Speckle intensity statistics for chromatic scattering media under partially polarized illumination

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Abstract: Intensity contrast in a fully developed speckle pattern resulting from the elastic scattering of a partially polarized light from a strongly scattering medium is theoretically and numerically studied. Simple expressions are derived when the illumination bandwidth is much smaller or larger than the chromatic length of the scattering medium.

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References and links

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1. Introduction

Light undergoes random fluctuations [1]. In time, these fluctuations are ultimately due to the quantum nature of optical sources. However, in the familiar context of a fully developed speckle

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pattern [2], the statistics of the instantaneous intensity I_t is perfectly known for thermal light [3]. At first-order, it depends on two parameters, namely the time-averaged intensity $I = \overline{I_t}$ and the degree of polarization of light P. The probability density function of the instantaneous intensity varies from a Gamma distribution of shape parameter 1 (negative exponential distribution) for totally polarized light to a Gamma distribution of shape parameter 2 for totally unpolarized light. As a consequence, the intensity contrast over time ranges from 1 to $1/\sqrt{2}$. Note that in this paper, we make use of the following definition for the probability density function of the Gamma distribution of shape parameter $k > 0$ and scale parameter $\theta > 0$, with Γ denoting the Gamma function [4]:

$$G[k,\theta](x \geq 0) = \frac{x^{k-1}e^{-x/\theta}}{\Gamma(k)\theta^k}. \tag{1}$$

However, in terms of applications, the speckle is first of all known as a source of noise in all imaging techniques with coherent light. In most optical setups, instantaneous intensity cannot be measured. Optical devices commonly used for imaging are sensitive to the time-averaged intensity. In a speckle pattern, this intensity I varies in space, with the speckle grain as characteristic dimension [5]. Techniques for reducing that speckle noise have been published, in optics [6] as well as in synthetic aperture radar and medical ultrasound imaging. In this paper, we are interested in the statistical analysis of the spatial fluctuations of the time-averaged intensity of light. Here, thermal light and laser light will not be distingued. This study is focused on fully developed speckle patterns. For this, we consider the scattering of a partially-polarized incident light by a linear, complex, disordered and strongly scattering medium [7], such as nonabsorbing bulks, or metallic rough surfaces with extreme slope statistics.

In that sense, the present work is the continuation of [8]. In this important paper, measurements of the spatial histogram of the time-averaged intensity of unpolarized light scattered from bulk Lambertian samples were analysed. An intensity contrast as low as 0.47 was measured, and the intensity histogram was identified as a Gamma distribution of shape parameter 4. This high value has to be connected to the presence of cross-scattering coefficients, as shown in [9]. We now theoretically extend these results in two ways.

First in section 2 we derive analytical expressions for the spatial histogram of the time-averaged scattered intensity when the incident light is partially polarized. Notice that these histograms were previously calculated for extreme states of polarization and for different kinds of inhomogeneities [10, 11], but not for partial polarization. Moreover the assumption of a gamma law was used (with an intermediate, non-integer shape parameter) to analyze the histograms versus the sample microstucture, while we here show that such assumption is not theoretically justified in the general case. Hence our analytical formula allows a better investigation of the speckle patterns.

Second in section 3 another key improvement is presented since we show how the spatial histograms are modified versus the chromatic properties of the scattering medium [7]. To our knowledge these effects were not considered until now, while the scattering coefficients may vary within the bandwidth of the incident beam. These effects, that are neglected in section 2, lead to a speckle reduction that is quantified with respect to the spectral characteristics of both the incident light and the scattering medium.

2. Incident state of polarization

The light state of polarization can be represented in several equivalent ways: Stokes vector, Poincaré sphere, and so on. Here, we will benefitly make use of the (time-averaged) intensity I, the polarization ratio β and the cross-correlation coefficient μ. These three parameters are

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defined as time averages over the two components E_S and E_P of the complex electric field.

$$I = \overline{|E_S|^2} + \overline{|E_P|^2} \qquad \beta = \frac{\overline{|E_P|^2}}{\overline{|E_S|^2}} \qquad \mu = \frac{\overline{E_S^* E_P}}{\sqrt{\overline{|E_S|^2}\ \overline{|E_P|^2}}} \tag{2}$$

The overbar denotes time-averaging. The intensity as defined in Eq. (2) is the total intensity, which can be directly measured, that is without any polarizing optical element. The degree of polarization (DoP) directly writes:

$$P = \sqrt{1 - \frac{4\beta}{(1+\beta)^2}(1-|\mu|^2)}. \tag{3}$$

The parameters for the incident light are tagged with a null underscript : I_0, β_0, μ_0 and finally P_0.

The scattered electric field in any far field direction is related to the incident electric field through the scattering matrix Σ

$$\begin{bmatrix} E_S \\ E_P \end{bmatrix} = \begin{bmatrix} \Sigma_{SS} & \Sigma_{SP} \\ \Sigma_{PS} & \Sigma_{PP} \end{bmatrix} \begin{bmatrix} E_S^0 \\ E_P^0 \end{bmatrix} \tag{4}$$

so that the scattered intensity I writes:

$$I = \frac{|\Sigma_{SS}|^2 + |\Sigma_{PS}|^2 + \beta_0(|\Sigma_{SP}|^2 + |\Sigma_{PP}|^2) + 2\sqrt{\beta_0}\Re e\{\mu_0(\Sigma_{SS}^*\Sigma_{SP} + \Sigma_{PS}^*\Sigma_{PP})\}}{1+\beta_0} I_0. \tag{5}$$

Here, we take another opportunity to emphasize the importance of the cross-scattering coefficients Σ_{SP} and Σ_{SP} [9]. In cases where those coefficients are negligible in front of the co-polarization coefficients Σ_{SS} , the total scattered intensity no longer depends on the incident light cross-correlation coefficient μ_0, and the statistics derivation gets trivial.

Following the fully developed speckle model, the scattering coefficients Σ_{XY} are four independent realizations of a circular complex Gaussian stochastic field. The standard deviation of the real part or imaginary part of this homogeneous stochastic field is denoted σ. Quantities such as the differential reflectivity and the scattering cross-section are proportional to σ^2. The statistics of the scattered intensity I can be analytically derived in some particular cases.

The simplest case is an example of totally polarized incident light, where $\mu_0 = e^{i\delta_0}$. Here, the scattered intensity writes as

$$I = \frac{|\Sigma_{SS} + \sqrt{\beta_0}e^{i\delta_0}\Sigma_{SP}|^2 + |\Sigma_{PS} + \sqrt{\beta_0}e^{i\delta_0}\Sigma_{PP}|^2}{1+\beta_0} I_0 \tag{6}$$

and its probability density function turns to a Gamma distribution of shape parameter 2 and scale parameter $2\sigma^2 I_0$ whatever the value of β_0.

$$p_I = G[2, 2\sigma^2 I_0] \tag{7}$$

Then comes the case where the incident cross-correlation coefficient vanishes: $\mu_0 = 0$. Note that all values of the incident DoP remain here reachable, with $P_0 = |1-\beta_0|/(1+\beta_0)$. In this case, we derived the scattered intensity distribution with classical statistics tools. We report here this distribution, that writes as a sum of Gamma distributions:

$$\begin{aligned} p_I \quad &= \quad \frac{1}{(1-\beta_0)^2}\left\{\beta_0^2 G\left[2, \frac{2\sigma^2\beta_0 I_0}{1+\beta_0}\right] + G\left[2, \frac{2\sigma^2 I_0}{1+\beta_0}\right]\right\} \\ &+ \quad \frac{2\beta_0}{(1-\beta_0)(1-\beta_0)^2}\left\{\beta_0 G\left[1, \frac{2\sigma^2\beta_0 I_0}{1+\beta_0}\right] - G\left[1, \frac{2\sigma^2 I_0}{1+\beta_0}\right]\right\}. \end{aligned} \tag{8}$$

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This rather complex distribution shows the following limits for particular values of the incident polarization ratio β_0. We find immediately that $p_I \to G[2,2\sigma^2 I_0]$ for $\beta_0 \to 0$ or $\beta_0 \to \infty$, thus leading to the same distribution as for the previous case. A more careful derivation implying derivatives of the Gamma distribution is required around $\beta_0 \to 1$, leading to the limit $p_I \to G[4,\sigma^2 I_0]$.

These two analytical cases illustrate the fact that the scattered intensity distribution in the strongly scattering regime is somehow bounded between the Gamma distributions of shape parameters 2 and 4, depending on the incident polarization. However, it should not be concluded that the scattered intensity is always Gamma-distributed, and this is a relevant point. This clearly appears in the expression of Eq. (8), where the probability density function is a weighted sum of Gamma distributions of shape parameters 1 and 2.

For a cross-correlation coefficient of arbitrary modulus $0 < |\mu_0| < 1$, the scattered intensity writes as the sum

$$I = |\gamma_1|^2 + |\gamma_2|^2 + |\gamma_3|^2 + |\gamma_4|^2 \tag{9}$$

of the squared modulus of four circular complex stochastic fields.

$$\gamma_1 = \left(\cos\theta\ \Sigma_{SS} + \cos\varphi\sqrt{\beta_0}e^{i\delta_0}\ \Sigma_{SP}\right)\sqrt{I_0/(1+\beta_0)} \tag{10}$$

$$\gamma_2 = \left(\sin\theta\ \Sigma_{SS} - \sin\varphi\sqrt{\beta_0}e^{i\delta_0}\ \Sigma_{SP}\right)\sqrt{I_0/(1+\beta_0)} \tag{11}$$

$$\gamma_3 = \left(\cos\theta\ \Sigma_{PS} + \cos\varphi\sqrt{\beta_0}e^{i\delta_0}\ \Sigma_{PP}\right)\sqrt{I_0/(1+\beta_0)} \tag{12}$$

$$\gamma_4 = \left(\sin\theta\ \Sigma_{PS} - \sin\varphi\sqrt{\beta_0}e^{i\delta_0}\ \Sigma_{PP}\right)\sqrt{I_0/(1+\beta_0)} \tag{13}$$

Here, δ_0 denotes the argument of the illumination cross-correlation coefficient μ_0, while angles θ and φ satisfy :

$$\cos(\theta+\varphi) = |\mu_0| \qquad \sin 2\theta = \beta_0 \sin 2\varphi. \tag{14}$$

However, the four fields of Eqs. (10) to (13) are uncorrelated, but they cannot be assumed independent. As a consequence, the full scattered intensity statistics cannot be derived in the general case. Still, the mean intensity $\langle I\rangle$, with brackets denoting space averaging, and associated standard deviation σ_I can easily be expressed

$$\langle I\rangle = 4\sigma^2 I_0 \qquad \sigma_I = \langle I\rangle\sqrt{\frac{1}{2} - \frac{\beta_0}{(1+\beta_0)^2}(1-|\mu_0|^2)} \tag{15}$$

in order to predict the speckle intensity contrast $\sigma_I/\langle I\rangle$ for partially polarized incident light. It is clearly related to the incident DoP (see Eq. (3)):

$$\frac{\sigma_I}{\langle I\rangle} = \sqrt{\frac{1+P_0^2}{4}}. \tag{16}$$

The scattered intensity contrast over space ranges from $1/\sqrt{2}$ to $1/2$. The expression of this intensity contrast in space appears quite similar to the classical relation in time: $\sqrt{\frac{1+P^2}{2}}$ [2, Eq. (3-88)]. Note however that this last expression connects the instantaneous intensity contrast in time and the DoP of a given light, while Eq. (16) relies the incident DoP to the time-averaged scattered intensity contrast in space. Equation (16) is also tied to a strongly scattering medium assumption.

To conclude this section, we present on Fig. 1 some simulations of the speckle intensity distribution for partially polarized illumination. When no analytical expression can be derived,

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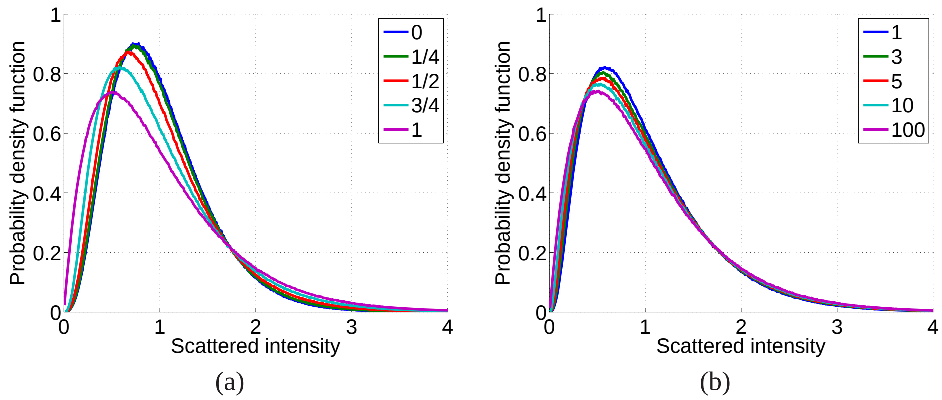

Fig. 1. Probability density function of the normalized scattered intensity $I/\langle I\rangle$ for (a) $\beta_0 = 1$ and five real values of μ_0 between 0 and 1, (b) $\mu_0 = 3/4$ and five values of β_0 between 1 and 100.

it is quite easy with a normally distributed pseudorandom generator [12] to compute and plot the sought distribution. For the class of strongly scattering media we study here, we find that the argument of the incident light cross-correlation coefficient has no influence on the intensity distribution. As a consequence, for this simulation and the following, this parameter μ_0 is set to real values. Moreover, only polarization ratios larger than or equal to unity are considered for the incident light, since a value β_0 and its inverse $1/\beta_0$ lead to the very same intensity distribution.

3. Chromatic scattering medium

The scattering matrix of Eq. (4) characterizes the response of the scatterer to an electromagnetic plane wave. For a given scattering medium, this matrix depends on incident and scattering directions, and on frequency. This last dependency comes from the fact that the ratio of distances inside the medium to electromagnetic wavelength varies with frequency. Therefore, the interference state of every point in the medium is changed with wavelength. For examples, a small particle in the Rayleigh regime has its scattering matrix varying as the second power of frequency. Dispersion also contributes to the chromatic feature of scattering media, in particular around resonant frequencies of constituting materials.

In this section, we consider the spectral variations of the scattering matrix coefficients in order to modelize configurations with intensity contrasts lower than one half. We stress the point that such configurations are commonly observed experimentally, even when very coherent and narrow bandwidth laser sources are used.

The time-averages are now expressed as integrals over temporal frequency. For the $X = S,P$ and $Y = S,P$ components of the electric field,

$$\overline{E_X^* E_Y} = \int \tilde{E}_X(\nu)^* \tilde{E}_Y(\nu)\, d\nu \tag{17}$$

$$= \int \left(\Sigma_{XS}(\nu)^* \Sigma_{YS}(\nu) |\tilde{E}_S^0(\nu)|^2 + \Sigma_{XS}(\nu)^* \Sigma_{YP}(\nu) \tilde{E}_S^0(\nu)^* \tilde{E}_P^0(\nu)\right) d\nu$$

$$+ \int \left(\Sigma_{XP}(\nu)^* \Sigma_{YS}(\nu) \tilde{E}_P^0(\nu)^* \tilde{E}_S^0(\nu) + \Sigma_{XP}(\nu)^* \Sigma_{YP}(\nu) |\tilde{E}_P^0(\nu)|^2\right) d\nu. \tag{18}$$

To go further, we make use of the cross-spectral purity assumption [13, 14] for the incident

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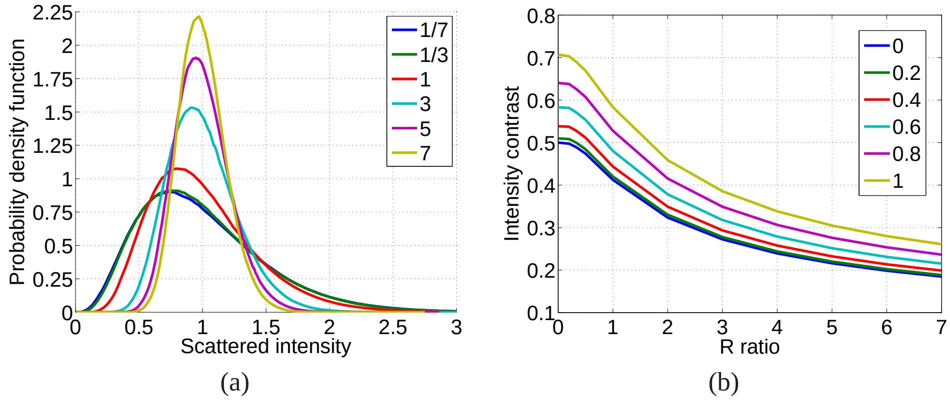

Fig. 2. (a) Scattered intensity probability density functions for unpolarized incident light and six values of the ratio R between $1/7$ and 7. (b) Scattered intensity contrast $\sigma_I/\langle I\rangle$ against the R ratio for six real values of the incident DoP P_0 between 0 and 1.

light. This leads to the following simplifications in the frequency domain,

$$\tilde{E}^0_X(\nu)^*\tilde{E}^0_Y(\nu) = \overline{E^{0*}_X E^0_Y}\, g(\nu) \qquad \int g(\nu)d\nu = 1 \tag{19}$$

with $g(\nu)$ a scalar function of frequency, centered over frequency ν_c and of bandwidth $\Delta\nu_i$. In numerical simulations, $g(\nu)$ will be a Gaussian function, under the assumption that the bandwidth is sufficiently small compared to the central frequency in order to neglect $g(\nu)$ for negative frequencies.

With simplifications of Eq. (19), we can now express the time-averages of the scattered field as the linear combination

$$\overline{E^*_X E_Y} = \frac{C_{XSYS} + \sqrt{\beta_0}\mu_0 C_{XSYP} + \sqrt{\beta_0}\mu_0^* C_{XPYS} + \beta_0 C_{XPYP}}{1+\beta_0} I_0 \tag{20}$$

with coefficients

$$C_{ABCD} = \int \Sigma_{AB}(\nu)^*\Sigma_{CD}(\nu)g(\nu)d\nu \tag{21}$$

where $A,B,C,D = S,P$. For computer simulations, we consider that the four scattering coefficients show independent spectral variations with the same chromatic length $\Delta\nu_\Sigma$ [7].

This approach is very comparable to the so-called ensemble average used to simulate Müller matrix for random media [14, 15]. As a matter of fact, the coefficients of Eq. (21) appear to be convex sum. The main difference here is that the scattering medium does not have to be assumed random or time-evolving. The electromagnetic wave theory perfectly enables the scattering matrix of a deterministic and static medium to be frequency-varying. With our approach, the ensemble average is driven by a parameter of obvious physical meaning, the ratio R of the incident light bandwidth $\Delta\nu_i$ over the scattering medium spectral length $\Delta\nu_\Sigma$.

$$R = \frac{\Delta\nu_i}{\Delta\nu_\Sigma} \tag{22}$$

With such a theory, the scattered intensity distribution can be numerically simulated for any incident state of polarization. Some examples are given in Fig. 2(a), where complete distributions are plotted for unpolarized light and various values of the ratio R. In a more exhaustive

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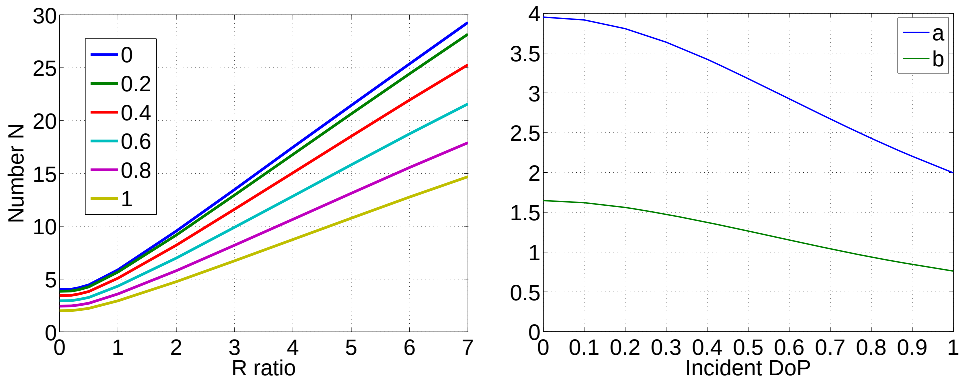

Fig. 3. (a) Number $N = (\sigma_I/\langle I\rangle)^{-2}$ against the R ratio for six real values of the incident DoP P_0 between 0 and 1. (b) Parameters a and b of the linear regression of $N = f(R)$ for large values of the ratio R against the incident DoP P_0.

way, the variation of the scattered intensity contrast $\sigma_I/\langle I\rangle$ with respect to the R ratio is presented in Fig. 2(b). Several values of the incident light DoP are tested in order to cover the whole range of partial polarization. Note that for a given incident light DoP, different values of the parameters μ_0 or β_0 lead to the same intensity distribution. This feature of the presently studied scattering regime has been systematically checked. For values of R much smaller than 1, that are for an illumination bandwidth much tighter than the scattering medium chromatic length, an intensity contrast between $1/2$ and $1/\sqrt{2}$ is retrieved, in agreement with the results of section 2. With the ratio R increasing, the scattered intensity constrast decreases for all incident light polarization states, as expected. Our simulations are given for the ratio R varying from 0 to 7, so that the intensity contrast is never smaller than $1/5$ in Fig. 2(b). However, the study is complete, as will appear shortly.

Low intensity contrasts are commonly read in a heuristic way as the sum of N independent speckle intensity patterns with equal exponential distributions [2]. In such cases, the contrast writes $\sigma_I/\langle I\rangle = 1/\sqrt{N}$, so that we define this number as $N = (\sigma_I/\langle I\rangle)^{-2}$. As can be observed in Fig. 3(a), the number N shows a linear variation with respect to R as soon as this ratio is larger than 2, say. It means that the contrast asymptotically meets the expression:

$$\frac{\sigma_I}{\langle I\rangle} \sim \frac{1}{\sqrt{aR+b}}. \tag{23}$$

This linear behaviour appears whatever the incident state of polarization (Fig. 3(a)), even if with different parametrization. The two asymptotic parameters a and b are plotted in Fig. 3(b) against the incident light DoP. They show a monotonous, decreasing variation with DoP. Note that the slope $a(P_0)$ varies from 4 to 2, which precisely correspond to the Gamma distribution shape parameter of the scattered intensity when the incident DoP is respectively 0 and 1.

Finally, with the relation $N = aR + b$ for large values of N, this heuristic number tends to be physically grounded by our theory, and its dependency to the incident light parameters, the DoP P_0 and bandwidth $\Delta\nu_i$, and to the scatterer parameter $\Delta\nu_\Sigma$, is now unveiled.

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4. Conclusion

In a speckle pattern, the time-averaged intensity presents spatial fluctuations. In presence of a strongly scattering medium, the scattering process for quasi-monochromatic light can be rather easily described from a statistical point of view. However, the scattered intensity statistics cannot be analytically derived for an incident light with arbitrary state of polarization. This distribution is yet provided when the cross-correlation coefficient of the two components of the incident light electric field is null or unity in modulus. It writes as a Gamma distribution of shape parameter 2 or 4, or as a linear combination of Gamma distributions of shape parameters 1 and 2. The spatial contrast of the total scattered intensity is a commonly measured parameters. We showed with Eq. (16) that this contrast does not depend on the exact state of polarization of the incident light, but only on its degree of polarization.

When the illumination bandwith and the scattering medium chromatic length come to be considered, and compared through their ratio R, the scattered intensity contrast is numerically estimated, and its variations with R and the incident light polarization degree are studied. Asymptotical expressions of this contrast for small (Eq. (16)) and large (larger than 2, Eq. (23)) values of R are of practical interest. As a matter of fact, in many applications and imaging techniques, intensity contrast due to speckle is a source of noise. The results of this paper can be used to determine, for given illumination bandwidth and polarization state, the minimum chromatic length of the scattering medium in order to ensure that the intensity contrast is kept below a fixed threshold. Finally, bearing informations on the scattering medium, the speckle can also be the signal of interest. In that sense, our future works will be turned toward connecting scattered intensity and scattered degree of polarization. We will also investigate other scattering regimes, in order to fill the gap between strong scattering as defined in this paper, and perturbative scattering.

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