Limiting amplitude principle for a two-layered medium composed of a dielectric material and a metamaterial

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Abstract

For wave propagation phenomena, the limiting amplitude principle (LAP) holds if the time-harmonic regime represents the large time asymptotic behavior of the solution of the evolution problem with a time-harmonic excitation. Considering a two-layered medium composed of a dielectric material and a Drude metamaterial separated by a plane interface, we prove that the LAP holds except for a critical situation related to a surface resonance phenomenon.

Keywords: Maxwell's equations, metamaterials, spectral theory.

1 Introduction

In the frequency domain, the permittivity and permeability of a non-dissipative dispersive material $\varepsilon(\omega)$ and $\mu(\omega)$ are real-valued functions of the frequency ω. For metamaterials, these coefficients may become negative in particular frequency ranges, which raises theoretical and numerical difficulties. In [1], the authors proved that for a transmission problem between a dielectric material and a metamaterial separated by a smooth interface, the time-harmonic problem is well-posed except when both ratios of ε and μ across the interface are equal to -1 (which is the case of the "perfect lens" [3]). Nevertheless, the associated time-dependent problem remains well-posed. What is the link between both problems, in particular when the harmonic problem is ill-posed? We answer here the question in the case of a planar transmission problem which involves a Drude metamaterial.

2 Formulation of the problem

We consider a two-layered medium composed of a standard dielectric material and a Drude material, both homogeneous and non-dissipative, which fill respectively the half planes $\mathbb{R}^3_- = \{\boldsymbol{x} = (x,y,z) \in \mathbb{R}^3 \mid x < 0\}$ and $\mathbb{R}^3_+ = \{\boldsymbol{x} = (x,y,z) \in \mathbb{R}^3 \mid x > 0\}$. $(\boldsymbol{e_x}, \boldsymbol{e_y}, \boldsymbol{e_z})$ will refer to the canonical basis of $\mathbb{R}^3$. We denote by $\boldsymbol{E}$ and $\boldsymbol{H}$ the electric and magnetic fields and by $\boldsymbol{D}$ and $\boldsymbol{B}$ the electric and magnetic inductions. In the presence of a source current density $\boldsymbol{J}_s$, the evolution of $(\boldsymbol{E}, \boldsymbol{D}, \boldsymbol{H}, \boldsymbol{B})$ is governed by Maxwell's equations:

$$\begin{aligned} \partial_t \boldsymbol{D} - \mathbf{Curl}\, \boldsymbol{H} &= -\boldsymbol{J}_s \\ \partial_t \boldsymbol{B} + \mathbf{Curl}\, \boldsymbol{E} &= 0, \end{aligned}$$

(where the usual transmission conditions at the interface $x = 0$ are implicitly understood). These equations must be supplemented by the constitutive laws of each material. In the dielectric material, they are simply expressed by

$$\boldsymbol{D} = \varepsilon_0\, \boldsymbol{E} \quad \text{and} \quad \boldsymbol{B} = \mu_0\, \boldsymbol{H},$$

for two positive constants ε_0 and μ_0. In a dispersive media, these laws involve two additional unknowns, the electric and magnetic polarizations $\boldsymbol{P}$ and $\boldsymbol{M}$:

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P} \quad \text{and} \quad \boldsymbol{B} = \mu_0 \boldsymbol{H} + \boldsymbol{M}.$$

For the Drude model, the fields $\boldsymbol{P}$ and $\boldsymbol{M}$ are related to $\boldsymbol{E}$ and $\boldsymbol{H}$ through

$$\begin{aligned} \partial_t \boldsymbol{P} = \boldsymbol{J} \quad &\text{and} \quad \partial_t \boldsymbol{J} = \varepsilon_0\, \Omega_e^2\, \boldsymbol{E} \\ \partial_t \boldsymbol{M} = \boldsymbol{K} \quad &\text{and} \quad \partial_t \boldsymbol{K} = \mu_0\, \Omega_m^2\, \boldsymbol{H}, \end{aligned}$$

where Ω_e and Ω_m are positive parameters. By eliminating $\boldsymbol{D}$, $\boldsymbol{B}$, $\boldsymbol{P}$ and $\boldsymbol{M}$ in the above equations, we obtain

$$(P) \begin{cases} \varepsilon_0\, \partial_t \boldsymbol{E} - \mathbf{Curl}\, \boldsymbol{H} + \mathbf{\Pi}\ \boldsymbol{J} = -J_s & \text{in } \mathbb{R}^3, \\ \mu_0\, \partial_t \boldsymbol{H} + \mathbf{Curl}\, \boldsymbol{E} + \mathbf{\Pi}\ \boldsymbol{K} = 0 & \text{in } \mathbb{R}^3, \\ \partial_t \boldsymbol{J} = \varepsilon_0\, \Omega_e^2\, \boldsymbol{E} & \text{in } \mathbb{R}^3_+, \\ \partial_t \boldsymbol{K} = \mu_0\, \Omega_m^2\, \boldsymbol{H} & \text{in } \mathbb{R}^3_+, \end{cases}$$

where $\mathbf{\Pi}$ denotes the operator of extension by 0 of a vectorial field defined on $\mathbb{R}^3_+$ to $\mathbb{R}^3$.

When looking for time-harmonic solutions of (P): $(\boldsymbol{\mathcal{E}}(\boldsymbol{x}), \boldsymbol{\mathcal{H}}(\boldsymbol{x}), \boldsymbol{\mathcal{J}}(\boldsymbol{x}), \boldsymbol{\mathcal{K}}(\boldsymbol{x}))\, e^{-i\omega t}$ for a periodic current density $\boldsymbol{\mathcal{J}}_s(\boldsymbol{x}) e^{-i\omega t}$, we can eliminate $\boldsymbol{\mathcal{J}}(\boldsymbol{x})$ and $\boldsymbol{\mathcal{K}}(\boldsymbol{x})$. In the half-plane $\mathbb{R}^3_+$ filled by the Drude material, we obtain

$$\begin{aligned} i\,\omega\varepsilon(\omega)\,\boldsymbol{\mathcal{E}} + \mathbf{Curl}\,\boldsymbol{\mathcal{H}} &= \boldsymbol{\mathcal{J}}_s \\ -i\,\omega\mu(\omega)\,\boldsymbol{\mathcal{H}} + \mathbf{Curl}\,\boldsymbol{\mathcal{E}} &= 0, \text{ where} \end{aligned}$$

$$\varepsilon(\omega) = \varepsilon_0 \left(1 - \frac{\Omega_e^2}{\omega^2}\right) \text{ and } \mu(\omega) = \mu_0 \left(1 - \frac{\Omega_m^2}{\omega^2}\right).$$

In the half-plane $\mathbb{R}^3_-$ filled by the dielectric material, we obtain the same equations with $\varepsilon(\omega)$ and $\mu(\omega)$ replaced by ε_0 and μ_0. Note that in the Drude material, $\varepsilon(\omega)$ and $\mu(\omega)$ become negative at low frequencies (which justifies the word "metamaterial"). Moreover, both ratios $\varepsilon(\omega)/\varepsilon_0$ and $\mu(\omega)/\mu_0$ are simultaneously equal to -1 at the same frequency if and only if $\Omega_e = \Omega_m$ $(:= \Omega^*)$ and $\omega = \pm\Omega^*/\sqrt{2}$ $(:= \pm\omega^*)$, where ω_* is called the *plasmonic frequency*.

3 Main results

We are interested in the long-time behavior of the solution of the transverse magnetic (TM) version of (P) for a time-harmonic source term $J_s(\boldsymbol{x}, t) = \mathcal{J}_s(x, y)\, e^{-i\omega t}\boldsymbol{e_z}$ with $\omega > 0$ and zero initial conditions. In this case, we have $\boldsymbol{E} = (0, 0, E_z)$ and $\boldsymbol{H} = (H_x, H_y, 0)$ where E_z, H_x and H_y do not depend on z, as well as the same properties for $\boldsymbol{J}$ and $\boldsymbol{K}$. We express below our main result in terms of the electrical field E_z but the same results hold for the other unknowns H_x, H_y, J_z, K_x, K_y.

Theorem 1 (i) *If $\Omega_e \neq \Omega_m$, the LAP holds at all frequencies, in the sense that for all $\omega > 0$, there exists a function $\mathcal{E}_z$ (related to the time-harmonic problem) such that*

$$E_z(\cdot, t) = \mathcal{E}_z(\cdot)\, e^{-i\,\omega t} + o(1) \ \text{as } t \to +\infty,$$

where $o(1)$ stands for a function which tends to 0 in $L^2_{loc}(\mathbb{R}^2)$.

(ii) *If $\Omega_e = \Omega_m$, the LAP never holds. More precisely, with the same notations as above,*

• *if $\omega \neq \omega_*$, then there exists functions $\mathcal{E}^*_{z,\pm}$ and $\mathcal{E}_z$ such that*

$$E_z(\cdot, t) = \sum_{\pm} \mathcal{E}^*_{z,\pm}(\cdot)\, \mathrm{e}^{\mp i\omega_*\, t} + \mathcal{E}_z(\cdot)\, \mathrm{e}^{-i\omega t} + o(1);$$

• *If $\omega = \omega_*$, then there exists functions $\mathcal{E}^*_z$ and $\mathcal{E}_z$ such that*

$$E_z(\cdot, t) = t\, \mathcal{E}^*_z(\cdot)\, \mathrm{e}^{-i\omega_* t} + \mathcal{E}_z(\cdot)\, \mathrm{e}^{-i\omega_* t} + o(1).$$

4 Method of Analysis

The (very technical) proof follows from standard arguments (see, e.g., [4]). The main difficulty here is related to the dependence of $\varepsilon(\omega)$ and $\mu(\omega)$ with respect to ω (see [2] for details). We first rewrite the original problem (P) as an abstract Schrödinger equation

$$\frac{d\boldsymbol{U}}{dt} + i\,\mathbb{A}\boldsymbol{U} = \boldsymbol{F}\, e^{-i\,\omega t} \ \text{ with } \boldsymbol{U}(0) = 0,$$

where $\mathbb{A}$ is an unbounded self-adjoint operator in an appropriate Hilbert space $\mathcal{H}$. The key of the analysis is the spectral theory of the operator $\mathbb{A}$. This permits a quasi-explicit representation of the solution via the (generalized) diagonalization of $\mathbb{A}$. This is achieved by combining a partial Fourier transform along the interface with Sturm-Liouville type techniques in the orthogonal direction. For $\Omega_e = \Omega_m$, the resonance phenomenon is linked to the fact that $\mathbb{A}$ admits at the plasmonic frequency ω_* an eigenvalue of infinite multiplicity.

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