NUMERICAL INVESTIGATION OF PHOTONIC CRYSTAL FIBERS BY MEANS OF SPECTRAL AND MULTIPOLE METHODS.

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Abstract

In this paper, we investigate the propagation of electromagnetic waves along microstructured optical fibres (MOF) with a crystal cladding of finite extension in the transverse plane. In these so-called Photonic Crystal Fibres (PCF), many interesting and unusual phenomena may be observed. For instance, it appears that, for some range of frequencies, a defect in the crystal cladding gives rise to localized waves in the low index core (i.e. in the defect). It is worth noting that the methods reducing the system of equations to the longitudinal electric or magnetic component cannot tackle these problems. We first achieve this study with a variational numerical method i.e. a Finite Element Formulation with the electric field as the variable, where edge elements are used for discretizing the transverse field and nodal elements for the longitudinal field [4]. Finally, we compare our finite element method - which leads to a generalized eigenvalue problem- with an other method co-developed by some of us [16, 11], which is based on the determination of poles of the scattering operator (multipole expansion method).

1. INTRODUCTION

In the last decade, some advances have been made in a deeper understanding of the propagating properties of a new class of electromagnetic waveguides. These structures [10], whose propagating properties depend on the geometry of a crystal lattice constituted of dielectric rods, have been called Photonic Crystal Fibers (PCF). PCF appear as being periodic what induces the so called photonic band gaps by analogy with the semiconductors: mode propagation is forbidden for certain frequencies in the crystal cladding and therefore light is confined in a central low index defect (PBG effect).

To study such structures, one approach is the supercell method [3] which is commonly employed for PCF (it assumes pseudo-periodic conditions on a cell which includes the waveguide cross-section): the main drawback comes from the assumption of periodicity in the transverse plane. In this paper we compare two alternative methods that do not rely on artificial periodicity. The first one is the Finite Element Method (FEM) which allows the modelling of complex geometries (e.g. PCF filled with rod cross-sections of different shapes) and takes into account the influence of the outer boundary of the guide [4]. Another approach which can take into account the finite size of the fibre cross-section, is the multipole method. In this method, the fields are expanded in Fourier-Bessel series, and a generalized scattering problem is solved[16, 11]. How can we justify the choice of these two methods? The advantages of having two methods to solve the same problem is that comparing the results gives some confidence in their accuracy if they match and also that the methods may have complementary advantages so that they can rescue each other if one of them meets difficulty on a particular problem. Therefore, it seems natural to choose the two methods as different as possible. It is quite the case between the multipole and the finite element methods. As a common feature, the two methods share the very classical fact that the unknown fields or functions are approximated using a linear combination of given basis functions. Nevertheless, one can say that the approach of the multipole method is global while the finite element method is local. On the one hand the basis functions in the multipole method are solutions of the Helmholtz equations to be solved (excepted at the source point but it is not included in the domain where the solution is considered) as they are the field produced by a multipolar radiating source. Their linear combinations are themselves solutions and the problem is to pick the good one by choosing the right coefficients (i.e. the values of the equivalent multipolar charges). This is performed thanks to the numerical expression of local conditions: continuity conditions of the field at the interfaces. On the other hand the basis functions in the finite element methods are simple (e.g. piecewise linear) functions with bounded support (of small size). In the case of Whitney elements [2], if basis functions do not match the Helmholtz equation, they locally respect the right continuity conditions at the interfaces e.g. tangential continuity for electric and magnetic fields represented by edge elements (that is the geometrical nature of the fields that determines the kind of element to be used) and so do their linear combinations. In this case, the Helmholtz equation itself (in weak variational form) is used to compute the numerical coefficients of the approximation. The purpose of this paper is to describe in the same framework the two methods, and to show that they give consistent results.

2. COMMON HYPOTHESIS AND NOTATIONS

We consider an inhomogoneous waveguide of constant section Ω invariant along the z axis, whose permittivity profile ε is a piecewise constant function and with constant permeability $\mu = \mu_0$ (non magnetic). We are looking for electromagnetic fields solutions of the vector Maxwell system. Choosing a time dependance in $e^{-i\omega t}$ with ω denoting the angular frequency, related to the free space wavenumber by $\omega = k_0 c$, and taking into account the invariance of the guide along its z axis, we define time-harmonic electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$. Each mode is characterized by the value of γ, its propagation constant along the z axis, and the dependence of the fields ($\mathbf{V} = \mathbf{E}$ or $\mathbf{V} = \mathbf{H}$) is given by:

$$\mathbf{V}(\vec{r}, t) = \widetilde{\mathbf{V}} e^{i(\gamma z - \omega t)} \tag{1.1}$$

In each constant permittivity region of the waveguide, the fields ($\widetilde{\mathbf{V}}$) satisfy an Helmholtz equation.

3. THE FINITE ELEMENT MODELING

In this formulation, $\widetilde{\mathbf{V}}$ depends on the cartesian coordinates x and y in equation (1.1). For the sake of simplicity, we consider that the boundary of the waveguide $\partial\Omega$ is infinite conducting, thereby avoiding to deal with an open problem (note that a judicious choice of coordinate transform allows this modeling [8]). We choose an electric field formulation, because the tangential trace of $\mathbf{E}$ is null, contrary to that of $\mathbf{H}$ on $\partial\Omega$. We achieve numerical computations with the help of finite elements. It involves both a transverse field in the section of the guide and a longitudinal field along its axis. The section of the guide is meshed with triangles and Whitney finite elements are used i.e. edge elements for the transverse field and nodal elements for the longitudinal field. The solution of the above problem is then given by the minimum of the following residual

$$\mathcal{R}(\gamma; \mathbf{E}, \mathbf{E}') = \int_\Omega \mathrm{curl}_\gamma\, \mathbf{E} \cdot \overline{\mathrm{curl}_\gamma\, \mathbf{E}'}\, dxdy - \mu_0 \omega^2 \int_\Omega \varepsilon \mathbf{E} \cdot \overline{\mathbf{E}'}\, dxdy \tag{1.2}$$

where $\mathrm{curl}_\gamma\, \mathbf{E}(x,y) = \mathrm{curl}\Big(\mathbf{E}(x,y)e^{i\gamma z}\Big)e^{-i\gamma z}$. It must be noticed that the penalty term involving the divergence of the field is not introduced in the discrete formulation because the use of Whitney elements guarantees the nullity of the divergence in weak sense [2, 8, 7]. The operator associated to this variational problem has a compact resolvent [8, 7] and its spectrum is thereofore a discrete set of eigenvalues belonging to $[\frac{\gamma^2}{\varepsilon}; +\infty[$ which gives us a numerical criterion to eliminate invalid modes. The GetDP software [4] has been used to set up the finite element problem. The Lanczos algorithm solves the generalized eigenvalue problem [7]. Since we study propagating modes in a low-index default, they are associated to complex frequencies: these so-called leaky modes are similar to classical propagating modes, except they are attenuated as energy is lost to the crystal cladding. These modes initially decrease in amplitude away from the PCF, but eventually increase at large distances and diverge infinitely far from the PCF. Because real PCF have a finite crystal cladding (contrary to the assumption of [3]), all the propagating modes induced by PBG effects are leaky. It is therefore important that the losses associated with leaky modes be considered when calculating the modal properties of PCF. For this, we embed our PCF in a metallic jacket: all the modes remain therefore bounded (associated with real eigenvalues) and we could take into account the losses of leaky modes via the computation of the current as the integral of the tangential magnetic field ($\mathbf{J}_S = n \wedge \mathrm{curl}_\gamma\, \mathbf{E}$) on the metallic cavity. Let us emphasize that $\mathbf{J}_S$ is an unknown of the problem which does not appear in the electric formulation ($n \wedge \mathbf{E} = 0$). The Dirichlet homogeneous boundary conditions of the electric field do not actually involve wave reflection, since the propagation occurs along the z-axis. Such a condition for the magnetic field is not *adequate*, since the tangential magnetic field has an unknown value on the boundary i.e. the surface currents. For the sake of completness, the reader can refer to [7] for a review on properties of PCF with a high central index (where a magnetic field formulation was used).

4. THE MULTIPOLE METHOD

Our method is inspired by the one developed in our laboratory in the framework of two-dimensional photonic crystal theory [5]. It is generally named as the multipole method. The first step of the method is to elaborate a theory of scattering of an infinitely long MOF in conical (off-plane) mounting. In this formulation, $\widetilde{\mathbf{V}}$ depends on polar coordinates r and θ in equation (1.1). This theory is based on the use of the three elements: the Fourier-Bessel series describing the electromagnetic fields, the translation properties of Fourier-Bessel functions (also called Graf's theorem) which appear in the series, and the scattering matrices of the holes pierced in silica. In addition, the notion of generalized scattering matrix is employed for the external boundary of the silica fibre. The second step of the method is to find homogeneous solutions of this scattering problem. Noting the appropriate matrix $\mathcal{M}$, we must solve:

$$\mathcal{M}\tilde{\mathcal{B}} = 0. \tag{1.3}$$

A non-zero vector $\tilde{\mathcal{B}}$ verifying this equation represents a non-zero field verifying the Helmholtz equation and the boundary conditions: the field associated with this vector is a mode of the structure. The matrix $\mathcal{M}$ is a function of the geometry, the wavelength and the effective index $n_{eff} = \gamma/k_0$. Finding the modes for a given structure and wavelength thus leads to the search of zeros of the complex function $\det(\mathcal{M})$ of the complex variable n_{eff}. The geometrical losses $\mathcal{L}$ in $dB.m^{-1}$ associated with the modes are directly obtained from the imaginary part of n_{eff}:

$$\mathcal{L} = \frac{20}{\ln(10)} \frac{2\pi}{\lambda} \Im(n_{\mathtt{eff}}) \times 10^6 \tag{1.4}$$

where λ is in μm.

For circular inclusions, the boundary conditions are implemented analytically, so that in this case the only approximations are the truncation of the Fourier-Bessel series as well as the fundamental hypothesis of the invariance of the fiber along its axis. For non circular inclusions, differential or integral methods can be used to compute the scattering matrices associated with the inclusions [14]. If the inclusions overlap, our method is no longer appropriate. Symmetry properties of the fibers are accurately satisfied: for an usual MOF with a rotational symmetry of order 6 (for example a triangular lattice of circular holes), the fundamental mode is two-fold degenerate as expected by Mc Isaac's theory[15]. It is worth noting that these symmetry properties can be used to reduce significantly the size of the matrix $\mathcal{M}$, and therefore the computational time. Our formulation also allows a self-consistency check of the computed results through a comparison between two different expansions of the fields. These expansions only match perfectly for untruncated series, and so their numerical difference can be used as an indicator of truncations errors and of the quality of the matrix null vector location.

With the MOF geometry and the wavelength as inputs, the method gives the modes of the MOF as an output. Material dispersion can thus be included in a natural way in the MOF geometry, using for instance Sellmeier expansions[6, 1]. The dispersion parameter D is computed through the usual formula from the real part of the effective index $\Re(n_{eff})$ [1]:

$$D = -\frac{\lambda}{c}\frac{\partial^2 \Re(n_{\tt eff})}{\partial \lambda^2} \tag{1.5}$$

The reader interested in applying this theory can refer to papers [16, 11], and results on dispersion engineering can be found in [12, 13].

5. NUMERICAL COMPARISONS

On one hand, we use a metallic jacket approach with Finite Elements i.e. the PCF is included in an outer metallic waveguide [8]. On the other hand, we involve a lossy jacket combined with our multipole method, and therefore deal with complex frequencies [11]. In both cases, the PCF consists of six air-holes ($n = 1$) of radius $2.5\mu m$ spaced in hexagonal arrangement by a pitch $\Lambda = 6.75\mu m$ in a matrix of silica ($n = 1.45$). The Getdp software [4] is used to set up the Finite Element Modeling (figure 1, left) and to retrieve the real part of the leaky mode's frequency calculated with the multipole method approach for the six holes honeycomb structure (figure 1, right). More precisely, the wavelength $\lambda = 1.45\mu m$ is set up in the multipole method and one obtains $n_{eff} = \gamma/k_0 = \gamma\lambda/2\pi = 1.445395345 + i\ 3.15\ 10^{-8}$. Then, for a given propagating constant $\gamma_{FEM} = 1.445395 k_0 = 6.263232\mu m^{-1}$, we catch a leaky mode (figure 1, left) whose eigenvalue $\omega_{FEM} = 2\pi c/\lambda$ corresponds to a wavelength $\lambda_{FEM} = 1.449996\mu m$ that matches the initial value. We illustrate on figure 2 the impact of a perturbation on the geometry of the PCF on this leaky mode.

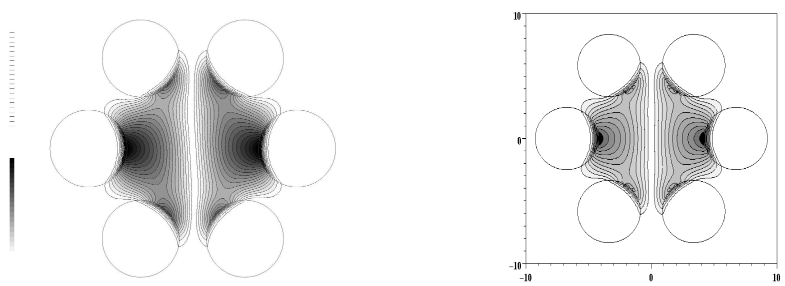

Figure 1 Longitudinal component of the electric field E in a 6-air-hole MOF (radii = $2.5\mu m$) of pitch $\Lambda = 6.75\mu m$ for a normalized propagating constant $\gamma\Lambda = 42.276817$ (left: finite element modeling, right: multipole method).

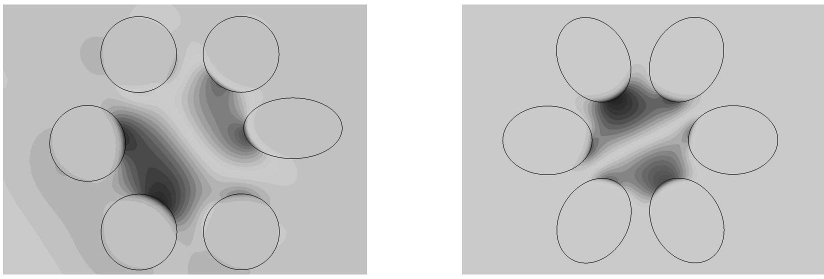

Figure 2 Behaviour of the eigenfield of figure 1 when perturbing the circular cross-section of the air holes (and thereby also breaking the hexagonal symmetry).

6. CONCLUSION

We have numerically investigated the propagating modes in a 6 air-hole honeycomb fiber with a finite element modeling and a multipole method, both of which proved to be fast and efficient algorithms. A possible extension of this work is to explore further the numerical comparisons on larger linear systems i.e. for a large bunch of fibers in various geometrical configurations (the multipole method taking into account the possible symmetries) and for higher index contrasts (the edge-element approach being well adapted).

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