Coupling of Global and Local Quantities in Various Finite Element Formulations and its Application to Electrostatics, Magnetostatics and Magnetoodynamics

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Abstract - A method for defining global quantities related to fluxes and circulations is proposed in the frame of the finite element method. The definition is in perfect accordance with the discretized weak formulations of the problems. It therefore enables a natural coupling between local and global quantities in various formulations, while keeping a symmetrical matrix for the system, and then is open to the coupling of physical problems. Applications are given for electrostatics, magnetostatics and magnetoynamics.

Index terms – Finite element method, coupled problems, electrostatics, magnetostatics, magnetoodynamics, electric circuit.

I. INTRODUCTION

A method is described for the treatment of global quantities in the frame of the finite element method. Thanks to the explicit characterization of the basis function spaces used for the approximation of local quantities — scalar or vector fields —, the test functions implied in the discrete weak formulations naturally appear. This method also has the advantage of directly characterizing global quantities associated with the studied problems, without the need of any intermediate computation method, e.g. differentiation. Consequently, it enables their direct consideration, while keeping a symmetrical matrix for the system. The method can also be applied to the computation of global quantities at the post-processing stage.

The method is applied to electrostatics, magnetostatics and magnetoodynamics, to naturally define electric charges and floating potentials, magnetic fluxes and magnetomotive forces, and currents and voltages. Illustrative examples are given to point out the main characteristics of the method.

II. POSITION OF THE WHOLE PROBLEM

Two kinds of Green formulae are generally involved in the establishment of weak formulations of partial differential equations. These are grad-div and curl-curl formulae applied to a domain $\Omega$ of boundary $\Gamma$, i.e.

$$\nabla \cdot (\nabla \phi) = \int_{\Omega} \rho \, d\Omega \quad \text{and} \quad \nabla \times \nabla \times \phi = \int_{\partial \Omega} \nabla \phi \cdot n \, d\Gamma,$$

where $\nabla = (\partial / \partial x, \partial / \partial y)$, $\nabla \cdot \phi = \int_{\Omega} \nabla \cdot \phi \, d\Omega$ and $\nabla \times \phi = \int_{\partial \Omega} \nabla \phi \times n \, d\Gamma$.

A suitable treatment of the surface integral terms in (1) and (2) can be made to naturally define global quantities of flux and circulation types in a weak sense, as it will appear in the following. Those weak global quantities will be associated with the strongly defined ones.

III. GLOBAL QUANTITIES OF FLUX TYPE

A. Discrete scalar potentials and their formulations

A discrete characterization is first developed for a scalar field $\phi \in H^2(\Omega)$ in grad-div formula (1). Such a field is generally discretized in a nodal finite element space, defined on a mesh of $\Omega$ and denoted $S^0(\Omega)$ [2], [4] — associated finite elements can be of various geometries and degrees, in 2D and 3D —, i.e.

$$\phi = \sum_{n \in N} \phi_n \, s_n, \quad \phi \in S^0(\Omega),$$

where $N$ is the set of nodes of $\Omega$, $s_n$ is the nodal basis function associated with node $n$ and $\phi_n$ is the value of $\phi$ at node $n$.

Functions $s_n$, $n \in N$, form a basis for the nodal finite element space without constraint, e.g. boundary conditions or fixed global quantities. In case constraints exist, functions $s_n$ in (3) are no longer linearly independent, i.e. relations exist between some of their coefficients. The direct expression of these constraints reveals the basis functions to consider, i.e. which can serve as test functions in the finite element method.

In particular, such a potential $\phi$ can be involved in a scalar potential electrostatic formulation ($v = -\nabla \phi$, $\nabla \cdot v = 0$), i.e.

$$\nabla \cdot (\nabla \phi) v = \int_{\Omega} \rho \, d\Omega = 0, \quad \nabla \cdot v = 0, \quad \nabla \times \nabla \times v = 0,$$

where $\nabla \cdot d_s$ is a constraint on the electric flux density associated with nonfixed potential boundaries $\Gamma_d$ of domain $\Omega$, e.g. on floating potential boundaries $\Gamma_d$, $\Gamma_d \in \partial \Omega$. $F_0(\Omega)$ is a function space of scalar fields defined in $\Omega$, with essential boundary conditions when subscripted.

A potential of form (3), noted $\phi$, can also be involved in a scalar potential magnetostrictive formulation ($h = -\nabla \phi$, $b = \mu h$, $\nabla \cdot b = 0$ in weak form using (1)), i.e.

$$\nabla \cdot (\nabla \phi) b = \int_{\Omega} \rho \, d\Omega = 0, \quad \nabla \cdot b = 0, \quad \nabla \times \nabla \times b = 0,$$

where the surface integral on $\Gamma_d$ contains constraints on the magnetic flux density. Parts of $\Gamma_d$ ($\Gamma_d \in \partial \Omega$) can be boundaries of perfectly magnetic region ($\mu \rightarrow \infty$) and surfaces crossed by given magnetic flux, for which $\phi$ is a floating potential. A source field can be added in case source currents are given.

B. Floating scalar potential constraints

In order to explicitly define constraints of floating potential type, the nodes of $\Omega$ are classified in complementary subsets: $\Omega_N$, which is the set of nodes inside $\Omega$, and $\Omega_N^0$, $\forall \Omega \in \Omega$, which are sets of nodes of parts $\Gamma_d$ (Fig. 1).
Floating potentials being constant on each $\Gamma_p$ expression (3) can then be decomposed as

$$v = \sum_{n \in N_v} v_n s_n + \sum_{f \in C_f} v_f s_f, \quad v \in S^0(\Omega),$$

with

$$s_f = \sum_{n \in N_f} s_n, \quad v_f \in C_f,$$

where $s_n, \forall n \in N_v$, and $s_f, \forall f \in C_f$, are basis functions for the constrained potential. Each function $s^f$ is associated with the group of nodes — a global geometrical entity, while nodes $n \in N_v$ are elementary entities — of boundary $\Gamma_f$ (Fig. 1).

![Fig. 1. Nodes and groups of nodes associated with the characterization of a scalar potential with floating values (6).](image)

C. Discrete global quantities of flux type

The discretization of weak formulations (4) and (5), by using test functions appearing in (6), gives regular symmetrical systems of equations. Test functions $s_n, \forall n \in N_v$, are classically treated, while test functions $s_f, \forall f \in C_f$, need attention.

The surface integral term in (4) has, for test function $s^f$, equal to one on $\Gamma_f$ [2], [4], a contribution equal to $\langle n \cdot d_{n_p} \rangle_{\Gamma_p} \rightarrow \Gamma_f$ and thus to the flux of $d_n$ through surface $f \in C_f$. This contribution is nothing else than the opposite of the total electric charge $Q_f$ of $\Omega_f(\partial \Omega_f = \Gamma_f)$. Then, to the surface integral on $\Gamma_f$ in (4) can be substituted the value of the total charge $Q_f$, i.e.,

$$\langle n \cdot d_{n_p} \rangle_{\Gamma_p} = -\langle n \cdot d_{n_p} \rangle_{\Gamma_f} = -Q_f,$$

for $v = s^f \in S^0(\Omega), f \in C_f.$

For a fixed potential region, (8) can also be used for an efficient computation of charges at the post-processing stage (this has similarities with the method in [1]). For that, it is sufficient to define also basis functions of type $s^f$ (7) for such regions.

Consequently, the computation of the electric charge can be performed in average by the volume integral in (4) in a transition layer (support of $s^f$; Fig. 1), i.e.,

$$Q_f = -\langle -\varepsilon \text{ grad } v, \text{ grad } s^f \rangle_{\Omega_f}, \quad f \in C_f.$$

This approach is in perfect accordance with the discretized weak formulation of the problem, i.e., with (4), and thus with an only weakly satisfied conservation of flux. In particular, this method enables an efficient computation of capacitivities in electrostatics thanks to a coherent definition of both electric potential and charge.

The computation of the charge based on the explicit surface integration of $-\varepsilon \cdot (\text{ grad } v)$ would be affected by the choice of the integration surface. There would be generally no reason for the so computed charge to be equal to the charge given by the volume integral in the transition layer, even if the surface is the actual boundary of the conductor.

Similarly, each test function $s^f$ gives a contribution to the surface integral term in (5) equal to the magnetic flux $\psi_f$ across $\Gamma_f$, i.e., $\langle n \cdot b_n \rangle_{\Gamma_f} = 0$ (equal to zero for a closed surface). Then, to the surface integral on $\Gamma_f$ can be substituted the value of the flux $\psi_f$, i.e.,

$$\langle n \cdot b_n \rangle_{\Gamma_f} = -\psi_f,$$

for $\psi = s^f \in S^0(\Omega), f \in C_f.$

A direct natural coupling between magnetic flux, magnetomotive force and local magnetic field is then obtained. This enables the modeling of magnetostatic circuits with definition of associate lumped parameters, i.e., reluctances.

Applications of floating potentials are really numerous. In this way, an electrokinetic formulation using a scalar potential can benefit from floating potentials to define global quantities such as electric voltages and currents. A thermal formulation can also use a floating temperature on boundaries of perfectly thermal conductive materials, naturally associated with heat flux as thermal source.

IV. GLOBAL QUANTITIES OF CIRCULATION TYPE

A. Discrete curl-conform vector fields and their formulations

Another discrete characterization is next developed for a curl-conform vector field $v = \mathbf{H}(\Omega)$ in curl−curl formula (2). Such a field can be discretized in an edge finite element space [2], [4], defined on a mesh of $\Omega$ and denoted $S^1(\Omega)$, i.e.,

$$v = \sum_{e \in E} v_e s_e, \quad v \in S^1(\Omega),$$

where $E$ is the set of edges of $\Omega$, $s_e$ is the edge basis function for edge $e$ and $v_e$ is the circulation of $v$ along edge $e$.

In particular, such a vector field $v$ can be involved in a vector potential magnetostatic formulation ($e = \mathbf{a}$ with gauge condition; $b = \text{curl } a, \mathbf{h} = \mu \mathbf{B}$, curl $e = 0$ in weak form using (2)) [2], i.e.,

$$\varepsilon \mu^{-1} \text{curl } a, \text{curl } a \rangle_{\Omega_f} = 0, \quad \forall a \in F^1(\Omega),$$

$$\langle n \times \mathbf{h}, N \rangle_{\Gamma_f} = 0,$$

where $\mathbf{h}$ is a constraint associated with certain boundaries $\Gamma_f$ of domain $\Omega$. $F^1(\Omega)$ is a function space of curl defined vector fields defined in $\Omega$, with essential boundary conditions when subscripted.

Vector field $v$ can also be involved in a $\mathbf{b}$−$\mathbf{p}$ magnetodynamic formulation ($v = \mathbf{H}$ with constraints of curl-free type for the definition of scalar potential $\phi$; curl $e = 0$, $\mathbf{h}$ in weak form using (2)) [2], i.e.,

$$\partial_t (\mu \mathbf{h}, \mathbf{n})_{\Omega_f} + (\sigma^{-1} \mathbf{curl} \mathbf{h}, \mathbf{curl h})_{\Omega_f} + <n \times e, n \times \mathbf{h}>_{\Gamma_f} = 0,$$

$$\forall \mathbf{n} \in F^1(\Gamma),$$

where $n \times e$ is a constraint associated with certain boundaries $\Gamma_f$ of domain $\Omega$. $\mathbf{h}$ contains the conducting regions.

B. Constraints of circulation type

Again, in case constraints exist in $S^1(\Omega)$, there are relations between some coefficients of functions $s_e$ in (11). When these constraints are relative to circulations of $v$ — or flux of curl $v$ — in a subdomain $\Omega_0^S$ of $\Omega$ (with $\Omega = \Omega_x \cup \Omega_0^S$), their direct expression can be obtained using a general characterization similar to the one given in [5], i.e.,

$$v = \sum_{k \in E^k} v_k s_k + \sum_{n \in N_v} v_n s_n + \sum_{f \in C_f} c_f s_f,$$

where $E^k$ is the set of inner edges of $\Omega_x$, $N_v^S$ is the set of nodes inside $\Omega_0^S$ and on its boundary $\partial \Omega_0^S$, and $C$ is a set of well defined cuts which make $\Omega_0^S$ simply connected (Fig. 2).
This characterization explicitly defines a coupling between field $v$ (in $\Omega e$; given by all three sums) and a scalar potential $\phi$ ($v = -\nabla \phi$ in $\Omega e^C$); the gradient of $\phi$ is given by the second and third sums; the third sum enables multivalued potentials to be considered. Actually, potential $\phi$ in $\Omega e^C$ is decomposed in continuous and discontinuous parts, of which the gradients are respectively given by the second and third sums in (14). Note that such a characterization enables function $v_n$, and thus the associated scalar potential, to be fully continuous in a multiply connected domain, the discontinuity being taken into account by functions $q_i$. Coefficients $\phi_n$ in (14) can usually be set to zero in case $v$ is a vector potential.

A very important point is that coefficients $I_i$ represent circulations of $v$ along well defined paths, which is a basis for definition of global quantities.

C. Discrete global quantities of circulation type

The discretization of weak formulation (12) is performed by using test functions defined in (14). Test functions $\psi_k$, $\forall k \in E_0$, and $v_n$, $\forall n \in N_0^C$, are classically treated.

The surface term in (12) has, for a test function $\psi_k$, by its definition [5], a contribution $\nabla \times h_q \cdot \hat{t}$, being a discretization of the magnetomotive force, i.e. a coherent average of the circulation of $h$ along flux tubes.

In order to demonstrate this property, first note vector function $q_i$ on $\partial \Omega e$, associated with a cut $C_q$, is the gradient of a discontinuous scalar field $q_i$ [5]. Such a field $q_i$ varies from 1 on one side of border $\partial \Omega$ of cut $C_q$ (on $\partial \Omega e^+$) to 0 on the other side (on $\partial \Omega e^-$) (Fig. 2). The way $q_i$ varies is not a matter at all. What is important is that its value is either 1 or 0, depending on the side of cut $C_q$. Particularly, the circulation of $q_i$ along any path surrounding $\Omega e$, i.e. going from any point on $\partial \Omega e^-$ to its counterpart on $\partial \Omega e^+$, is then equal to 1.

The duality between weak formulations (5) and (12) is well known. This duality also exists as far as global quantities, fluxes and circulations, are concerned. For (5), the magnetomotive force is directly and strongly expressed with $\phi$, while the flux is given in a weak sense. On the other hand, for (12), the flux is directly expressed with $a$, while the magnetomotive force is given in a weak sense. But the way each weak quantity is defined seems to be the best one, coherent with each weak formulation. Here again, for circulations, on the choice of the integration path would depend the results if a post-processing numerical integration, of $\mu^{-1} \text{curl} a \cdot \text{d}l$, is done.

The same method can be applied to $h \cdot \phi$ magnetodynamic formulation (13) to substitute electromotive forces to surface integrals on $\Gamma e$. This enables to take source voltages into account in a weak sense, while currents are directly expressed by coefficients $I_i$ in (14).

For that, consider the inductor in Fig. 3 with a source of electromotive force located between two sections, being two electrodes very near to each other. The studied domain $\Omega e$ contains all the regions but the very thin one $\Omega e_i$, separating the electrodes; the boundary of $\Omega e_i$ is then a part of surface $\Gamma e$.

The electric field $e \cdot \hat{e}_i$ in $\Omega e_i$ can be considered as being known and its circulation along any path from one electrode to the other in $\Omega e_i$ is actually the applied voltage $V_i$. Consequently, the surface integral in (13) for test function $q_i$ is equal to this voltage, i.e.

$$<n \times e_i \cdot \hat{e}_i >_{\Gamma e} = \oint_{\partial \Gamma e} q_i e_i \cdot d\ell = \oint_{\partial \Gamma e} e_i \cdot d\ell = V_i,$$

(16)

because only the part $\gamma$ of oriented contour $\partial \Gamma e$ in contact with $\partial \Omega e^+$ gives a nonzero contribution (Fig. 3).

Other magnetodynamic formulations — e.g., $\alpha \cdot \alpha$, $a \cdot j$ and $\alpha \cdot j$ formulations [6] — can also be expressed in a similar way to (13) and can benefit from the same rigor of definition of global quantities, such as currents, voltages, magnetic fluxes and electromotive forces.

Discrete spaces defined by (6) and (14) concern both test and shape functions and therefore, matrices of systems of equations are symmetrical. This is a key feature of the method which enables efficient system solving.

V. APPLICATIONS

An efficient practical implementation of the method can be performed. It consists of defining, in addition to elementary entities — nodes, edges, facets and volumes —, some global entities such as groups of nodes, groups of edges, ..., to explicitly build the characterization of function spaces [7].

Applications of computation of global quantities are very numerous. Some examples are presented here for basic 2D or
3D electrostatic, magnetostatic and magnetodynamic problems to illustrate and validate the method.

A. Electrostatic problem

A capacitor with mixed dielectric is studied with formulation (4) in 2D (Fig. 4). Constraints are relative to electric charges on both plates, while electric potentials of plates are unknown. The computed capacitance is 188.24 pF. The same geometry with condition n·d=0 on the borders of the dielectrics, which neglects the fringing at edges, gives a lower capacitance 168.65 pF, also equal to the theoretical value. Note that in case constraints are set potentials on plates, charges being unknown, the computed capacitance is the same as the previously computed one, because discrete equations are the same, the only difference being relative to the constraints.

![Fig. 4. Mixed dielectric capacitor (plate width 0.1 m, plate separation 0.01 m, ε₁=1, ε₂=20).](image)

B. Magnetostatic problem

A part of a magnetic circuit (Fig. 5; μ₀=1000) is studied to obtain its reluctance, i.e., the ratio of magnetomotive force and magnetic flux. Both formulations (5) and (12) are used with a 2D model. This particularly points out the formulation of (12), (15) from 3D to 2D, and therefore helps to understand the kind of generalization done in 3D compared with what appears natural in 2D.

For formulation (5), φ has floating values on Γ₀₀ and Γ₈₈ while homogeneous natural condition n·b=0 is defined on Γ₀₀ and Γ₀₁ (Fig. 5; scalar potential lines). With these floating values are associated weak values of magnetic fluxes Ψ₅₅ and Ψ₈₆ by (10). Then, for example, taking φ₁₀₀=0 and φ₃₀₁=1 A/m, computation gives Ψ₅₅₀ and Ψ₈₆₀. For formulation (12), a₞ₓₙ=0 and aₙₓₗ=1 Wb/m enables to set a magnetic flux across Γ₀₀ and Γ₀₁ (Fig. 5; field lines). Homogeneous natural condition n·h=0 is defined on Γ₀₀ and Γ₀₁. Computed results are given in Table 1; two positions of the rotating part of the circuit are considered with three meshes composed of first order triangular elements.

![Fig. 5. A 2D magnetic circuit element (potential lines (5), field lines (12)).](image)

### Table 1

<table>
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<th>Mesh (type)</th>
<th>Reluctance with φ = 0°</th>
<th>Reluctance with φ = 30°</th>
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C. Magnetodynamic problem

A 3D solid inductor [5] is studied with formulation (13) to obtain its impedance at 50 Hz (Fig. 6; the inductor and the air are meshed with hexahedra). For an imposed voltage (16) of e.g. 0.08 V (the voltage for 1/8 of the inductor is then 0.01 V), computed current is 543.2 A≤-46.0° and 792.7 A≤-81.0° for conductivities respectively equal to 5 10⁶ and 5 10⁷ Ω⁻¹ m⁻¹. Such a problem is at the basis of circuit coupling.

![Fig. 6. Inductor (1/8 because of symmetry), its cut and its boundary.](image)

VI. CONCLUSIONS

A method has been proposed which comes into a general frame of definition of global quantities related to fluxes and circulations. It can be applied to a wide range of formulations of various physical problems. The method has been applied to electrostatics, magnetostatics and magnetodynamics, to naturally define the implied global quantities in both weak and strong senses. The generality of the method, which is set at the formulation level, enables its application to all kinds of geometrical models (2D or 3D), with linear or nonlinear material characteristics. The method is also independent of the characteristics of the finite elements used (geometry and degree). All the advantages of the method appear when local and global quantities have to be coupled, either within a finite element problem or through external lumped circuits, which directly opens the method to the coupling of physical problems.

REFERENCES