \Theta(t)\varepsilon(t-t')\,\boldsymbol{E}(\boldsymbol{r},t')dt'$$

$\Theta(t)$



The Heaviside Function

Fourier transform

$$\tilde{\varepsilon}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\Theta(t)\varepsilon(t) \qquad \widetilde{\boldsymbol{D}}(\boldsymbol{r},\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\boldsymbol{D}(\boldsymbol{r},t) \qquad \widetilde{\boldsymbol{E}}(\boldsymbol{r},\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\boldsymbol{E}(\boldsymbol{r},t)$$

$$\boldsymbol{D}(t) = \int_{-\infty}^{\infty} \Theta(t)\varepsilon(t-t')\,\boldsymbol{E}(\boldsymbol{r},t')dt' \;\Rightarrow\; \widetilde{\boldsymbol{D}}(\boldsymbol{r},\omega) = \tilde{\varepsilon}(\omega)\widetilde{\boldsymbol{E}}(\boldsymbol{r},\omega) = \{1+\tilde{\chi}_e(\omega)\}\widetilde{\boldsymbol{E}}(\boldsymbol{r},\omega)$$

Susceptibility : $\tilde{\chi}_e(\omega)$ $\tilde{\varepsilon}(\omega) = 1 + \tilde{\chi}_e(\omega)$

Fourier transform

$$\tilde{\chi}_e(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t} \Theta(t) \chi_e(t)$$

$$\widetilde{\boldsymbol{D}}(\boldsymbol{r},\omega) = \tilde{\varepsilon}(\omega)\widetilde{\boldsymbol{E}}(\boldsymbol{r},\omega) = \{1 + \tilde{\chi}_e(\omega)\}\widetilde{\boldsymbol{E}}(\boldsymbol{r},\omega)$$

$\chi_e(t)$ is a real – valued function ⟹ $\tilde{\chi}_e(-\omega^*) = \tilde{\chi}_e^*(\omega)$

Inverse Fourier transform

$$\chi_e(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} \tilde{\chi}_e(\omega)$$

$$\boldsymbol{E}(\boldsymbol{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{-i\omega t} \widetilde{\boldsymbol{E}}(\boldsymbol{r},\omega)$$

Time harmonic formalism : $\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_0(\boldsymbol{r})\cos(\omega t - \varphi) = \frac{1}{2}\left\{\boldsymbol{E}_0(\boldsymbol{r})e^{i\varphi}e^{-i\omega t} + \boldsymbol{E}_0(\boldsymbol{r})e^{-i\varphi}e^{i\omega t}\right\}$

$$= \mathrm{Re}\left\{\boldsymbol{E}_0(\boldsymbol{r})e^{i\varphi}e^{-i\omega t}\right\}$$

Magnetic constitutive relations

(adimensioned units)

$$\boldsymbol{B}(\boldsymbol{r},t)=\int_{-\infty}^{t}\overleftrightarrow{\mu}(\boldsymbol{r},t-t')\cdot\boldsymbol{H}(\boldsymbol{r},t')dt'$$

Homogeneous media

$$\boldsymbol{B}(\boldsymbol{r},t)=\int_{-\infty}^{t}\overleftrightarrow{\mu}(t-t')\cdot\boldsymbol{H}(\boldsymbol{r},t')dt'$$

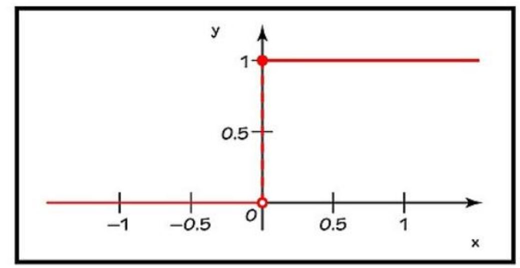
Isotropic media

$$\boldsymbol{B}(\boldsymbol{r},t)=\int_{-\infty}^{t}\mu(t-t')\,\boldsymbol{H}(\boldsymbol{r},t')dt'$$

Convolution integral
(with Heaviside $\Theta(t)$)

$$\boldsymbol{B}(\boldsymbol{r},t)=\int_{-\infty}^{\infty}\Theta(t)\mu(t-t')\,\boldsymbol{H}(\boldsymbol{r},t')dt'$$

$\Theta(t)$



The Heaviside Function

Fourier transform

$$\tilde{\mu}(\omega)=\int_{-\infty}^{\infty}d\omega\,e^{i\omega t}\Theta(t)\mu(t)\qquad\widetilde{\boldsymbol{B}}(\boldsymbol{r},\omega)=\int_{-\infty}^{\infty}d\omega\,e^{i\omega t}\boldsymbol{B}(\boldsymbol{r},t)\qquad\widetilde{\boldsymbol{H}}(\boldsymbol{r},\omega)=\int_{-\infty}^{\infty}d\omega\,e^{i\omega t}\boldsymbol{H}(\boldsymbol{r},t)$$

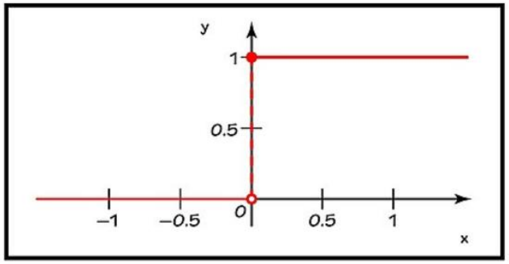
$$\widetilde{\boldsymbol{B}}(\boldsymbol{r},\omega)=\tilde{\mu}(\omega)\widetilde{\boldsymbol{H}}(\boldsymbol{r},\omega)=\{1+\tilde{\chi}_m(\omega)\}\widetilde{\boldsymbol{H}}(\boldsymbol{r},\omega)$$

In most materials at optical frequencies $\chi_m \sim 0$

Kramers-Krönig relations : causality

Theory of distributions

$$\Theta(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\Theta(t) = \int_{0}^{\infty} dt\, e^{i\omega t}\Theta(t) = \pi\delta(\omega) + \mathrm{P.V.}\frac{i}{\omega}$$



The Heaviside Function

$\Theta(t)$

Causality: $\chi(t) = \chi(t)\Theta(t)$ ⟹ $$\chi(\omega) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi}\chi(\omega')\Theta(\omega-\omega')$$

⟹ $$\chi(\omega) = \frac{1}{i\pi}\mathrm{P.V.}\int_{-\infty}^{\infty}\frac{\chi(\omega')}{\omega'-\omega}d\omega'$$

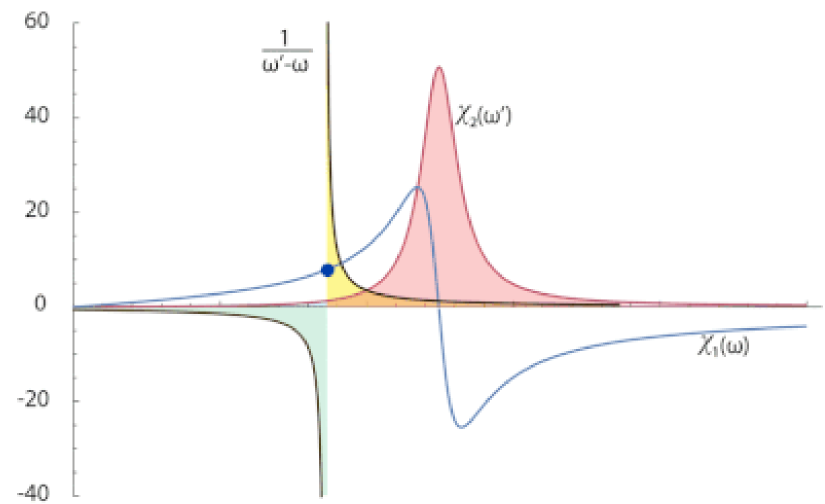
$$\chi(\omega) = \chi'(\omega) + i\chi''(\omega)$$

Kramers-Krönig relations : causality

$$\varepsilon(\omega) = 1 + \chi(\omega) \qquad \chi(\omega) = \chi'(\omega) + i\chi''(\omega) = \chi_1(\omega) + i\chi_2(\omega)$$

$$\chi'(\omega) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\chi''(\omega')}{\omega' - \omega}d\omega'$$

$$\chi''(\omega) = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\chi'(\omega')}{\omega' - \omega}d\omega'$$

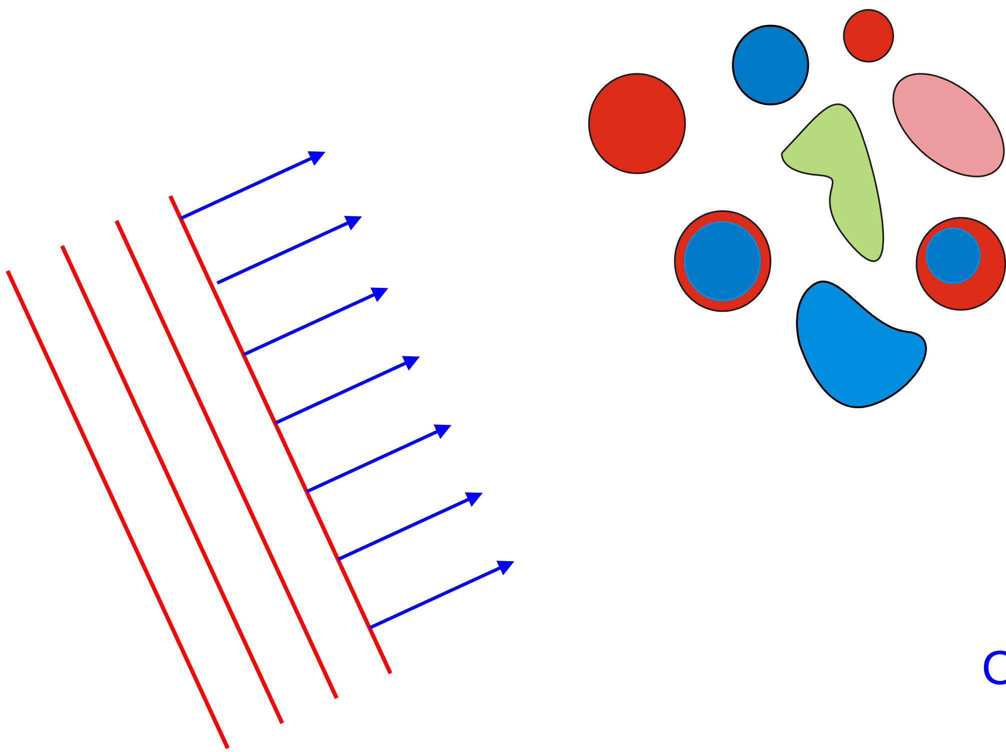


$$\tilde{\chi}_e(-\omega^*) = \tilde{\chi}_e^*(\omega) \quad \Rightarrow \quad \chi'(\omega) = \frac{2}{\pi}\mathcal{P}\int_{0}^{\infty}\frac{\omega'\chi''(\omega')}{\omega'^2 - \omega^2}d\omega'$$

How do simulate light scattering in the “real world” ?

“Seeing is believing and all we see is scattered light” J.C. Stover

Simplification : piecewise continuous media



Constitutive parameters

$$\boldsymbol{D}_{\omega}(\vec{\mathbf{r}}) = \overleftrightarrow{\varepsilon}(\omega) \cdot \boldsymbol{E}_{\omega}(\vec{\mathbf{r}})$$

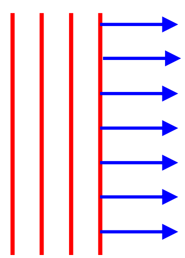
$$\boldsymbol{H}_{\omega}(\vec{\mathbf{r}}) = \overleftrightarrow{\mu}(\omega) \cdot \boldsymbol{B}_{\omega}(\vec{\mathbf{r}})$$

Scattering by a spherically symmetric object

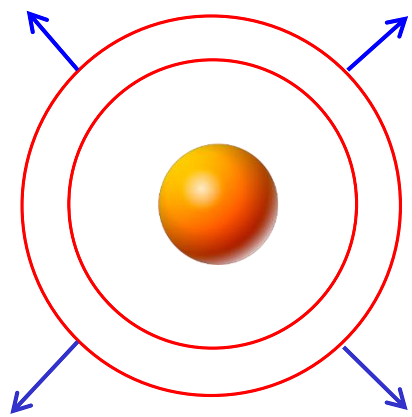
Lorenz - Mie - Debye theory

(1890) (1908) (1909)

Incident -`excitation' field :

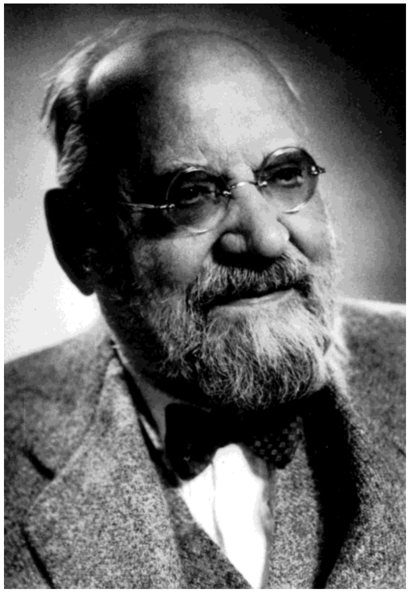


Outgoing scattered field



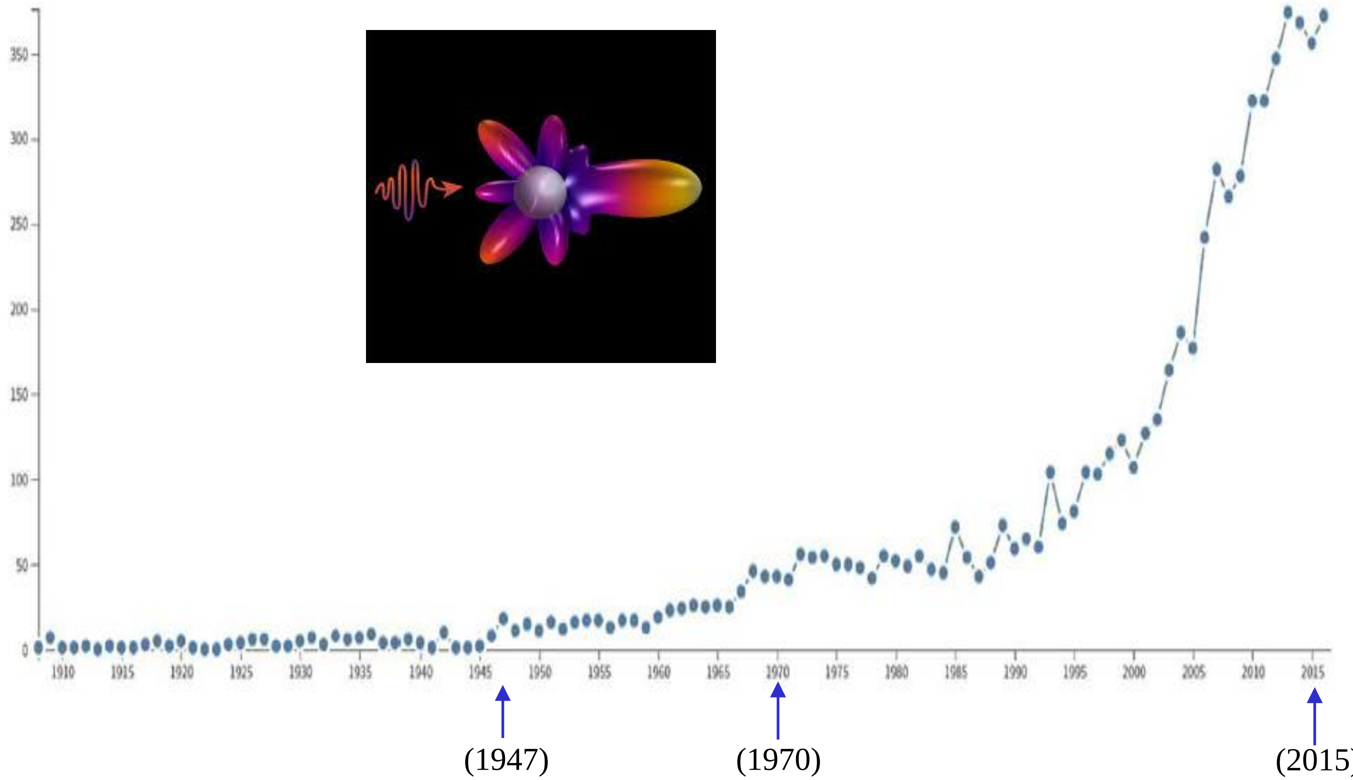
Exact solution !

Lorenz(1890)-Mie(1908)-Debye(1909) theory ?



Gustav Mie

Citations per year



Gustav Mie (1868-1957) **“Contributions to the Optics of Turbid Media, particularly of colloidal metal solutions”** Translation (Royal Aircraft Establishment (1976). **(1908)**

Ludvig Lorenz (1829–91) **“Light scattering and reflection by a transparent sphere (surface)”** in Oeuvres scientifiques de **L. Lorenz**. 1898, p 403-529 **(1890)**.

A lot of physics is hidden in the Mie coefficients

Long winded to derive but easy to use!

$$a_n = \frac{\frac{\varepsilon_s}{\varepsilon_b} j_n(k_s R)\psi'_n(kR) - \psi'_n(k_s R) j_n(kR)}{\frac{\varepsilon_s}{\varepsilon_b} j_n(k_s R)\xi'_n(kR) - \psi'_n(k_s R) h_n(kR)}$$

$$b_n = \frac{\frac{\mu_s}{\mu_b} j_n(k_s R)\psi'_n(kR) - \psi'_n(kR) j_n(kR)}{\frac{\mu_s}{\mu_b} \psi_n(k_s R)\xi'_n(kR) - \psi'_n(k_s R) h_n(kR)}$$

$$\psi_n(x) \equiv x j_n(x)$$

$$\xi_n(x) \equiv x h_n(x)$$

$$\frac{k_s}{k} \equiv \frac{N_s}{N}$$

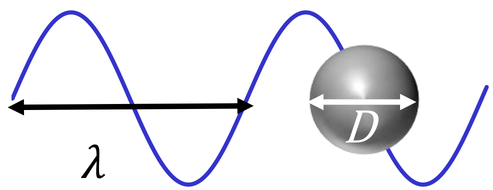
Cross sectios : σ

Extinction : $$\sigma_e = \frac{2\pi}{k^2}\sum_{n=1}^{\infty}(2n+1)\mathrm{Re}\{a_n + b_n\}$$

Scattering : $$\sigma_s = \frac{2\pi}{k^2}\sum_{n=1}^{\infty}(2n+1)(|a_n|^2 + |b_n|^2)$$

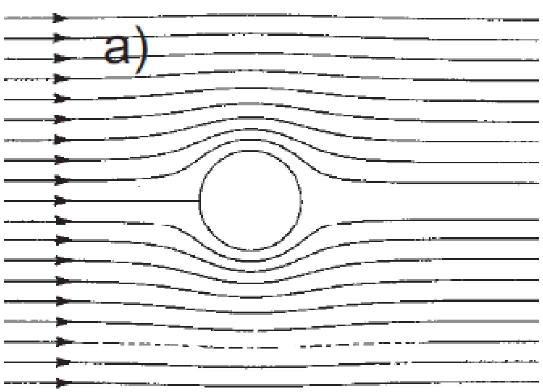
Absorption : $$\sigma_a = \sigma_e - \sigma_s$$

Interaction of light with subwavelength structures



$D < \lambda$

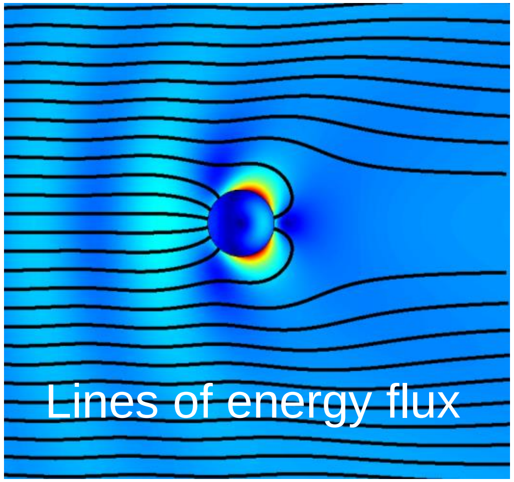
Weak interaction



Bohren, C. F., & Huffman, D. R. (2008). *Absorption and scattering of light by small particles*. John Wiley & Sons.

Extinction cross sections

Resonant optical interaction



$$\mathbf{S}_{\mathrm{ext}} + \mathbf{S}_{\mathrm{inc}} = \mathbf{S}_{\mathrm{tot}} - \mathbf{S}_{\mathrm{scat}}$$

Mie theory from nano to milli-metric particles

Rayleigh scattering

$$\sigma_p = \sigma_e + \frac{4\pi}{k^2}\left[\frac{n(n+2)}{n+1}\mathrm{Re}\{a_n a_{n+1}^* + b_n b_{n+1}^*\} + \frac{2n+1}{n(n+1)}\mathrm{Re}\{a_n b_n^*\}\right]$$

Radiation pressure

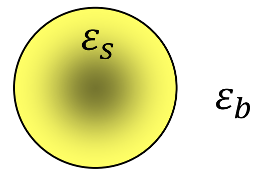
Glory – backscattering

Rainbows, clouds

Photonics : polarizability approach light interacting with a small material sphere)

Electric dipole moment and polarizability, $\alpha(\omega)$

$$\vec{p} = \epsilon_0 \varepsilon_b \alpha(\omega) \vec{E}_{\text{exc}}$$



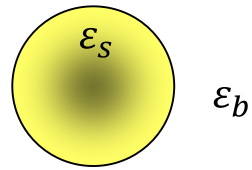
Quasi-static polarizability

$$\alpha_0(\omega) \equiv \lim_{kR \to 0} \alpha(\omega) = 4\pi R^3 \frac{\varepsilon_s - \varepsilon_b}{\varepsilon_s + 2\varepsilon_b}$$

Photonics : polarizability approach to cross section (case of a material sphere)

Electric dipole moment and polarizability, $\alpha(\omega)$

$$\boldsymbol{p} = \epsilon_0 \varepsilon_b \alpha(\omega) \boldsymbol{E}_{\text{exc}}$$



$$k = \frac{2\pi}{\lambda_b} = \frac{\omega}{c_b}$$

$$\alpha_0(\omega) \equiv \lim_{kR \to 0} \alpha(\omega) = 4\pi R^3 \frac{\varepsilon_s - \varepsilon_b}{\varepsilon_s + 2\varepsilon_b}$$

$$P_{\text{s,e,a}} = \sigma_{s,e,a} I_{\text{inc}} \qquad I_{\text{inc}} \propto \|\boldsymbol{E}_{\text{inc}}\|^2$$

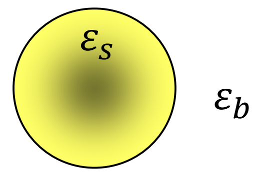
Cross sections :

$$\sigma_{\text{ext}} = k\,\text{Im}\{\alpha(\omega)\} \qquad \sigma_{\text{scat}} = \frac{k^4}{6\pi} |\alpha(\omega)|^2 \qquad \sigma_{\text{abs}} \equiv \sigma_{\text{ext}} - \sigma_{\text{scat}}$$

Despite first appearances there are limits to cross sections !

Electric dipole moment and polarizability, $\alpha(\omega)$

$$\boldsymbol{p} = \epsilon_0 \varepsilon_b \alpha(\omega) \boldsymbol{E}_{\text{exc}}$$



$$\alpha_0(\omega) \equiv \lim_{kR \to 0} \alpha(\omega) = 4\pi R^3 \frac{\varepsilon_s - \varepsilon_b}{\varepsilon_s + 2\varepsilon_b}$$

Infinite ? dielectric response
When $\varepsilon_s = -2\varepsilon_b$

No !

Unitarity (energy conservation) imposes : $\alpha(\omega) = \dfrac{A(\omega)}{1 - i\dfrac{k^3}{6\pi}A(\omega)}$ $\Rightarrow$ $\lim_{\omega \to 0} \alpha(\omega) = \dfrac{\alpha_0}{1 - i\dfrac{k^3}{6\pi}\alpha_0}$

$$k = \frac{2\pi}{\lambda_b} = \frac{\omega}{c_b}$$

$$|\alpha(\omega)| \leq \frac{6\pi}{k^3}$$

$$\sigma_{\text{scat}} = \frac{k^4}{6\pi}|\alpha(\omega)|^2 \leq \frac{6\pi}{k^2} = \frac{3\lambda^2}{2\pi} \cong \frac{\lambda_b^2}{2}$$

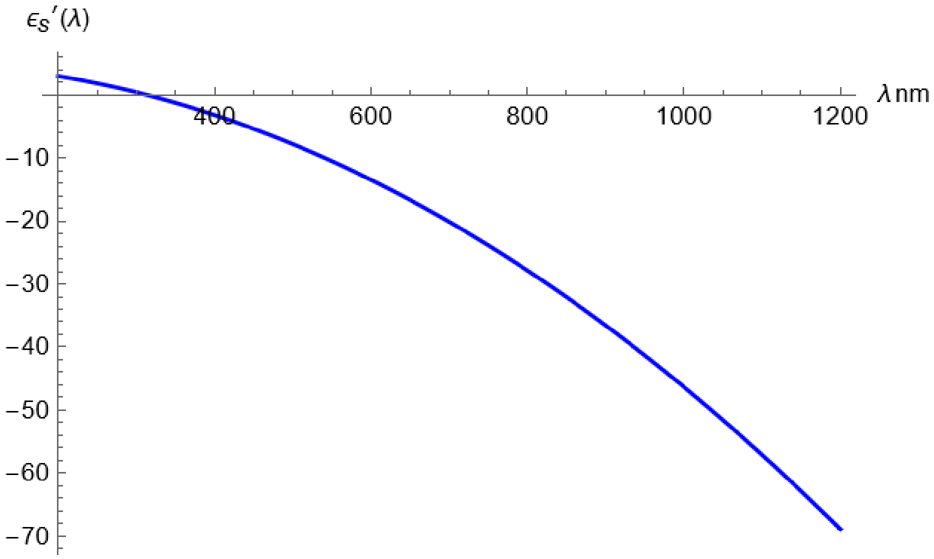
Unitary limit !

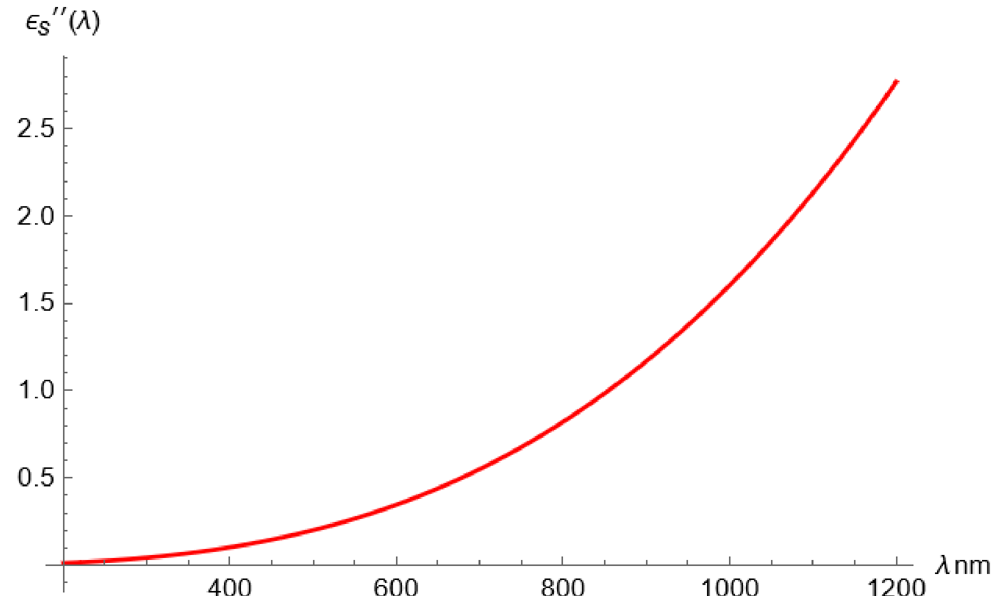
Localized surface plasmon resonances (recap)

$$\sigma_{\text{ext}} = k\,\text{Im}\{\alpha(\omega)\} \qquad \sigma_{\text{scat}} = \frac{k^4}{6\pi}|\alpha(\omega)|^2 \qquad \sigma_{\text{abs}} \equiv \sigma_{\text{ext}} - \sigma_{\text{scat}}$$

$$\alpha_0(\omega) \equiv \lim_{kR\to 0} \alpha(\omega) = 4\pi R^3 \frac{\varepsilon_s - \varepsilon_b}{\varepsilon_s + 2\varepsilon_b}$$

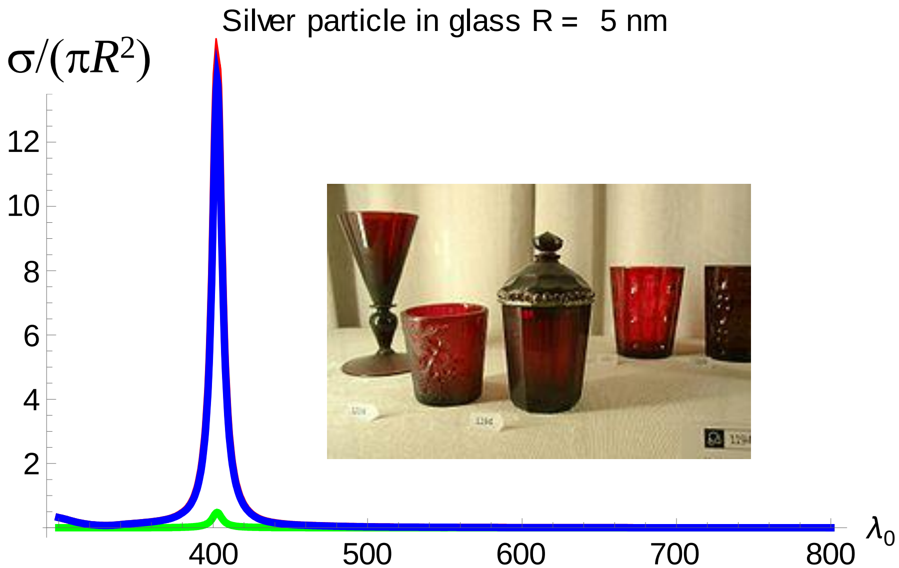
$$\varepsilon_s(\omega) = \varepsilon'_s(\omega) + i\varepsilon_s''(\omega)$$

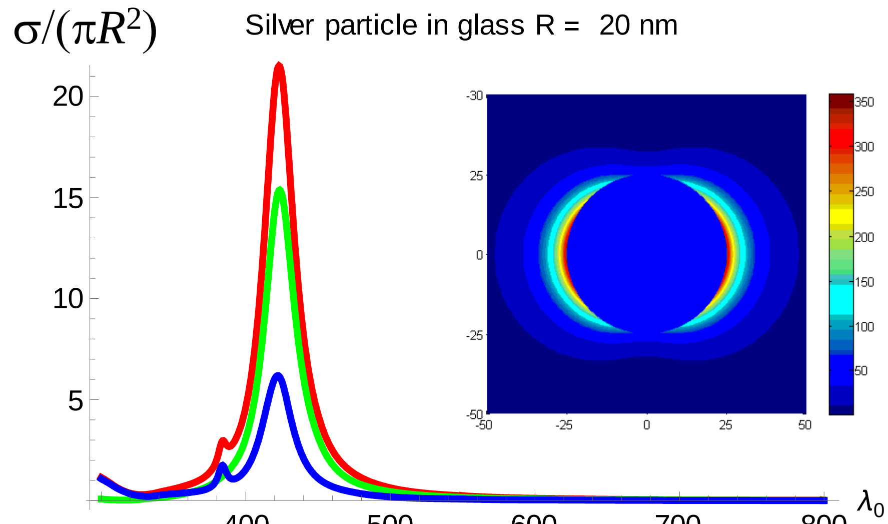


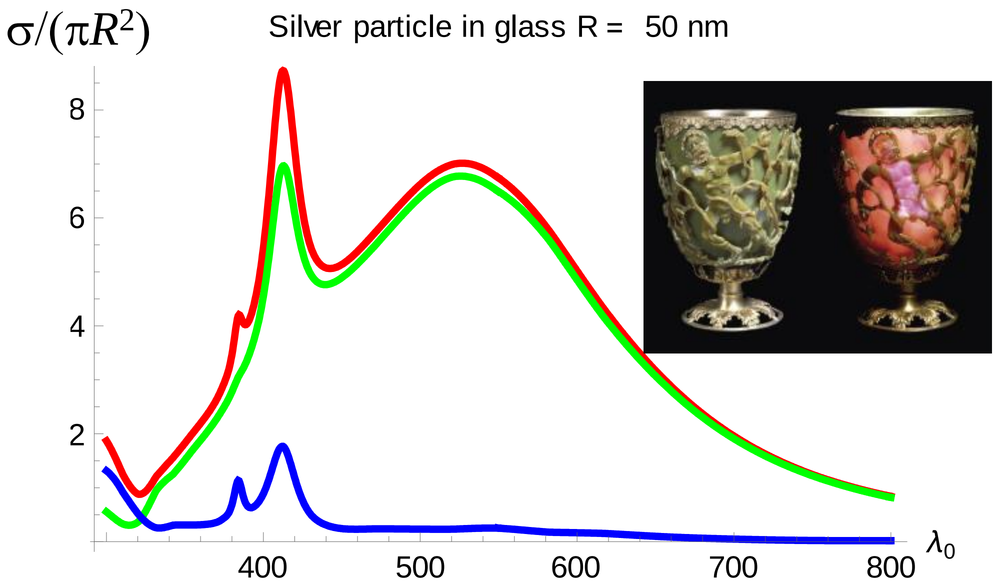


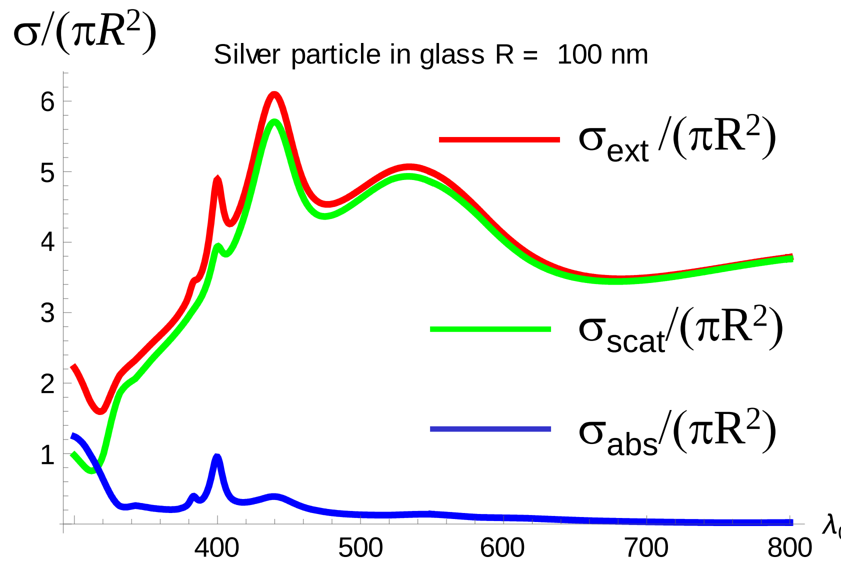
Plasmonics : silver spheres in glass

(Localized surface plasmon resonances)









Scattering and absorption by nano-particles (plasmonics)

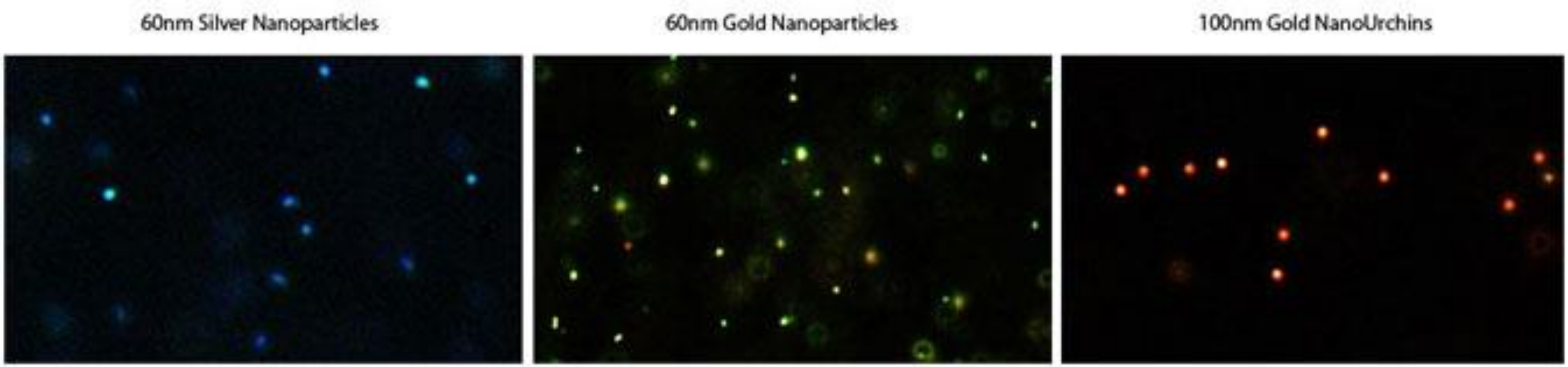
Mie(1908) - Contributions to the optics of turbid media, particularly of colloidal metal solutions

Stain glass ~ 17th century

Gold “Ruby” glass

Lycurgus cup ~ 4th century

Dark field imaging of plasmonic particles
(imaging "below the diffraction limit")



Multipole theory

Homogeneous wave equations in 3D
(spherically symmetric isotropic potentials)

Schrödinger's equation:

$$\Delta\psi + \frac{2m}{\hbar^2}E\psi - \frac{2m}{\hbar^2}V(r)\psi = 0$$

Acoustic waves:

$$\boldsymbol{\nabla}\cdot\frac{1}{\rho(r)}\boldsymbol{\nabla}\psi - \frac{1}{B(r)}\left(\frac{\omega}{c}\right)^2\psi = 0$$

Electromagnetic waves:

$$\boldsymbol{\nabla}\times\frac{1}{\mu(r)}\boldsymbol{\nabla}\times\boldsymbol{E} - \varepsilon(r)\left(\frac{\omega}{c}\right)^2\boldsymbol{E} = \vec{\mathbf{0}}$$

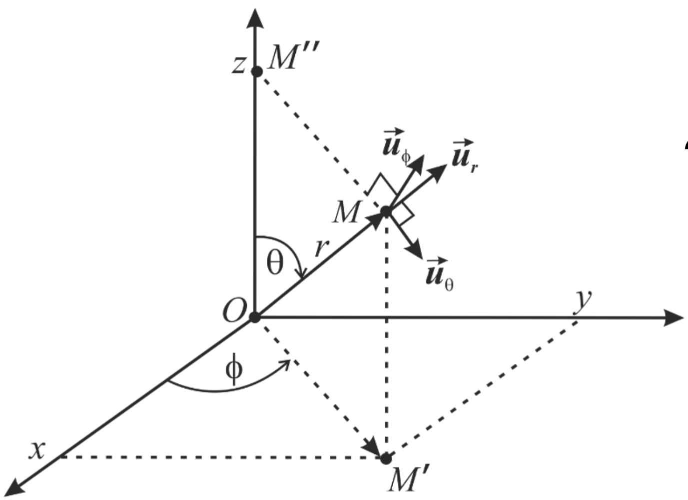
Scalar "light" (acoustics : $\rho = 1\,, \varepsilon(r) = 1/B(r)$):

$$\Delta\psi + \varepsilon(r)\left(\frac{\omega}{c}\right)^2\psi = 0$$

Optical fields ($\mu = 1$) :

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{E} - \varepsilon(r)\left(\frac{\omega}{c}\right)^2\boldsymbol{E} = \vec{\mathbf{0}}$$

Scalar Helmholtz equation
in a spherically symmetric potential:



$$\Delta\psi + \varepsilon(r)\left(\frac{\omega}{c}\right)^2 \psi = 0$$

$$\frac{1}{r}\frac{\partial^2(r\psi)}{\partial r^2} + \frac{1}{r^2\sin^2\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial^2\psi}{\partial\phi^2}\right] + \left(\frac{\omega}{c}\right)^2 \varepsilon(r)\psi = 0$$

$$\frac{1}{r}\frac{\partial^2(r\psi)}{\partial r^2} - \frac{\vec{\mathbb{L}}_{\mathrm{cl}}^2\psi}{r^2} = 0$$

Separation of variables : $\psi(r,\theta,\phi) = \psi_r(r)\psi_\theta(\theta)\psi_\phi(\phi)$

$$\left[\frac{d^2}{d\phi^2} + a\right]\psi_\phi(\phi) = 0 \implies a = m^2 \quad , \quad \psi_\phi(\phi) \propto e^{im\phi} \implies \begin{array}{l} m \in \mathbb{Z} \\ m = -\infty, \dots, -1,0,1, \dots \infty \end{array}$$

$$\left[\frac{1}{\sin^2\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) - \frac{m^2}{\sin^2\theta}\right]\psi_\theta(\theta) = -\ell(\ell+1)\psi_\theta(\theta) \implies \psi_\theta(\theta) \propto P_\ell^m(\cos\theta) \qquad \ell = 0,1,\dots,|m|$$

Ref : Jackson 3rd edition – chapter 3

Legendre Polynomials : $P_\ell(x)$
Associated Legendre functions : $P_\ell^m(x)$

$$\underbrace{-\left[\frac{1}{\sin^2\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)-\frac{m^2}{\sin^2\theta}\right]\psi_\theta(\theta)e^{im\phi}}_{} = \ell(\ell+1)\psi_\theta(\theta)e^{im\phi}$$

$$\vec{\mathbb{L}}^2_{\mathrm{cl}}\psi_\theta(\theta)e^{im\phi} = \ell(\ell+1)\psi_\theta(\theta)e^{im\phi} \Longrightarrow \psi_\theta(\theta) = P_\ell^m(\cos\theta) \quad \ell = 0,1,2,\ldots,\infty$$

Legendre Polynomials: $P_\ell(x) = P_\ell^0(x) = \frac{1}{2^n\ell!}\frac{d^\ell}{dx^\ell}\left[(x^2-1)^\ell\right]$ (Rodrigues' Formula)

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}\left(3x^2-1\right)$$

$$P_3(x) = \frac{1}{2}\left(5x^2-3x\right)$$

$\vdots$

Associated Legendre functions:

$$P_\ell^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_\ell(x)$$

$$= (-1)^m(\sin\theta)^{m/2}\frac{d^m}{dx^m}P_\ell(x)$$

Scalar Spherical harmonics

$$Y_{\ell,m}(\theta,\phi) = \left[\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2} e^{im\phi} P_\ell^m(\cos\theta) \qquad \ell = 0,1,\ldots,\infty \qquad m = -n,\ldots,n$$

$$\equiv e^{im\phi}\bar{P}_\ell^m(\cos\theta)$$

Angular momentum operator: $\vec{\mathbb{L}}_{\mathrm{cl}} \equiv \frac{1}{i}(\boldsymbol{r}\times\boldsymbol{\nabla})$

$$-\left[\frac{1}{\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] Y_{\ell,m}(\theta,\phi)$$

$$\equiv \vec{\mathbb{L}}_{\mathrm{cl}}^2 Y_{\ell,m}(\theta,\phi) = \ell(\ell+1)Y_{\ell,m}(\theta,\phi)$$

Orthogonal basis:

$$\int_0^{4\pi} Y_{\ell,m}^*(\theta,\phi)\Psi_{\ell',\mu}(\theta,\phi)d\Omega = \delta_{\ell,\ell'}\delta_{m,\mu}$$

$$n=0 \quad Y_{00}(x) = \frac{1}{\sqrt{4\pi}}$$

$$n=1 \quad \begin{cases} Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi} \\ Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta \end{cases}$$

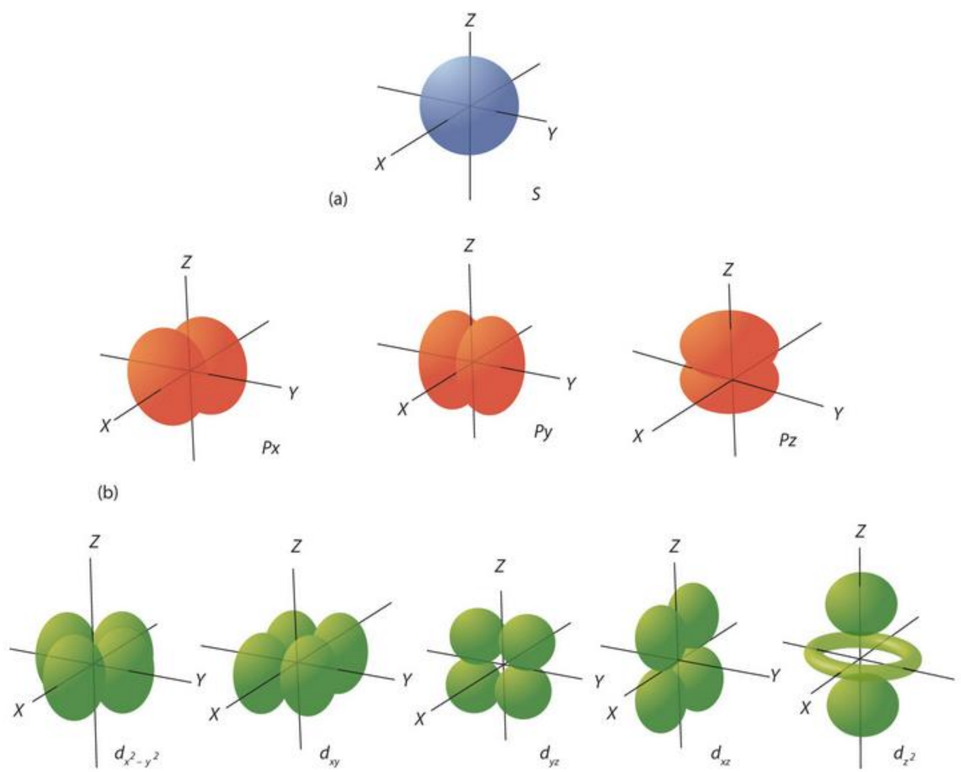
$$n=2 \quad \begin{cases} Y_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta e^{i2\phi} \\ Y_{21} = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{i\phi} \\ Y_{20} = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) \end{cases}$$

The scalar harmonics determine the electron `orbitals'

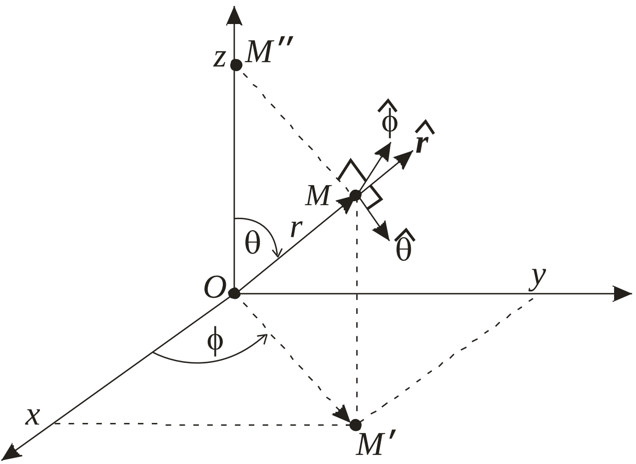
$Y_{0,0}$

$Y_{1,-1}, Y_{1,0}, Y_{1,1}$

$Y_{2,-2}, Y_{2,-1}, Y_{2,0}, Y_{2,1}, Y_{2,2}$



Scalar Helmholtz equation in a spherically symmetric potential:



$$\Delta\psi + \varepsilon(r)\left(\frac{\omega}{c}\right)^2 \psi = 0$$

$$\frac{1}{r}\frac{\partial^2(r\psi)}{\partial r^2} - \frac{\vec{\mathbb{L}}_{\text{cl}}^2}{r^2}\psi + \varepsilon(r)\left(\frac{\omega}{c}\right)^2 \psi = 0$$

$$\psi(r,\theta,\phi) \equiv \psi_r(r) Y_{\ell,m}(\theta,\phi) \qquad \vec{\mathbb{L}}_{\text{cl}}^2 Y_{\ell,m} = \ell(\ell+1) Y_{\ell,m}$$

$$Y_{\ell,m}(\theta,\phi) = e^{im\phi}\bar{P}_\ell^m(\cos\theta) \qquad \begin{cases} \ell = 0,1,2,\ldots,\infty \\ m = -\ell,\ldots,\ell \end{cases}$$

$$\boxed{\frac{d^2u_\ell}{dr^2} - \frac{\ell(\ell+1)}{r^2}u_\ell + \varepsilon(r)\left(\frac{\omega}{c}\right)^2 u_\ell = 0} \qquad u_\ell(r) \equiv r\psi_r(r)$$

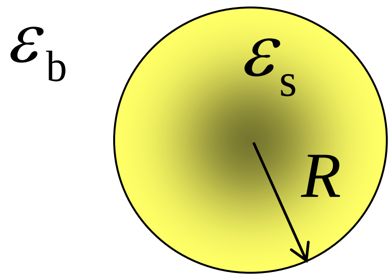
Scattering from a spherically symmetric potential
(Quantum mechanical analogy)

Scalar light :

$$k^2 \equiv \varepsilon_b \left(\frac{\omega}{c}\right)^2$$

$$\frac{d^2}{dr^2} u_\ell + k^2 u_\ell - \left[V_{\text{eff}}^{(\text{opt})}(r,\omega)\right]_\ell u_\ell = 0$$

$$\left[V_{\text{eff}}^{(\text{opt})}(r,\omega)\right]_\ell \equiv \frac{\ell(\ell+1)}{r^2} - (\varepsilon_s - \varepsilon_b)\left(\frac{\omega}{c}\right)^2 \theta(R-r)$$



Schrödinger equation :

$$\frac{d^2}{dr^2} u_\ell + \frac{2m}{\hbar^2} E u_\ell - \left[V_{\text{eff}}^{(\text{Sch})}(r)\right]_\ell u_\ell = 0$$

$$\left[V_{\text{eff}}^{(\text{Sch})}(r)\right]_\ell \equiv \frac{\ell(\ell+1)}{r^2} - \frac{2m}{\hbar^2}\frac{e^2}{4\pi\epsilon_0 r}$$

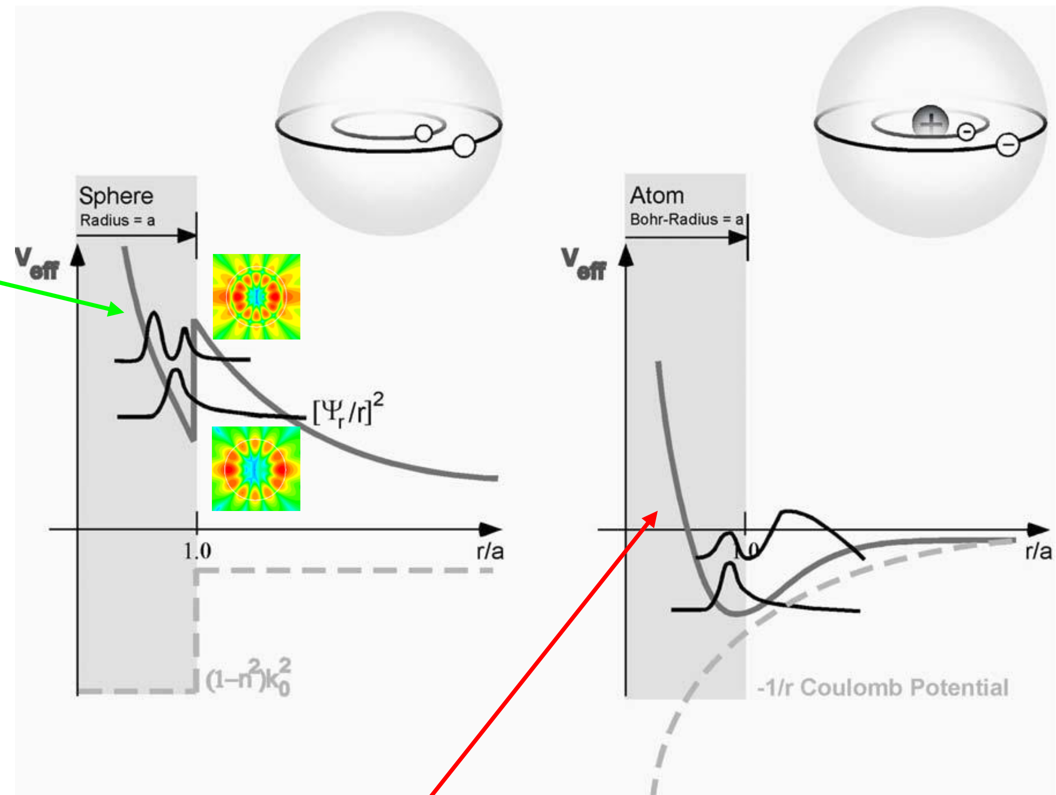
Effective potential – Photonic “atom”

Scalar light : $k^2 \equiv \varepsilon_b \left(\frac{\omega}{c}\right)^2$

Whispering Gallery modes $\varepsilon_s > \varepsilon_b$

“attractive” potential with radiative loss :

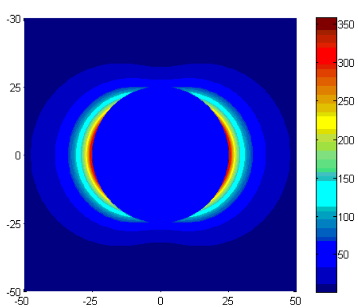
$$\left[V_{\text{eff}}^{(\text{opt})}(r)\right]_\ell \equiv \frac{\ell(\ell+1)}{r^2} - (\varepsilon_s - \varepsilon_b)\left(\frac{\omega}{c}\right)^2 \theta(R-r)$$



Plasmonic resonances

“repulsive” potential :

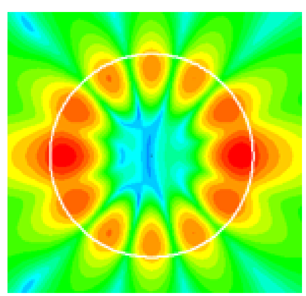
$\varepsilon_s < \varepsilon_b$



“Morphologic” resonances

“attractive” potential :

$\varepsilon_s > \varepsilon_b$

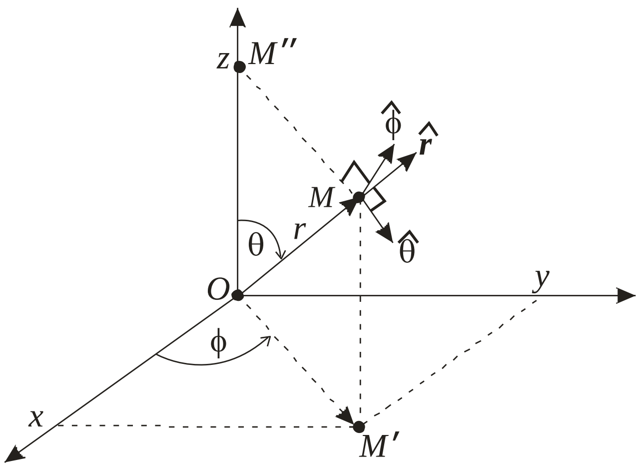


$$\left[V_{\text{eff}}^{(\text{Sch})}(r)\right]_\ell \equiv \frac{\ell(\ell+1)}{r^2} - \frac{2m}{\hbar^2}\frac{e^2}{4\pi\epsilon_0 r}$$

Homogeneous media Helmholtz equation

$$\Delta\psi + k^2\psi = 0$$

Separation of variables:



$$\psi(r,\theta,\phi) = \psi_r(r)Y_{\ell,m}(\theta,\phi) \quad \ell = 0,1,2,\dots,\infty \quad m = -\ell,\dots,\ell$$

$$\Delta\psi + k^2\psi = 0 \rightarrow \left[\frac{1}{r}\frac{d^2(r\psi_r)}{dr} + \frac{\psi_r\vec{\mathbb{L}}_{\mathrm{cl}}^2}{r^2}\psi_r + k^2\psi_r\right]Y_{\ell,m} = 0$$

$$\left[r\frac{d^2(r\psi_r)}{dr} + \ell(\ell+1)\psi_r + k^2r^2\psi_r\right]Y_{\ell,m} = 0$$

Change of variables: $\dfrac{z_\ell(r)}{(kr)^{1/2}} \equiv \psi_r(r)$ $\qquad k^2 \equiv \varepsilon_b\left(\dfrac{\omega}{c}\right)^2$

Spherical Bessel function equation:

$$r^2\frac{d^2z_\ell}{dr^2} + r\frac{dz_\ell}{dr} + \left[k^2r^2 - \left(\ell + \frac{1}{2}\right)^2\right]z_\ell = 0$$

Homogeneous Helmholtz equation in spherical coordinates

$$r^2 \frac{d^2 z_\ell}{dr^2} + r\frac{dz_\ell}{dr} + \left[k^2r^2 - \left(\ell + \frac{1}{2}\right)^2\right] z_\ell = 0$$

$$\frac{z_\ell(r)}{(kr)^{1/2}} \equiv \psi_r(r)$$

Spherical Bessel, Neumann and Hankel functions:

$$\psi_r(r) = \begin{cases} j_\ell(x) \equiv \sqrt{\frac{\pi}{2x}} J_{\ell+1/2}(x) \\ y_\ell(x) \equiv \sqrt{\frac{\pi}{2x}} Y_{\ell+1/2}(x) \\ h_\ell^{(+)}(x) \equiv \sqrt{\frac{\pi}{2x}} H_{\ell+1/2}^{(+)}(x) = j_\ell(x) + i y_\ell(x) \\ h_\ell^{(-)}(x) \equiv \sqrt{\frac{\pi}{2x}} H_{\ell+1/2}^{(-)}(x) = j_\ell(x) - i y_\ell(x) \end{cases}$$

Bessel (1):

$$j_0(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)!} x^{2s} = \frac{\sin x}{x}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

$$\vdots$$

Neumann (2):

$$y_0(x) = -\frac{\cos x}{x}$$

$$y_1(x) = \left(\frac{\cos^2 x}{x^2} - \frac{\sin x}{x}\right)$$

$$\vdots$$

Hankel + (3):

$$h_0^{(+)}(x) = -\frac{i}{x} e^{ix}$$

$$h_1^{(+)}(x) = -e^{ix}\left(\frac{1}{x} + \frac{i}{x^2}\right)$$

$$\vdots$$

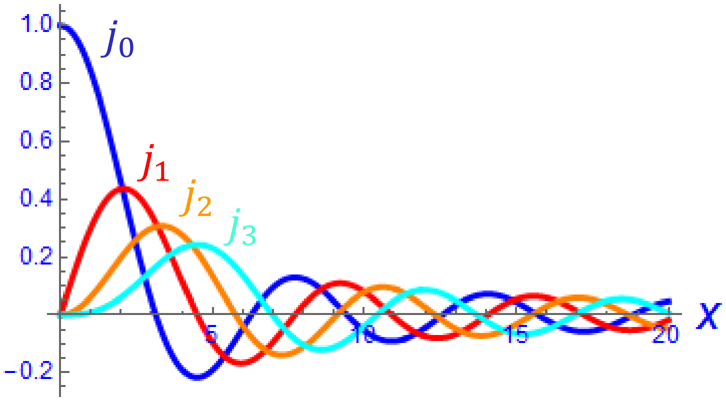
Linearly independent solutions

Spherical Bessel functions (1)

$$j_0(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{(2s+1)!} x^{2s} = \frac{\sin x}{x}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x}$$

$$\vdots$$

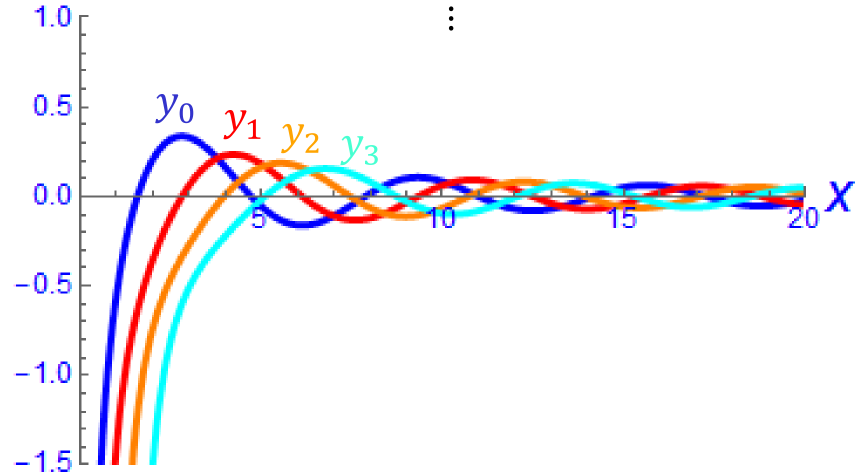


Spherical Neumann functions (2)

$$y_0(x) = -\frac{\cos x}{x}$$

$$y_1(x) = \left(\frac{\cos^2 x}{x^2} - \frac{\sin x}{x}\right)$$

$$\vdots$$



Outgoing spherical Hankel functions (+)

$$h_0^{(+)}(x) = -\frac{i}{x} e^{ix}$$

$$h_1^{(+)}(x) = -e^{ix}\left(\frac{1}{x} + \frac{i}{x^2}\right)$$

$$\vdots$$

Incoming spherical Hankel functions (-)

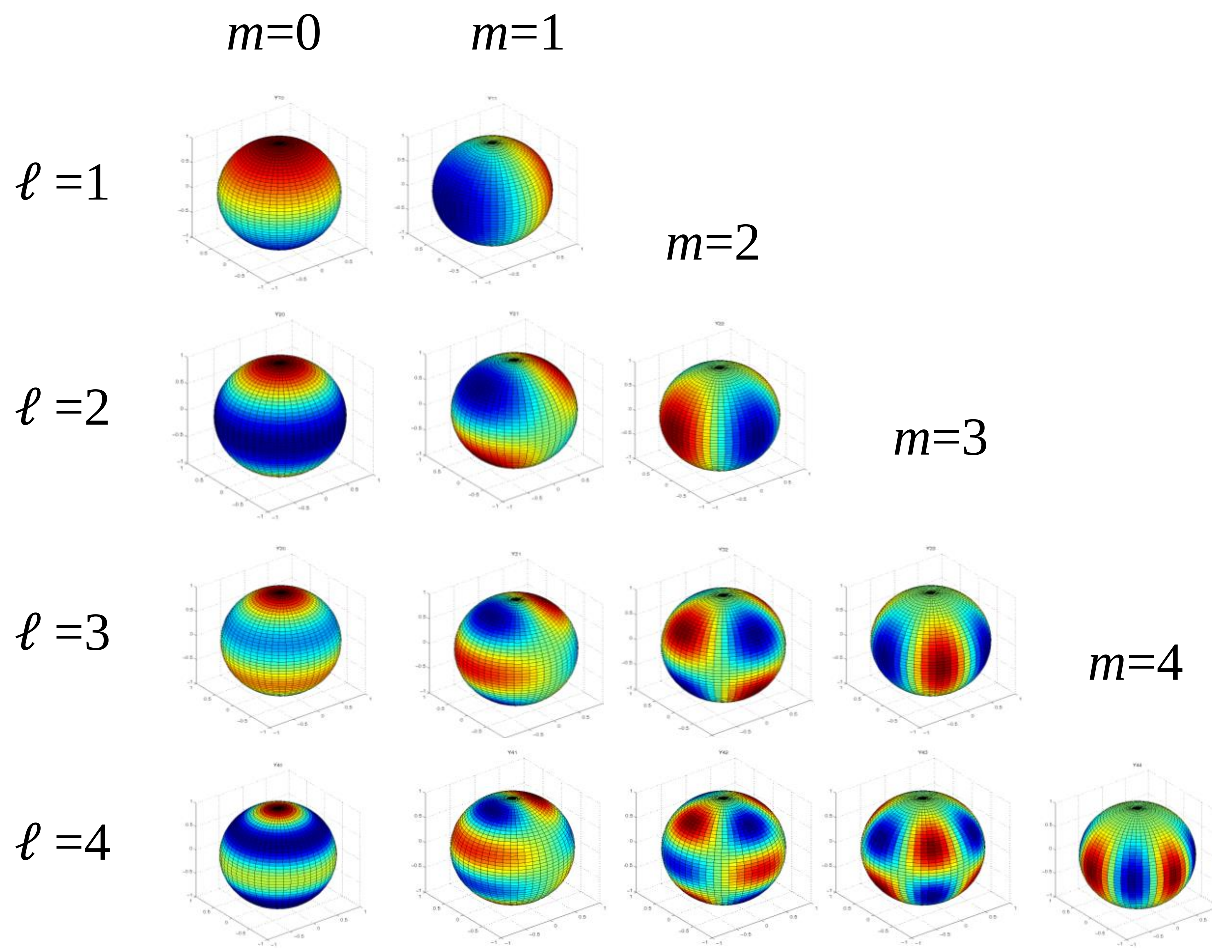
$$h_0^{(-)}(x) = \frac{i}{x} e^{-ix}$$

$$h_1^{(-)}(x) = -e^{-ix}\left(\frac{1}{x} - \frac{i}{x^2}\right)$$

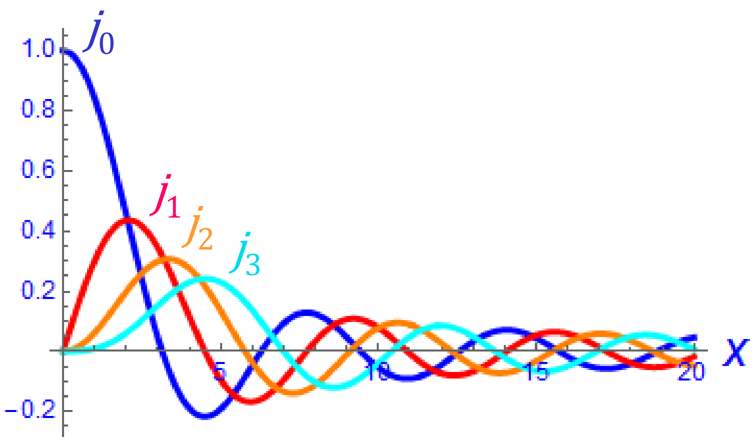
$$\vdots$$

Regular partial wave basis : $\Psi_{\ell,m} \equiv i^{\ell} k^{3/2} j_{\ell}(kr) Y_{\ell,m}(\theta,\phi)$

Spherical Harmonics : $Y_{\ell,m}(\theta,\phi)$



Bessel functions : $j_{\ell}(kr)$



Regular partial waves form a basis for source-free fields

(scalar waves)

$$\Psi_{\ell,m}(k\boldsymbol{r}) \equiv i^{\ell} k^{3/2} j_{\ell}(kr) Y_{\ell,m}(\theta,\phi) \qquad \ell = 0,1,2,\dots,\infty \quad m = -\ell,\dots,\ell$$

Orthogonality

$$\frac{2}{\pi}\int \Psi_{\ell,m}(k\boldsymbol{r})\Psi_{\ell',\mu}(k'\boldsymbol{r})d\boldsymbol{r} = \delta_{\ell,\ell'}\delta_{m,\mu}\delta(k-k')$$

Closure

$$\frac{2}{\pi}\int_0^{\infty}\sum_{\ell,m=0}^{\infty}\Psi_{\ell,m}(k\boldsymbol{r})\Psi_{\ell,m}(k\boldsymbol{r}')\frac{dk}{k} = \delta^{(3)}(\boldsymbol{r}-\boldsymbol{r}')$$

Partial wave basis (scalar waves)

Plane wave expansion :

$$k^{3/2}e^{i\boldsymbol{k}\cdot\boldsymbol{r}} = \sum_{\ell=0}^{\infty}\sum_{m=-n}^{n} p_{\ell,m}\Psi_{\ell,m}(k\boldsymbol{r})$$

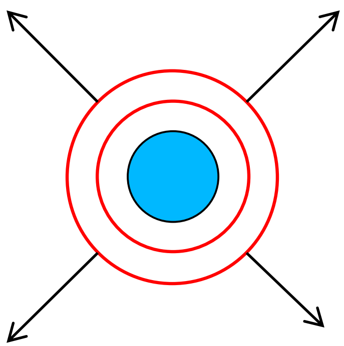
Plane wave Coefficients :

$$p_{\ell,m} = 4\pi Y^{*}_{\ell,m}(\boldsymbol{u_k})$$

$$\boldsymbol{u_k} \equiv \frac{\boldsymbol{k}}{k}$$

Regular partial wave :

$$\Psi_{\ell,m}(k\boldsymbol{r}) \equiv i^{\ell}k^{3/2}j_{\ell}(kr)Y_{\ell,m}(\theta,\phi)$$



Outgoing "partial" wave :

$$\Psi^{(+)}_{\ell,m}(k\boldsymbol{r}) \equiv i^{\ell}k^{3/2}h^{(+)}_{\ell}(kr)Y_{\ell,m}(\theta,\phi)$$

Green's function (scalar waves)
Complete solutions to the wave equations valid for arbitrary sources

We want to solve the inhomogeneous wave equation for an arbitrary source $s(\vec{\boldsymbol{r}})$ in a homogeneous medium :

$$k^2 \equiv \varepsilon \left(\frac{\omega}{c}\right)^2$$

$$\Delta\psi(\boldsymbol{r}) + k^2\psi(\boldsymbol{r}) = -s(\boldsymbol{r})$$

This can be achieved by solving for the Green's function $g(\boldsymbol{r}, \boldsymbol{r}')$

$$\Delta g(\boldsymbol{r}, \boldsymbol{r}') + k^2 g(\boldsymbol{r}, \boldsymbol{r}') = -\delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}')$$

This solution to the wave equation, $\psi(\vec{\boldsymbol{r}})$ for an arbitrary source, $s(\vec{\boldsymbol{r}})$ is then found by:

$$\psi(\boldsymbol{r}) = \int g(\boldsymbol{r}, \boldsymbol{r}') s(\boldsymbol{r}') d\boldsymbol{r}'$$

Green's function in a homogeneous media can be constructed from the `partial' waves: $\Psi^{(+)}_{\ell,m}$, $\Psi_{\ell,m}$

$$\Delta g(\boldsymbol{r},\boldsymbol{r}') + k^2 g(\boldsymbol{r},\boldsymbol{r}') = \delta^{(3)}(\boldsymbol{r}-\boldsymbol{r}') \qquad k^2 \equiv \varepsilon\left(\frac{\omega}{c}\right)^2$$

$$g(\boldsymbol{r},\boldsymbol{r}') = \frac{i}{k^2}\sum_{\ell\geq 0,m}(-)^l \Psi^{(+)}_{\ell,m}(k\boldsymbol{r}_>)\Psi_{\ell,m}(k\boldsymbol{r}_<)$$

$$|\boldsymbol{r}| > |\boldsymbol{r}'| \Rightarrow \{\boldsymbol{r}_> = \boldsymbol{r} \text{ and } \boldsymbol{r}_< = \vec{\boldsymbol{r}}'\}$$

$$|\boldsymbol{r}| < |\boldsymbol{r}'| \Rightarrow \{\boldsymbol{r}_> = \boldsymbol{r}' \text{ and } \boldsymbol{r}_> = \boldsymbol{r}\}$$

Regular partial waves:

$$\Psi_{\ell,m}(k\boldsymbol{r}) = i^\ell k^{3/2} j_\ell(kr) Y_{\ell,m}(\theta,\phi)$$

Outgoing "partial" waves:

$$\Psi^{(+)}_{\ell,m}(k\boldsymbol{r}) = i^\ell k^{3/2} h^{(+)}_\ell(kr) Y_{\ell,m}(\theta,\phi)$$

Closed form for the Green's function :

The infinite sum can be simplified by letting $\vec{\boldsymbol{r}}' \rightarrow \mathbf{0}$ and remarking that only $\Psi_{0,0}(k\mathbf{0}) = \frac{1}{\sqrt{4\pi}}$ is non-zero in this limit.

The Green's function then simplifies to:

$$g(\vec{\boldsymbol{r}}, \vec{\mathbf{0}}) = \frac{i}{k^2} \sum_{\ell \geq 0, m} \Psi_{\ell,m}^{(+)}(k\boldsymbol{r}_>) \Psi_{\ell,m}(k\boldsymbol{r}_<)$$

$$\longrightarrow \frac{i}{k^2} \Psi_{0,0}^{(+)}(k\boldsymbol{r}) \Psi_{0,0}(k\mathbf{0}) = \frac{e^{ikr}}{4\pi r}$$

Famous result for Green's function of the scalar Helmholtz equation:

$$g(\boldsymbol{r}_1, \boldsymbol{r}) = \frac{e^{ik|\boldsymbol{r}_1 - \boldsymbol{r}_2|}}{4\pi|\boldsymbol{r}_1 - \boldsymbol{r}_2|} \equiv \frac{e^{ikr_{12}}}{4\pi r_{12}}$$

Homogenous Maxwell equation in spherical coordinates

$$\nabla \times \nabla \times \boldsymbol{E} - k^2 \boldsymbol{E} = \boldsymbol{0}$$

$$k^2 = \varepsilon\mu \left(\frac{\omega}{c}\right)^2$$

$$\Delta\psi + k^2\psi = 0$$

Transverse vector "partial" waves : $\boldsymbol{\nabla} \cdot \boldsymbol{M}_{n,m} = \boldsymbol{\nabla} \cdot \boldsymbol{N}_{n,m} = 0$

Scalar "partial" waves

$$\Psi_{\ell,m}(k\boldsymbol{r}) \equiv i^{\ell} k^{3/2} j_{\ell}(kr) Y_{\ell,m}(\theta,\phi)$$

$$\boldsymbol{M}_{n,m}(k\boldsymbol{r}) \equiv \frac{\nabla \times \left[\boldsymbol{r}\Psi_{n,m}(k\boldsymbol{r})\right]}{\sqrt{n(n+1)}}$$ magnetic

$$\boldsymbol{N}_{n,m}(k\boldsymbol{r}) \equiv \frac{\nabla \times \boldsymbol{M}_{n,m}(k\boldsymbol{r})}{ik}$$ electric

$$\ell = 0,1,2,\dots,\infty \quad m = -\ell,\dots,\ell$$

Longitudinal vector "partial" waves

$$\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{L}) + k^2 \boldsymbol{L} = \boldsymbol{0}$$

$$\boldsymbol{L}_{\ell,m}(k\boldsymbol{r}) = \frac{\nabla\left[\Psi_{\ell,m}(k\boldsymbol{r})\right]}{k\sqrt{\ell(\ell+1)}}$$

$$\Delta\boldsymbol{A} + k^2\boldsymbol{A} = \boldsymbol{0} \qquad \boldsymbol{A} = \boldsymbol{L},\boldsymbol{M},\boldsymbol{N}$$

Bouwkamp-Casimir (Jackson) approach

$$\Delta(\boldsymbol{r} \cdot \boldsymbol{H}) + k^2 \boldsymbol{r} \cdot \boldsymbol{H} = \boldsymbol{0}$$

$$\Delta(\boldsymbol{r} \cdot \boldsymbol{E}) + k^2 \boldsymbol{r} \cdot \boldsymbol{E} = \boldsymbol{0}$$

Vector spherical harmonics

$$Y_{\ell,m}(\theta,\phi) = \left[\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2} e^{im\phi} P_\ell^m(\cos\theta) \equiv e^{im\phi} \bar{P}_\ell^m(\cos\theta)$$

Angular momentum operator : $\vec{\mathbb{L}}_{\text{op}} \equiv \frac{1}{i}(\boldsymbol{r} \times \boldsymbol{\nabla})$

3 types of vector spherical harmonics :

$$\boldsymbol{W}^{(1)}_{\ell,m}(\theta,\phi) \equiv \boldsymbol{Y}_{\ell,m}(\theta,\phi) \equiv \boldsymbol{u}_r Y_{\ell,m}(\theta,\phi)$$

ℓ =0,1,2,…,∞ m=- ℓ,…, ℓ

$$\boldsymbol{W}^{(2)}_{n,m}(\theta,\phi) \equiv \boldsymbol{X}_{n,m}(\theta,\phi) \equiv \frac{\vec{\mathbb{L}}_{\text{op}}}{i} Y_{n,m}(\theta,\phi) = \boldsymbol{Z}_{n,m}(\theta,\phi) \times \boldsymbol{u}_r$$

n=1,2,…,∞ m=-n,…,n

$$\boldsymbol{W}^{(3)}_{n,m}(\theta,\phi) \equiv \boldsymbol{Z}_{n,m}(\theta,\phi) \equiv \frac{r\vec{\boldsymbol{\nabla}} Y_{n,m}(\theta,\phi)}{\sqrt{n(n+1)}} = \boldsymbol{u}_r \times \vec{\boldsymbol{X}}_{n,m}(\theta,\phi)$$

$$\int_0^{4\pi} d\Omega\, \boldsymbol{W}^{(j),*}_{\nu,\mu}(\theta,\phi) \cdot \boldsymbol{W}^{(k)}_{n,m}(\theta,\phi) = \delta_{j,k}\delta_{n,\nu}\delta_{m,\nu}$$

Vector spherical harmonics

$$\boldsymbol{Y}_{\ell,m}(\theta,\phi) = \boldsymbol{u}_r Y_{\ell,m}(\theta,\phi)$$

$\ell = 0,1,2,\ldots,\infty$ $\quad m = -\ell, \ldots, \ell$

$$\boldsymbol{X}_{n,m}(\theta,\phi) = \boldsymbol{e}^{im\phi}\left[\boldsymbol{u}_\theta i\bar{u}_n^m(\theta) - \boldsymbol{u}_\phi \bar{s}_n^m(\theta)\right]$$

$n = 1,2,\ldots,\infty$ $\quad m = -n, \ldots, n$

$$\boldsymbol{Z}_{n,m}(\theta,\phi) = \boldsymbol{e}^{im\phi}\left[\boldsymbol{u}_\theta \bar{s}_n^m(\theta) - \boldsymbol{u}_\phi i\bar{u}_n^m(\theta)\right]$$

$$\bar{u}_n^m(\theta) = \frac{1}{\sqrt{n(n+1)}}\frac{m}{\sin\theta}\bar{P}_n^m(\theta)$$

$$\bar{s}_n^m(\theta) = \frac{1}{\sqrt{n(n+1)}}\frac{d}{d\theta}\bar{P}_n^m(\theta)$$

Vector ‘partial waves’ $\nabla \times \nabla \times \boldsymbol{E} - k^2 \boldsymbol{E} = \mathbf{0}$

Ricatti-Bessel functions

$$\psi_n(x) \equiv x j_n(x)$$

$$\psi_n'(x) = \left[j_n(x) + x j_n'(x) \right]$$

$$\boldsymbol{M}_{n,m}(k\boldsymbol{r}) = i^n k^{3/2} j_n(kr) \boldsymbol{X}_{n,m}(\theta,\phi)$$

$$\boldsymbol{N}_{n,m}(k\boldsymbol{r}) = i^{n-1} k^{3/2} \left[\frac{1}{kr} j_n(kr) \sqrt{n(n+1)} \boldsymbol{Y}_{n,m}(\theta,\phi) + \psi'_n(kr) \boldsymbol{Z}_{n,m}(\theta,\phi) \right]$$

$n = 1, 2, \ldots, \infty$ $\quad$ $-n < m < n$

$$\nabla \cdot \boldsymbol{M}_{n,m} = \nabla \cdot \boldsymbol{N}_{n,m} = 0 \qquad \nabla \times \boldsymbol{M}_{n,m} = k \boldsymbol{N}_{n,m} \qquad \nabla \times \boldsymbol{N}_{n,m} = k \boldsymbol{M}_{n,m}$$

$$\Delta \boldsymbol{M}_{n,m} + k^2 \boldsymbol{M}_{n,m} = \vec{\mathbf{0}} \qquad \Delta \boldsymbol{N}_{n,m} + k^2 \boldsymbol{N}_{n,m} = \mathbf{0}$$

Longitudinal partial waves

$$\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\boldsymbol{L}_{\ell,m}\right)+k^2\boldsymbol{L}_{\ell,m}=\boldsymbol{0}$$

$$\boldsymbol{L}_{\ell,m}(k\boldsymbol{r})=\frac{\nabla\left[\Psi_{\ell,m}(k\boldsymbol{r})\right]}{ik\sqrt{\ell(\ell+1)}}$$

$$\boldsymbol{\nabla}\times\boldsymbol{L}_{\ell,m}=\boldsymbol{0}$$

$$\boldsymbol{L}_{\ell,m}(k\boldsymbol{r})=i^{\ell-1}k^{3/2}\left[j_\ell'(kr)\boldsymbol{Y}_{\ell,m}(\theta,\phi)+\sqrt{\ell(\ell+1)}\frac{j_\ell(kr)}{kr}\boldsymbol{Z}_{\ell,m}(\theta,\phi)\right]$$

$\ell = 0,1,2,\ldots,\infty$ $\quad$ $-\ell < m < \ell$

$$\Delta\boldsymbol{L}_{\ell,m}+k^2\boldsymbol{L}_{\ell,m}=\boldsymbol{0}$$

Regular partial waves form a basis for source-free fields
(vector waves)

Orthogonality

$$\int \boldsymbol{M}_{n,m}(k\vec{\boldsymbol{r}}) \cdot \boldsymbol{M}_{\nu,\mu}(k'\boldsymbol{r})d\vec{\boldsymbol{r}} = (-)^n k\delta_{n,\nu}\delta_{m,\mu}\delta(k-k')$$

$$n,\nu = 1,2,\dots,\infty \qquad \begin{matrix} m = -n,\dots,n \\ \mu = -\nu,\dots,\nu \end{matrix}$$

$$\int \boldsymbol{N}_{n,m}(k\vec{\boldsymbol{r}}) \cdot \boldsymbol{N}_{\nu,\mu}(k'\boldsymbol{r})d\vec{\boldsymbol{r}} = (-)^{n-1} k\delta_{n,\nu}\delta_{m,\mu}\delta(k-k')$$

$$\int \boldsymbol{L}_{\ell,m}(k\boldsymbol{r}) \cdot \boldsymbol{L}_{\ell',\mu}(k'\boldsymbol{r})d\vec{\boldsymbol{r}} = (-)^{n-1} k\delta_{\ell,\ell'}\delta_{m,\mu}\delta(k-k')$$

$$\ell = 0,1,2,\dots,\infty \qquad \begin{matrix} m = -\ell,\dots,\ell \\ \mu = -\ell',\dots,\ell \end{matrix}$$

Closure

$$\frac{2}{\pi}\int_0^\infty \sum_{n,m=0}^{\infty} (-)^n \boldsymbol{M}_{n,m}(k\boldsymbol{r})\boldsymbol{M}_{n,m}(k\boldsymbol{r}')\frac{dk}{k} + \frac{2}{\pi}\int_0^\infty (-)^{n-1} \sum_{n,m=0}^{\infty} (-)^n \boldsymbol{N}_{n,m}(k\boldsymbol{r})\boldsymbol{N}_{n,m}(k\boldsymbol{r}')\frac{dk}{k}$$

$$+\frac{2}{\pi}\int_0^\infty \sum_{\ell,m=0}^{\infty} (-)^{\ell-1} \boldsymbol{L}_{\ell,m}(k\boldsymbol{r})\boldsymbol{L}_{\ell,m}(k\boldsymbol{r}')\frac{dk}{k} = \overleftrightarrow{\mathbb{I}}\delta^{(3)}(\boldsymbol{r}-\boldsymbol{r}')$$

Development of a vector plane wave (Vector partial wave basis)

Plane wave expansion :

$$k^{3/2} e^{i\boldsymbol{k}_\mathrm{i}\cdot\boldsymbol{r}}\boldsymbol{u}_\mathrm{i} = \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left[p_{n,m}^{(h)}\boldsymbol{M}_{n,m}(k\boldsymbol{r}) + p_{n,m}^{(e)}\boldsymbol{N}_{n,m}(k\boldsymbol{r})\right]$$

$$\boldsymbol{u}_\mathrm{i} \equiv \frac{\boldsymbol{k}_\mathrm{i}}{k}$$

Plane wave coefficients :

$$p_{n,m}^{(h)} = 4\pi \boldsymbol{X}_{n,m}^{*}(\boldsymbol{u}_\mathrm{i})\cdot\hat{\boldsymbol{e}}_\mathrm{i} \qquad p_{n,m}^{(e)} = 4\pi \boldsymbol{Z}_{n,m}^{*}(\boldsymbol{u}_\mathrm{i})\cdot\hat{\boldsymbol{e}}_\mathrm{i}$$

Green's function (vector waves)
Complete solutions to the wave equations valid for arbitrary sources

We want to solve the inhomogeneous wave equation for an arbitrary source $\boldsymbol{j}(\boldsymbol{r})$ in a homogeneous medium:

$$\nabla \times \nabla \times \boldsymbol{E}_\omega(\boldsymbol{r}) - \frac{\omega^2}{c^2}\varepsilon_b \mu_b \boldsymbol{E}_\omega(\boldsymbol{r}) = \frac{i\omega}{\varepsilon_b \epsilon_0 c^2}\boldsymbol{j}_\omega(\boldsymbol{r})$$

This can be achieved by solving for the dyadic Green's function $\overleftrightarrow{\boldsymbol{g}}(\boldsymbol{r},\boldsymbol{r}')$:

$$\nabla \times \frac{1}{\mu_b}\nabla \times \overleftrightarrow{\boldsymbol{g}}(\boldsymbol{r},\boldsymbol{r}') - \frac{\omega^2}{c^2}\varepsilon_b \overleftrightarrow{\boldsymbol{g}}(\boldsymbol{r},\boldsymbol{r}') = \overleftrightarrow{\mathbb{I}}\delta^{(3)}(\boldsymbol{r}-\boldsymbol{r}')$$

This solution to the wave equation, $\vec{\boldsymbol{E}}(\boldsymbol{r})$, for an arbitrary source, $\vec{\boldsymbol{j}}(\boldsymbol{r})$, is then:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{i\omega}{\varepsilon_b \epsilon_0 c^2}\int \overleftrightarrow{\boldsymbol{g}}(\boldsymbol{r},\boldsymbol{r}') \cdot \vec{\boldsymbol{j}}(\boldsymbol{r}')d\boldsymbol{r}'$$

Green's function dyadic constructed from the vector 'partial' waves:

$$\boldsymbol{M}_{n,m}^{(+)}, \boldsymbol{N}_{n,m}^{(+)}, \boldsymbol{M}_{n,m}, \boldsymbol{N}_{n,m}$$

Green's function solution satisfying `outgoing' field conditions

$$\overleftrightarrow{\boldsymbol{g}}(\boldsymbol{r},\boldsymbol{r}') = \frac{i}{k^2}\sum_{n\geq 1,m}\left\{\boldsymbol{M}_{n,m}^{(+)}(k\boldsymbol{r}_>)\boldsymbol{M}_{n,m}(k\boldsymbol{r}_<) + \boldsymbol{N}_{n,m}^{(+)}(k\boldsymbol{r}_>)\boldsymbol{N}_{n,m}(k\boldsymbol{r}_<)\right\} - \frac{\boldsymbol{u}_r\boldsymbol{u}_r}{k^2}\delta^{(3)}(\boldsymbol{r}-\boldsymbol{r}')$$

$$\boldsymbol{u}_r \equiv \frac{\vec{\boldsymbol{r}}}{r}$$

$$|\boldsymbol{r}| > |\boldsymbol{r}'| \Rightarrow \{\boldsymbol{r}_> = \boldsymbol{r} \text{ and } \boldsymbol{r}_< = \boldsymbol{r}'\}$$

$$|\boldsymbol{r}| < |\boldsymbol{r}'| \Rightarrow \{\boldsymbol{r}_> = \boldsymbol{r}' \text{ and } \boldsymbol{r}_> = \boldsymbol{r}\}$$

Often, the distribution $-\frac{\boldsymbol{u}_r\boldsymbol{u}_r}{k^2}\delta^{(3)}(\boldsymbol{r}-\boldsymbol{r}')$ can be safely ignored, but sometimes must be included in certain applications:

The distribution $-\frac{\boldsymbol{u}_r\boldsymbol{u}_r}{k^2}\delta^{(3)}(\boldsymbol{r}-\boldsymbol{r}')$ compensates for strongly divergent fields that are only defined outside an infinitesimal exclusion volume around the origin

(3D principal volume : cf. Jackson, Chew, Tsang end Kong, …)

Closed form for the Green's function :

Letting $\boldsymbol{r}' \to \mathbf{0}$ in $\overleftrightarrow{\boldsymbol{g}}(\boldsymbol{r},\boldsymbol{r}')$ and remarking that only $\boldsymbol{N}_{1,m}(k\mathbf{0})$ is non-zero in this limit so the Green's function then simplifies to:

$$\overleftrightarrow{\boldsymbol{g}}(\boldsymbol{r},\mathbf{0}) \longrightarrow \frac{i}{k^2}\sum_{m=-1,0,1}(-)^m\left\{\boldsymbol{N}_{1,m}^{(+)}(k\vec{\boldsymbol{r}})\boldsymbol{N}_{1,-m}(k\mathbf{0})\right\} - \frac{\boldsymbol{u}_r\boldsymbol{u}_r}{k^2}\delta^{(3)}(\boldsymbol{r})$$

$$= \frac{e^{ikr}}{4\pi r^3\varepsilon_b\left(\frac{\omega}{c}\right)^2}\left\{(1-ikr)\left[3\boldsymbol{u}_r\boldsymbol{u}_r - \overleftrightarrow{\mathbb{I}}\right] + k^2r^2\left[\overleftrightarrow{\mathbb{I}} - \boldsymbol{u}_r\boldsymbol{u}_r\right]\right\} - \frac{\overleftrightarrow{\mathbb{I}}}{3k^2}\delta^{(3)}(\boldsymbol{r})$$

Electromagnetic Green's function :

$$\overleftrightarrow{\boldsymbol{g}}(\boldsymbol{r}_1,\boldsymbol{r}_2) = \frac{e^{ikr_{12}}}{4\pi r_{12}^3\varepsilon_b\left(\frac{\omega}{c}\right)^2}\left\{k^2r_{12}^2\left[\overleftrightarrow{\mathbb{I}} - \boldsymbol{u}_{12}\boldsymbol{u}_{12}\right] + (1-ikr_{12})\left[3\boldsymbol{u}_{12}\boldsymbol{u}_{12} - \overleftrightarrow{\mathbb{I}}\right]\right\} - \frac{\overleftrightarrow{\mathbb{I}}}{3k^2}\delta^{(3)}(\boldsymbol{r}_1-\boldsymbol{r}_2)$$

Example: Electric dipole emission:

A 'point' electric dipole, $\boldsymbol{p}$ at the position, $\boldsymbol{r}_2$, oscillating at angular frequency of ω i.e. characterized by a current : $\boldsymbol{j}(\boldsymbol{r}',t) = -i\omega\boldsymbol{p}e^{-i\omega t}\delta^{(3)}(\boldsymbol{r}'-\boldsymbol{r}_2)$.

The electric field measured at a position $\boldsymbol{r}_1$ is :

$$\boldsymbol{E}(\boldsymbol{r}_1) = \frac{i\omega}{\epsilon_0 c^2}\int \overleftrightarrow{\boldsymbol{g}}(\boldsymbol{r}_1,\boldsymbol{r}')\cdot\boldsymbol{j}(\boldsymbol{r}')d\boldsymbol{r}' \rightarrow \frac{\omega^2}{\epsilon_0 c^2}\overleftrightarrow{\boldsymbol{g}}(\boldsymbol{r}_1,\boldsymbol{r}_2)\cdot\boldsymbol{j}(\boldsymbol{r}_2)$$

The electric field by a point electric dipole at the position $\vec{\boldsymbol{r}}_2$ is thus :

"intermediate" field

$$\boldsymbol{E}(\boldsymbol{r}_1) = \frac{e^{ikr_{12}}}{4\pi r_{12}^3\varepsilon_b\epsilon_0}\{k^2 r_{12}^2[\boldsymbol{p}-\boldsymbol{u}_{12}(\boldsymbol{u}_{12}\cdot\boldsymbol{p})] + (1-ikr_{12})[3\boldsymbol{u}_{12}(\boldsymbol{u}_{12}\cdot\boldsymbol{p})-\boldsymbol{p}]\} - \frac{\boldsymbol{p}}{3\varepsilon_b\epsilon_0}\delta^{(3)}(\boldsymbol{r}_{12})$$

"far" field

"near" field

$$\boldsymbol{r}_{12} \equiv \boldsymbol{r}_1 - \boldsymbol{r}_2$$
$$r_{12} \equiv |\boldsymbol{r}_1 - \boldsymbol{r}_2|$$
$$\vec{\boldsymbol{u}}_{12} \equiv \frac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{r_{12}}$$

Electric dipole emission:

Alternative expression:

“intermediate” field

$$\boldsymbol{E}(\boldsymbol{r}_1) = \frac{e^{ikr_{12}}}{4\pi r_{12}^3 \varepsilon_b \epsilon_0}\{k^2 r_{12}^2[(\boldsymbol{u}_{12} \times \boldsymbol{p}) \times \boldsymbol{u}_{12}] + (1 - ikr_{12})[3\boldsymbol{u}_{12}(\boldsymbol{u}_{12} \cdot \boldsymbol{p}) - \boldsymbol{p}]\} - \frac{\boldsymbol{p}}{3\varepsilon_b \epsilon_0}\delta^{(3)}(\boldsymbol{r}_{12})$$

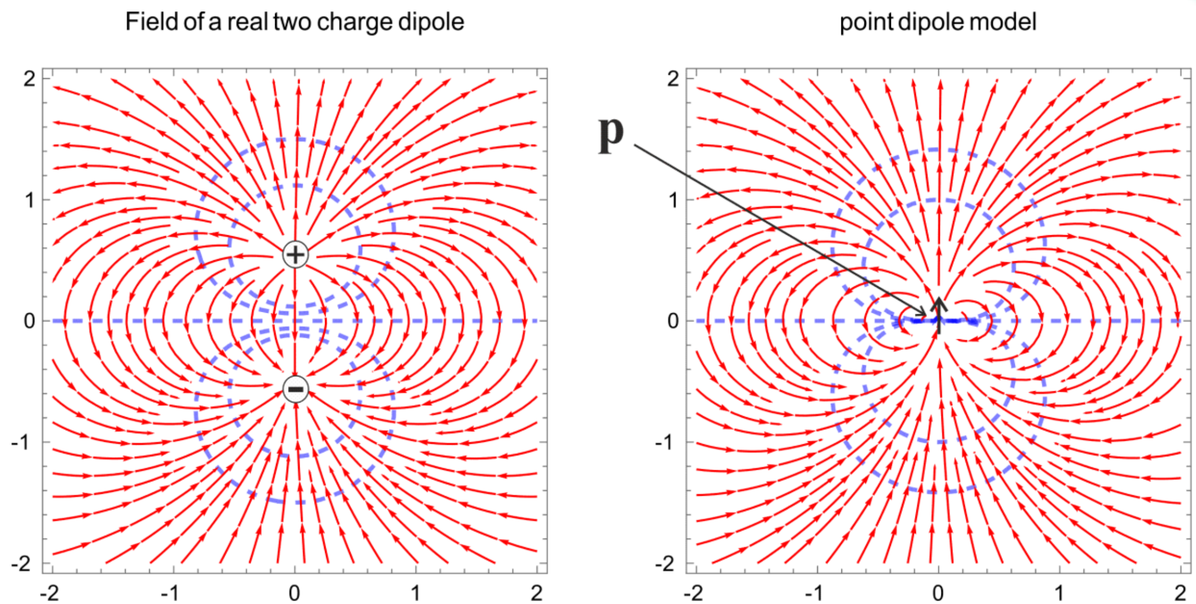
“far” field

“near” field

Such expressions for electric dipole emission are very important but notoriously difficult to derive and have a rich physical interpretation !

Origin of the delta function contribution:

$$\boldsymbol{E}(\boldsymbol{r}_1) = \frac{e^{ikr_{12}}}{4\pi r_{12}^3 \varepsilon_b \epsilon_0}\{k^2 r_{12}^2[(\boldsymbol{u}_{12}\times\boldsymbol{p})\times\boldsymbol{u}_{12}] + (1 - ikr_{12})[3\boldsymbol{u}_{12}(\boldsymbol{u}_{12}\cdot\boldsymbol{p}) - \boldsymbol{p}]\} - \frac{\boldsymbol{p}}{3\varepsilon_b\epsilon_0}\delta^{(3)}(\boldsymbol{r}_{12})$$

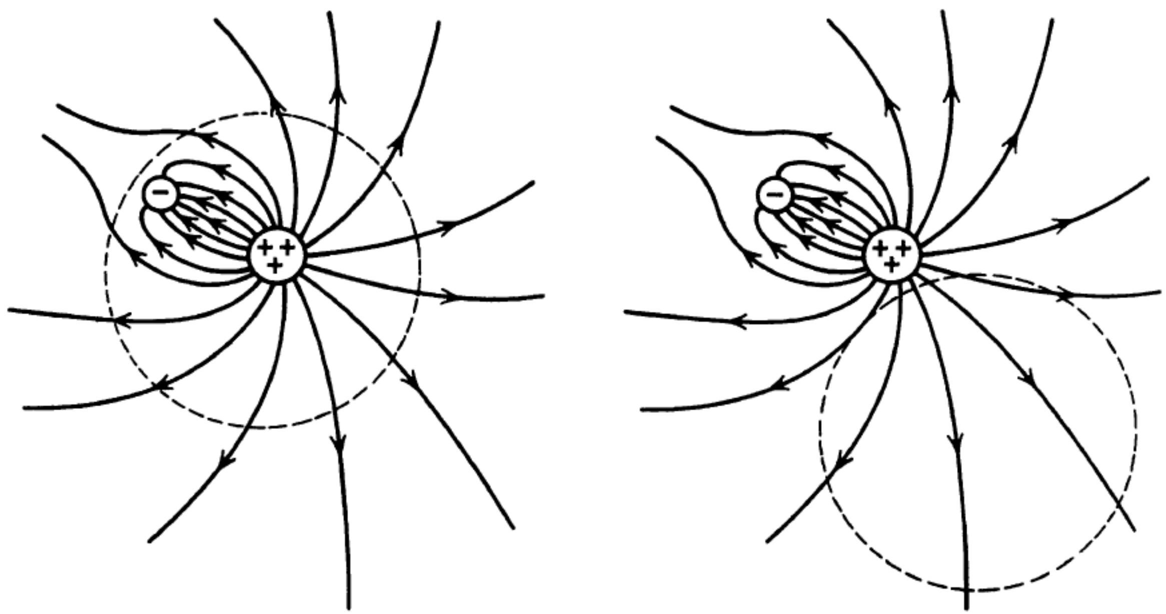


Distribution theory:

The 3D delta function accounts for the strong fields between charges in the “point-like” limit

Other references : $\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{3\mathbf{n}(\mathbf{p}\cdot\mathbf{n}) - \mathbf{p}}{|\mathbf{x}-\mathbf{x}_0|^3} - \frac{4\pi}{3}\mathbf{p}\delta(\mathbf{x}-\mathbf{x}_0)\right]$

Jackson (3rd edition: pp 148-150) explains that the delta function contribution to the dipole field corrects for the fact that in the electrostatic limit, the average field inside a spherical (charge-free) region is equal to the field at the center of the sphere, but if the sphere surrounds an electric dipole then the average field is the value at the center of the sphere **plus** $\frac{\boldsymbol{p}}{3\varepsilon_b\epsilon_0}$. The symmetry of the field around a "point" dipole would naively suggest that the volume average in a sphere centered on the dipole vanishes (but this reasoning ignores the strong fields between the positive and negative charges). The 3D delta function $\delta^{(3)}(\boldsymbol{r}_{12})$ is consequently a necessary addition to the dipole field.

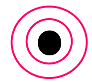


Part 3:

Light-matter interactions – Radiative emission

Classical radiation by a point-source

(monopolar `scalar' light source)

O  $s(O,t) = s_c \,\mathrm{Re}\left[e^{-i\omega_0 t}\right]$

$$g(\boldsymbol{r}_1,\boldsymbol{r}_2) = \frac{e^{ik_b|\boldsymbol{r}_1-\boldsymbol{r}_2|}}{4\pi|\boldsymbol{r}_1-\boldsymbol{r}_2|} \equiv \frac{e^{ik_b r_{12}}}{4\pi r_{12}}$$

$$\phi(\vec{\boldsymbol{r}},t) \propto g(\boldsymbol{r},O\,)s(t) \qquad \frac{d\boldsymbol{P}_s}{dS} \propto \phi\boldsymbol{\nabla}\phi \longrightarrow c_b k_b \boldsymbol{u}_r \phi^2$$

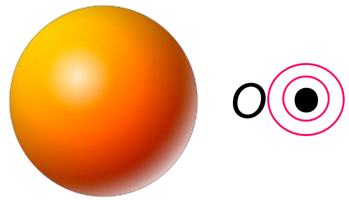
$$P_{\mathrm{r},0} \equiv \langle P_{\mathrm{r}}\rangle_0 \propto \frac{1}{T}\int_0^T dt \oiint d\boldsymbol{S}\cdot\frac{d\boldsymbol{P}_s}{dS} \longrightarrow \frac{c_b}{2}\oiint d\boldsymbol{S}\cdot k_b\vec{\boldsymbol{u}}_r|\phi|^2 \longrightarrow \frac{c_b k_b |s_c|^2}{4\pi}$$

$$P_{\mathrm{e},0} \equiv \langle P_{\mathrm{e}}\rangle_0 \propto \frac{1}{T}\int_0^T dt\,\phi(\vec{\boldsymbol{0}},t)s(\vec{\boldsymbol{0}},t) \longrightarrow \frac{c_b}{2}\mathrm{Im}\left[g(\vec{\boldsymbol{0}},\vec{\boldsymbol{0}})\right]|s_c|^2 \longrightarrow \frac{c_b \mathrm{Re}(k_b)|s_c|^2}{4\pi}$$

In a lossless host medium : $P_{r,0} = P_{e,0} \equiv P_0$

Modifying the decay rate by the environment

(`weak' coupling model - scalar theory)



$$s(O,t) = \mathrm{Re}\left[s_c e^{-i\omega_0 t}\right]$$

$$G(\boldsymbol{r}_1,\boldsymbol{r}_2) = g(\boldsymbol{r}_1,\boldsymbol{r}_2) + G_s(\boldsymbol{r}_1,\boldsymbol{r}_2) \qquad g(\boldsymbol{r}_1,\boldsymbol{r}_2) = \frac{e^{ikr_{12}}}{4\pi r_{12}}$$

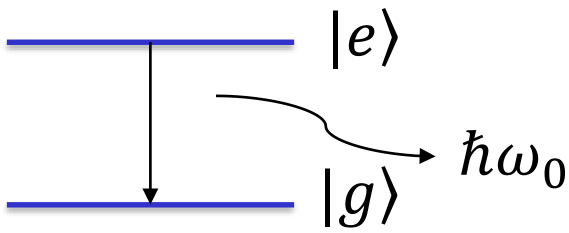
$$P_{\mathrm{e},0} \equiv \langle P_\mathrm{e}\rangle_0 \propto \frac{c_b}{2}\mathrm{Im}[g(\mathbf{0},\mathbf{0})]|s_c|^2 \longrightarrow \frac{c_b \mathrm{Re}(k_b)|s_c|^2}{4\pi} = P_0$$

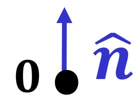
$$P_\mathrm{e}^* \equiv \langle P_\mathrm{e}\rangle \propto \frac{c_b}{2}\mathrm{Im}[g(\mathbf{0},\mathbf{0}) + G_s(\mathbf{0},\mathbf{0})]|s_c|^2$$

$$\frac{P_e^*}{P_{\mathrm{e},0}} = 1 + \frac{2\pi}{\mathrm{Re}(k_b)}\mathrm{Im}[G_s(\mathbf{0},\mathbf{0})]$$

Unlike $g(\vec{\mathbf{0}},\vec{\mathbf{0}})$, – the scattering feedback $G_s(\vec{\mathbf{0}},\vec{\mathbf{0}})$ is regular (no divergence)

Modeling quantum emission as a classical "point-like" antenna far from resonance





$$\boldsymbol{j}_s(\vec{\boldsymbol{r}}, t) = \mathrm{Re}\left[-i\omega_0 \boldsymbol{p} e^{-i\omega_0 t}\right]$$

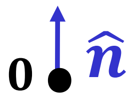
$$\vec{\boldsymbol{E}}(\vec{\boldsymbol{r}}, t) = \frac{i\omega_0^2}{\epsilon_0 c^2} \overleftrightarrow{\mathbf{g}}(\boldsymbol{r}, \mathbf{0}) \cdot \boldsymbol{j}_s(t)$$

$$\langle P_{\boldsymbol{e}} \rangle = \frac{1}{T} \int_0^T d\boldsymbol{x}\, \vec{\boldsymbol{E}}(\boldsymbol{r}, t) \cdot \boldsymbol{j}_s(\boldsymbol{r}, t)$$

$$= \frac{\omega_0^3}{2\epsilon_0 c^2} |\boldsymbol{p}|^2 \mathrm{Im}\left[\hat{\boldsymbol{n}}^* \cdot \overleftrightarrow{\mathbf{g}}(\mathbf{0}, \mathbf{0}; \omega) \cdot \hat{\boldsymbol{n}}\right] = \gamma_{0,\mathrm{cl}}$$

$$\tau \propto \frac{1}{\gamma_{0,\mathrm{cl}}}$$

Radiation by “point-like” dipole in free space
(Weak coupling - high impedance mismatch)



$$\vec{j}_s(\boldsymbol{r},t) = \frac{d}{dt}\boldsymbol{p}_e(\boldsymbol{0},t) = \mathrm{Re}\left[-i\omega_0 p_e \hat{\boldsymbol{n}} e^{-i\omega_0 t}\right]$$

$$\overleftrightarrow{\boldsymbol{g}}_0(\boldsymbol{r}) = \frac{c^2 e^{ikr}}{4\pi\omega^2\epsilon_0\varepsilon_b r^3}\mathrm{P.V.}\left\{(-1+ik_b r)\left(\overleftrightarrow{\mathbf{I}} - 3\boldsymbol{u}_r\boldsymbol{u}_r\right) + k_b^2 r^2\left(\overleftrightarrow{\mathbf{I}} - \boldsymbol{u}_r\boldsymbol{u}_r\right)\right\} - \frac{c^2\delta(\boldsymbol{r})}{3\omega^2\epsilon_0\varepsilon_b}\overleftrightarrow{\mathbf{I}}$$

$$\boldsymbol{E}(\vec{\boldsymbol{r}},t) = \frac{\omega_0^3}{2\epsilon_0 c^2}\overleftrightarrow{\boldsymbol{g}}(\boldsymbol{r},\boldsymbol{0})\cdot\boldsymbol{p}_e(t)$$

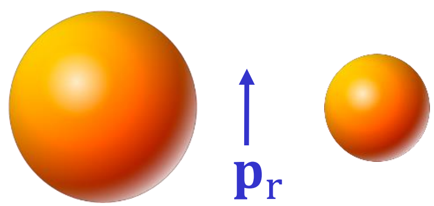
$$P_{\mathrm{e},0} \equiv \langle P_e\rangle_0 = \frac{1}{T}\int_0^T dt\,\boldsymbol{E}(\boldsymbol{0}\,,t)\cdot\vec{j}_s(\vec{\boldsymbol{0}},t) = \frac{\omega_0^3}{2\epsilon_0 c^2}|\vec{\boldsymbol{p}}_e|^2\,\mathrm{Im}\left[\hat{\boldsymbol{n}}^*\cdot\overleftrightarrow{\boldsymbol{g}}(\vec{\boldsymbol{0}},\boldsymbol{0};\omega)\cdot\hat{\boldsymbol{n}}\right]$$

$$= \frac{\omega_0^3\mathrm{Re}(k_b)}{12\pi\epsilon_0 c^2}|\vec{\boldsymbol{p}}_e|^2 \equiv P_0 \qquad \tau \propto \frac{1}{P_{\mathrm{e},0}}$$

"Nano" antenna

Sub-wavelength "nano" elements will modify the :

Decay rate, emission frequency, and directivity of quantum emissions

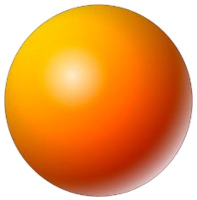


$$\boldsymbol{E}(\boldsymbol{r}) = \frac{\omega_0^3}{\epsilon_0 c^2} \overleftrightarrow{\mathbf{G}}(\boldsymbol{r}, \mathbf{0}) \cdot \boldsymbol{p}$$

$$\langle P_{\boldsymbol{e}} \rangle = \frac{\omega_0^3}{2\epsilon_0 c^2} \mathrm{Im}\left[\boldsymbol{p}^* \cdot \overleftrightarrow{\mathbf{G}}(\mathbf{0}, \mathbf{0}; \boldsymbol{\omega}) \cdot \boldsymbol{p}\right] \propto \gamma^* \propto \mathrm{Re}\left[Z_{\vec{0}}\right]$$

Decay rate : $\tau \propto \dfrac{1}{\gamma^*}$

Modifying the emission rate with a nano-environment



$\uparrow \hat{\boldsymbol{n}}$

$$\boldsymbol{j}_s(\boldsymbol{r},t) = \frac{d}{dt}\boldsymbol{p}_e(\boldsymbol{0},t) = \mathrm{Re}\left[-i\omega_0 p_e \hat{\boldsymbol{n}} e^{-i\omega_0 t}\right]$$

$$\boldsymbol{E}(\boldsymbol{r},t) = \frac{k_b^2}{\epsilon_0 \varepsilon_b c^2} \overleftrightarrow{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{0}) \cdot \boldsymbol{p}_e(t)$$

$$\overleftrightarrow{\boldsymbol{G}}(\boldsymbol{r}_1,\boldsymbol{r}_2) = \overleftrightarrow{\boldsymbol{g}}(\boldsymbol{r}_1,\boldsymbol{r}_2) + \overleftrightarrow{\boldsymbol{G}}_s(\boldsymbol{r}_1,\boldsymbol{r}_2)$$

$$P_e^* \equiv \langle P_\mathrm{e} \rangle = \frac{1}{T}\int_0^T dt\, \vec{\boldsymbol{E}}(\boldsymbol{0},t) \cdot \boldsymbol{j}_s(\boldsymbol{0},t) = \frac{\omega_0^3}{2\epsilon_0 c^2} |\boldsymbol{p}_e|^2 \mathrm{Im}\left[\hat{\boldsymbol{n}}^* \cdot \left(\overleftrightarrow{\boldsymbol{g}}(\boldsymbol{0},\boldsymbol{0}) + \overleftrightarrow{\boldsymbol{G}}_s(\boldsymbol{0},\boldsymbol{0})\right) \cdot \hat{\boldsymbol{n}}\right]$$

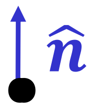
$$P_{\mathrm{e},0} \equiv \langle P_\mathrm{e} \rangle_0 = \frac{\omega_0^3 \mathrm{Re}(k_b)}{12\pi\epsilon_0 c^2} |\vec{\boldsymbol{p}}_e|^2$$

$$\boxed{\frac{P_e^*}{P_{\mathrm{e},0}} = 1 + 6\pi \frac{\mathrm{Im}\left[\hat{\boldsymbol{n}}^* \cdot \overleftrightarrow{\boldsymbol{G}}_s(\boldsymbol{0},\boldsymbol{0}) \cdot \hat{\boldsymbol{n}}\right]}{\mathrm{Re}(k_b)}}$$

$$\tau \propto \frac{1}{P_e^*}$$

Unlike $\overleftrightarrow{\boldsymbol{g}}(\vec{\boldsymbol{0}},\vec{\boldsymbol{0}})$, the scattering feedback, $\overleftrightarrow{\boldsymbol{G}}_s(\vec{\boldsymbol{0}},\vec{\boldsymbol{0}})$ is regular (no divergence)

Radiative emission – “point-like” antenna model
(high impedance mismatch)



$$\boldsymbol{E}(\boldsymbol{r},t) = \frac{k_b^2}{\epsilon_0 \varepsilon_b c^2} \overleftrightarrow{\boldsymbol{g}}(\boldsymbol{r},\boldsymbol{0}) \cdot \boldsymbol{p}_e(t)$$

$$\lim_{r\to\infty} \boldsymbol{H}(\boldsymbol{r},t) = \frac{\epsilon_0 c^2 k_b}{\omega_0} \boldsymbol{u}_r \times \boldsymbol{E}(\boldsymbol{r},t)$$

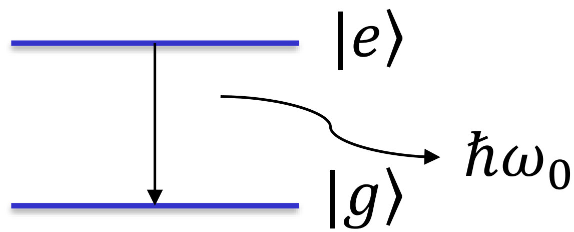
$$P_{r,0} \equiv \langle P_r \rangle_0 = \lim_{r\to\infty} \frac{1}{T}\int_0^T dt \oiint \boldsymbol{E}(\boldsymbol{r},t) \times \boldsymbol{H}(\boldsymbol{r},t) \cdot d\boldsymbol{S} \xrightarrow{\text{f.f. free-space}} \frac{\omega_0 k_b^3}{12\pi\epsilon_0\varepsilon_b} |\boldsymbol{p}_e|^2$$

$$\frac{P_r^*}{P_{r,0}} = \lim_{r\to\infty} \frac{1}{P_{r,0}} \oiint \mathrm{Re}[\boldsymbol{E}^*(\boldsymbol{r},\omega_0) \times \boldsymbol{H}(\boldsymbol{r},\omega_0)] \cdot d\boldsymbol{S}$$

lossless host medium : $P_{r,0} = P_{e,0} \equiv P_0$

Lossy local environment : $P_r^* < P_e^*$

Quantum emission



$$\tau = \frac{1}{\gamma_{r,\mathrm{q}}}$$

"Classical" result

$$P_0 = \frac{\omega_0^4 \sqrt{\varepsilon_b}}{12\pi\epsilon_0 c^3} |\boldsymbol{p}|^2$$

"Quantum" – Fermi golden-rule result

$$\gamma_{r,\mathrm{q}} = \frac{4}{\hbar\omega_0} P_0 = \frac{\omega_0^3 \sqrt{\varepsilon_b}}{3\pi\epsilon_0 \hbar c^3} |\langle g|\boldsymbol{p}|e\rangle|^2$$

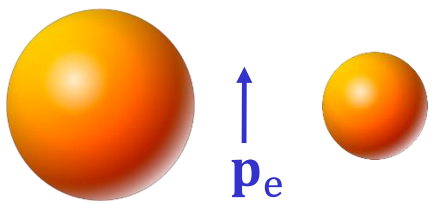
$$= \frac{\pi\omega_0 \sqrt{\varepsilon_b}}{3\epsilon_0 \hbar} \frac{\omega_0^2}{\pi^2 c^3} |\langle g|\boldsymbol{p}|e\rangle|^2$$

Properties of quantum emission modified by “nano” antennas

Weak coupling approximation $\gamma^* \ll \omega_0$

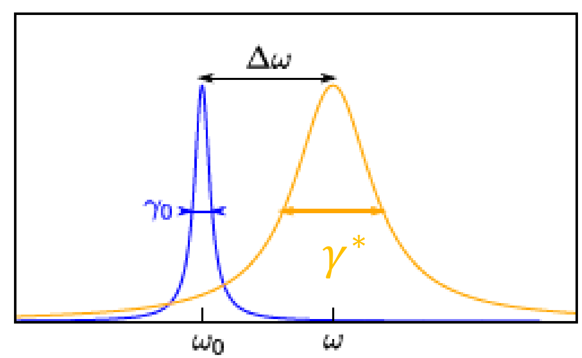
Sub-wavelength “nano” elements will modify the :

Decay rate, emission frequency, and directivity of quantum emissions



Quantum efficiency : $\eta \equiv \frac{\gamma_r}{\gamma_r + \gamma_{n.r.}} \equiv \frac{\gamma_r}{\gamma_{0,t}}$

lifetime modification : $$\frac{\tau_0}{\tau^*} = \frac{\gamma^*}{\gamma_{0,t}} = \frac{P_e^*}{P_{e,0}} = 1 + \eta 6\pi \frac{\mathrm{Im}\left[\hat{\boldsymbol{n}}^* \cdot \overleftrightarrow{\boldsymbol{G}}_s(\mathbf{0},\mathbf{0}) \cdot \hat{\boldsymbol{n}}\right]}{\mathrm{Re}(k_b)}$$



“Lamb” shift : $$\frac{\Delta\omega}{\gamma_{0,t}} = \eta 3\pi \frac{\mathrm{Re}\left[\hat{\boldsymbol{n}}^* \cdot \overleftrightarrow{\boldsymbol{G}}_s(\mathbf{0},\mathbf{0}) \cdot \hat{\boldsymbol{n}}\right]}{\mathrm{Re}(k_b)}$$

Radiation modification : $$\frac{\tau_0}{\tau_r^*} = \frac{\gamma_r^*}{\gamma_{0,t}} = 1 + \lim_{r\to\infty} \frac{\eta}{P_0} \oiint \mathrm{Re}\left[\boldsymbol{E}^*(\boldsymbol{r},\omega_0) \times \boldsymbol{H}(\boldsymbol{r},\omega_0)\right] \cdot d\boldsymbol{S}$$