MULTIPLE ACOUSTIC SCATTERING BY SMALL OBSTACLES IN TWO DIMENSIONS

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Talk Abstract

We are concerned with a two-dimensional problem which models the scattering of a time-harmonic acoustic wave by an arbitrary number of sound-soft circular obstacles. Assuming that their radii are small compared to the wavelength, we propose a mathematical justification of different levels of asymptotic models available in the physical literature.

1 Introduction

We consider the scattering of an acoustic time-harmonic wave by an arbitrary number of small circular sound-soft obstacles located in a two-dimensional homogeneous medium. Since its discovery, the Foldy-Lax model [3] has been used in numerous physical and numerical applications to approximate the scattered wave. But to our knowledge, there is no mathematical justification of this approximation. Our purpose is to propose such a justification and to provide error estimates.

For the sake of simplicity, we are concerned with a non-dimensional model which amounts to a constant celerity $c = 1$ in the propagative medium. Let w be a given incident field of circular frequency ω. We denote by $O_1^\epsilon, ..., O_P^\epsilon$ a family of P disjoint obstacles located in the medium, and we call respectively $s_1, ..., s_P$ their centers and $r_1^\epsilon, ..., r_P^\epsilon$ their radii. We suppose that each obstacle has a small radius compared to the wavelength $2\pi/\omega$ and that they are all of the same order of magnitude, i.e.,

$$\forall p \in \{1, ..., P\},\ \omega r_p^\epsilon = O(\epsilon),$$

where ϵ is a small positive parameter. The scattered field u^ϵ is the solution to the problem:

$$(P)\begin{cases} \Delta u^\epsilon + \omega^2 u^\epsilon = 0 \quad \text{in } \mathbb{R}^2 \setminus \left(\overline{\bigcup_{p=1}^P O_p^\epsilon}\right), \\ u^\epsilon = -w \text{ on } \partial\left(\bigcup_{p=1}^P O_p^\epsilon\right), \\ u^\epsilon \text{ satisfies the Sommerfeld radiation condition.} \end{cases}$$

In order to approximate the above system of equations, we consider a family of asymptotic models. They are based on the fact that in the case of one scatterer ($P = 1$), the solution u^ϵ of (P) can be approximated (see [5]) by

$$\sigma_1^\epsilon w(s_1) G(x - s_1),$$

where $G(x) = H_0^{(1)}(\omega |x|)/4i$ is the outgoing Green function of the Helmholtz equation ($H_0^{(1)}$ is the Hankel function of the first kind of order 0) and σ_1^ϵ is the reflection coefficient on the scatterer, which is given by $\sigma_1^\epsilon = -4i/H_0^{(1)}(\omega r_1^\epsilon)$ for a circular obstacle.

If $P > 1$, we can consider different levels of approximation $u^{\epsilon,0}, u^{\epsilon,1}, \ldots, u^{\epsilon,\infty}$ of u^ϵ which consist in superpositions of the form

$$u^{\epsilon,k}(x) := \sum_{p=1}^P \sigma_p^\epsilon\, w_p^{\epsilon,k}(s_p)\, G(x - s_p), \tag{1}$$

where $w_p^{\epsilon,k}(s_p)$ represents different approximations of an "exciting field" on the p-th scatterer. In the simplest model ($k = 0$), we choose $w_p^{\epsilon,0}(s_p) = w(s_p)$, which amounts to neglecting the interactions between the obstacles. The case $k = \infty$ corresponds to the Foldy-Lax model [3], which takes into account these interactions. In this case, the exciting field is the superposition of the incident field and the waves scattered by all the other obstacles, i.e., for $p = 1, \ldots, P$,

$$w_p^{\epsilon,\infty}(s_p) = w(s_p) + \sum_{q \neq p} \sigma_q^\epsilon w_q^{\epsilon,\infty}(s_q) G(s_p - s_q). \tag{2}$$

If we denote by $W^{\epsilon,\infty}$ and W the vectors of $\mathbb{C}^P$ with components $w_p^{\epsilon,\infty}(s_p)$ and $w(s_p)$ respectively, this coupling between the exciting fields can be written equivalently as the following linear system:

$$(\mathbb{I} + \mathbb{M}^\epsilon)\, W^{\epsilon,\infty} = W, \tag{3}$$

where $\mathbb{M}^\epsilon$ is the $P \times P$ matrix defined by

$$\mathbb{M}_{pq}^\epsilon = -\sigma_q^\epsilon G(s_p - s_q) \text{ if } q \neq p, \text{ and } \mathbb{M}_{pp}^\epsilon = 0.$$

Between the cases $k = 0$ and $k = \infty$, one can consider intermediate models which take into account the successive reflections between the scatterers. Instead of (2), the exciting field is defined for $p = 1, \ldots, P$ recursively by

$$w_p^{\epsilon,k+1} = w(s_p) + \sum_{q \neq p} \sigma_q^\epsilon w_q^{\epsilon,k}(s_q) G(s_p - s_q).$$

It is readily seen that this relation amounts to approximating the inverse of operator $\mathbb{I} + \mathbb{M}^\epsilon$ involved in (3) by a truncated Neumann series, so that we can summarize these different models by the formula

$$W^{\epsilon,k} = \sum_{\ell=0}^{k} (-\mathbb{M}^\epsilon)^\ell \, W \quad \text{for } k = 0, 1, \ldots, \infty. \qquad (4)$$

2 Mathematical justification

The link between the above asymptotic models and our initial problem (P) is easily made using standard tools for multiple scattering [5]. The first step is to give a representation u^ϵ which makes clear the notion of "exciting field" in problem (P).

Proposition 1. *Let u^ϵ be the solution to problem (P). Then, the family of P coupled single scattering problems*

$$(P_p) \begin{cases} \Delta u_p^\epsilon + \omega^2 u_p^\epsilon = 0 \quad \textit{in } \mathbb{R}^2 \setminus \overline{O_p^\epsilon} \\ \\ u_p^\epsilon = -w - \sum_{q=1, q\neq p}^{P} u_q^\epsilon \ \textit{on } \partial O_p^\epsilon \\ \\ u_p^\epsilon \ \textit{satisfies the Sommerfeld radiation condition} \end{cases}$$

(for $p \in \{1, ..., P\}$) admits a unique solution $(u_1^\epsilon, u_2^\epsilon, ..., u_P^\epsilon)$ and moreover the following decomposition holds:

$$u^\epsilon = \sum_{p=1}^{P} u_p^\epsilon \,. \qquad (5)$$

See [2] for the proof. Each wave u_p^ϵ is the wave scattered only by the p-th obstacle illuminated by the exciting field.

We equip $\mathbb{R}^2$ with the Cartesian coordinate system: (O, e_1, e_2) and define for each obstacle $p \in \{1, ..., P\}$ the local polar coordinates by : (ρ_p, θ_p) where $\rho_p = |x - s_p|$ and θ_p is the angle $(e_1, x - s_p)$. Let us introduce for $m \in \mathbb{Z}$ the local outgoing cylindrical wavefunctions associated with the p-th scatterer:

$$\psi_{p,m}(x) = H_m^{(1)}(\omega \rho_p) e^{im\theta_p} \ \text{ for } \rho_p > 0.$$

As u_p^ϵ is an outgoing solution of the homogeneous Helmholtz equation outside $\overline{O_p^\epsilon}$, it admits a modal decomposition on the $\psi_{p,m}$:

$$u_p^\epsilon(x) \;=\; \sum_{m\in\mathbb{Z}} \frac{c_{p,m}^\epsilon}{H_m^{(1)}(\omega r_p^\epsilon)} \psi_{p,m}(x), \qquad (6)$$

where $c_{p,m}^\epsilon$ is the m-th Fourier coefficient of $u_p^\epsilon(r_p^\epsilon, \cdot)$ on the circle ∂O_p^ϵ:

$$c_{p,m}^\epsilon = \frac{1}{2\pi} \int_0^{2\pi} u_p^\epsilon \left(r_p^\epsilon, \theta_p\right) e^{-im\theta_p} d\theta_p.$$

Similarly, w is a solution of the homogeneous Helmholtz equation on a ball containing the p-th obstacle, therefore it has a modal decomposition on the local Bessel functions:

$$w(x) = \sum_{m\in\mathbb{Z}} d_{p,m} J_m(\omega \rho_p) e^{im\theta_p}.$$

From the addition formula:

$$\psi_{q,m}(x) = \sum_{n\in\mathbb{Z}} \psi_{q,m-n}(s_p) J_n(\omega\rho_p) e^{in\theta_p}, \qquad (7)$$

which is valid for $p, q \in \{1, ..., P\}$ with $p \neq q$ and for $\rho_p < |s_p - s_q|$, and from the Dirichlet conditions of problems (P_p), one can easily verify that the family of problems (P_p) is equivalent to the following linear system:

$$(\mathbb{I} + \mathbb{K}^\epsilon) c^\epsilon = f^\epsilon, \qquad (8)$$

where $c^\epsilon = (c_1^\epsilon, \ldots, c_P^\epsilon)^\top$ and each c_p^ϵ denotes the sequence of the Fourier coefficients $(c_{p,m}^\epsilon)_{m\in\mathbb{Z}}$ of the p-th obstacle. $\mathbb{K}^\epsilon$ is defined by:

$$\mathbb{K}^\epsilon = \begin{pmatrix} 0 & \mathbb{K}_{12}^\epsilon & \ldots & \mathbb{K}_{1P}^\epsilon \\ \mathbb{K}_{21}^\epsilon & 0 & \ldots & \mathbb{K}_{2P}^\epsilon \\ \vdots & \ldots & \ddots & \vdots \\ \mathbb{K}_{P1}^\epsilon & \mathbb{K}_{P2}^\epsilon & \ldots & 0 \end{pmatrix}$$

where for $p, q \in \{1, ..., P\}, p \neq q$, $\mathbb{K}_{pq}^\epsilon$ is an operator which represents the interactions of the q-th obstacle on the p-th obstacle:

$$\mathbb{K}_{pq}^\epsilon \;:\; c_q \;\; \longmapsto \;\; \left(J_m(\omega r_p^\epsilon) \sum_{n\in\mathbb{Z}} \frac{\psi_{q,n-m}(s_p)}{H_n^{(1)}(\omega r_q^\epsilon)} c_{q,n} \right)_{m\in\mathbb{Z}},$$

and for $m \in \mathbb{Z}$,

$$f_{p,m}^\epsilon = -J_m(\omega r_p^\epsilon) d_{p,m}.$$

Asymptotically, we have for $p, q \in \{1, ..., P\},\ p \neq q$:

- $\forall (m, n) \in \mathbb{Z}^2 - \{(0, 0)\}$,
  $J_m(\omega r_p^\epsilon) \dfrac{\psi_{q,n-m}(s_p)}{H_n^{(1)}(\omega r_q^\epsilon)} = O(\epsilon)$ and $f_{p,m}^\epsilon = O(\epsilon)$
- $(m, n) = (0, 0)$,
  $J_0(\omega r_p^\epsilon) \dfrac{\psi_{q,0}(s_p)}{H_0^{(1)}(\omega r_q^\epsilon)} = \dfrac{\psi_{q,0}(s_p)}{H_0^{(1)}(\omega r_q^\epsilon)} + O(\epsilon^2/\log(\epsilon))$
  and $f_{p,0}^\epsilon = -w(s_p) + O(\epsilon^2)$.

So the dominant coefficient of $\mathbb{K}_{pq}^\epsilon$ is reached for $(m, n) = (0, 0)$ and is $O(1/\log(\epsilon))$. Formally, an approximation of order ϵ of (8) is given by the following system:

$$(\mathbb{I} + \widetilde{\mathbb{K}}^\epsilon) c^{\epsilon,\infty} = f^0. \qquad (9)$$

Here $\widetilde{\mathbb{K}}^\epsilon$ has the same block structure as $\mathbb{K}^\epsilon$. For $p, q \in \{1, ..., P\}$, $\widetilde{\mathbb{K}}^\epsilon_{pp} = 0$ and for $p \neq q$, $\widetilde{\mathbb{K}}^\epsilon_{pq}$ is a finite rank operator defined by:

$$\widetilde{\mathbb{K}}^\epsilon_{pq} \,:\, c_q \quad\longmapsto\quad \left(\delta_{m,0}\frac{\psi_{q,0}(s_p)}{H_0^{(1)}(\omega r_q^\epsilon)}c_{q,0}\right)_{m\in\mathbb{Z}}$$

and for $m \in \mathbb{Z}$,

$$f^0_{p,m} = -\delta_{m,0}w(s_p).$$

It follows from (9) that all the Fourier coefficients with $m \neq 0$ are equal to zero. Hence, the system of equations which involves the other coefficients ($m = 0$) is equivalent to (3) with

$$W_p^{\epsilon,\infty} = -c_{p,0}^{\epsilon,\infty}, \text{ for } p = 1, \dots, P. \tag{10}$$

Considering the truncated Neumann series associated with the inverse of the operator $\mathbb{I} + \widetilde{\mathbb{K}}^\epsilon$ involved in (9), we can define

$$c^{\epsilon,k} = \sum_{l=0}^{k}(-\widetilde{\mathbb{K}}^\epsilon)^l f^0.$$

Similarly, for $m \neq 0$ $c_{p,m}^{\epsilon,k} = 0$ and the $W_p^{\epsilon,k}$ defined in (4) are given by the formula:

$$W_p^{\epsilon,k} = -c_{p,0}^{\epsilon,k}, \text{ for } p = 1, \dots, P. \tag{11}$$

In order to obtain error estimates, consider the Hilbert space $\ell^2(\mathbb{C})^P$ with its scalar product:

$$\left\langle c|c' \right\rangle = \sum_{p=1}^{P}\sum_{m\in\mathbb{Z}} c_{p,m}\overline{c'_{p,m}}.$$

Theorem 1. *There exists a constant $C > 0$ such that for ϵ small enough,*

$$\|\mathbb{K}^\epsilon\|_{\mathcal{L}(\ell^2(\mathbb{C})^P)} \leq \frac{C}{|\log(\epsilon)|} \tag{12}$$

$$\left\|\mathbb{K}^\epsilon - \widetilde{\mathbb{K}}^\epsilon\right\|_{\mathcal{L}(\ell^2(\mathbb{C})^P)} \leq C\,\epsilon \tag{13}$$

$$\left\|f^\epsilon - f^0\right\|_{\ell^2(\mathbb{C})^P} \leq C\,\epsilon.$$

As a direct consequence of theorem 1, for ϵ small enough, (9) is well-posed and the Neumann series of $(\mathbb{I} + \widetilde{\mathbb{K}}^\epsilon)^{-1}$ is well-defined. Using the theorem 10.1 of [4], we can now estimate the error between c^ϵ and $c^{\epsilon,\infty}$.

Corollary 1. *There exits a constant $C > 0$ such that for ϵ small enough,*

$$\|c^\epsilon - c^{\epsilon,\infty}\|_{\ell^2(\mathbb{C})^P} \leq C\,\epsilon \tag{14}$$

$$\left\|c^\epsilon - c^{\epsilon,k}\right\|_{\ell^2(\mathbb{C})^P} \leq \frac{C}{|\log(\epsilon)|^{k+1}}. \tag{15}$$

The inequality (15) is a direct consequence of (14) with an estimation of the remainder of the Neumann's series associated with $(\mathbb{I} + \widetilde{\mathbb{K}}^\epsilon)^{-1}$. Following (5) and (6), we consider for $k = 0, \cdots, \infty$ the approximations

$$u^{\epsilon,k}(x) = \sum_{p=1}^{P}\frac{c_{p,0}^{\epsilon,k}}{H_0^1(\omega r_p^\epsilon)}H_0^1(\omega\,|x - s_p|)$$

of the solution to problem (P). From the equalities (10) and (11), the definitions of the coefficients of reflection σ_p^ϵ and of the Green function, it is easy to see that these $u^{\epsilon,k}$ coincide with the functions $u^{\epsilon,k}$ defined in (1). It is easy to deduce from corollary 1 the following error estimates in a local L^2 norm.

Corollary 2. *Let K be a compact subset of $\mathbb{R}^2 \setminus \bigcup_{p=1}^{P}\{s_p\}$ and $k \in \mathbb{N} \cup \{\infty\}$, there is a constant C_K independent of ϵ such that for ϵ small enough,*

$$\text{if } k \in \mathbb{N}, \quad \left\|u^\epsilon - u^{\epsilon,k}\right\|_{L^2(K)} \leq \frac{C_K}{|\log(\epsilon)|^{k+2}}$$

$$\text{if } k = \infty, \quad \|u^\epsilon - u^{\epsilon,\infty}\|_{L^2(K)} \leq \frac{C_K\,\epsilon}{|\log(\epsilon)|}.$$

Note that this result is valid for any distribution of scatterers. However the constant C_K involved in the above estimates depends on their localization and may become large in certain situations (close scatterers).

3 Proof of Theorem 1

We will give the main ideas of the proof of the inequalities (12) and (13). It is enough to show these properties for each block of the operators $\mathbb{K}^\epsilon$ and $\widetilde{\mathbb{K}}^\epsilon$. Using the addition formula (7), it is easier to prove formula (12) for the operator

$$\mathbb{L}^\epsilon_{pq} \,:\, c_q \longmapsto \left(\frac{\sum_{n\in\mathbb{Z}}\overline{\psi_{q,m-n}(s_p)J_n(\omega r_p^\epsilon)}c_{q,n}}{\overline{H_m^{(1)}(\omega r_q^\epsilon)}}\right)_{m\in\mathbb{Z}}$$

which is readily seen to be the adjoint of $\mathbb{K}^\epsilon_{pq}$ once this formula is proved. For $c_q \in \ell^2(\mathbb{C})$, we have

$$\left\|\mathbb{L}^\epsilon_{pq}c_q\right\|^2_{\ell^2(\mathbb{C})} = \sum_{m\in\mathbb{Z}}\frac{\left|\sum_{n\in\mathbb{Z}}\overline{\psi_{q,m-n}(s_p)J_n(\omega r_p^\epsilon)}c_{q,n}\right|^2}{\left|H_m^{(1)}(\omega r_q^\epsilon)\right|^2}.$$

Hence, using the Cauchy-Schwarz inequality, the addition formula (7) and the Parseval identity, we deduce that

$$\left\|\mathbb{L}_{pq}^{\epsilon} c_q\right\|_{\ell^2(\mathbb{C})}^2 \leq \left(\sum_{m\in\mathbb{Z}} \frac{\left\|\psi_{q,m}(r_p^{\epsilon},\cdot)\right\|_{L^2(0,2\pi)}^2}{\left|H_m^{(1)}(\omega r_q^{\epsilon})\right|^2} \right) \|c_q\|_{\ell^2(\mathbb{C})}^2 ,$$

where we have denoted

$$\left\|\psi_{q,m}(r_p^{\epsilon},\cdot)\right\|_{L^2(0,2\pi)}^2 = \frac{1}{2\pi}\int_0^{2\pi} \left|\psi_{q,m}(r_p^{\epsilon},\theta_p)\right|^2 d\theta_p.$$

The estimation of this quantity for large m is easily deduce from classical asymptotic properties of the Hankel functions (see [1]). On the other hand, we need a bound of the function $1/\left|H_m^1\right|$ which holds uniformly in ϵ and m. We have proved the following apparently non usual inequality:

$$\left|\frac{1}{H_m^{(1)}(x)}\right| \leq \frac{2x^{m-2}}{m!\left|H_2^{(1)}(x)\right|}, \text{ for } 0<x<1 \text{ and } m\geq 3,$$

from which we finally deduce that

$$\left\|\mathbb{L}_{pq}^{\epsilon}\right\|_{\mathcal{L}(\ell^2(\mathbb{C}))} \leq \frac{C}{|\log(\epsilon)|}.$$

In order to prove formula (13), we follow a similar idea. First notice that the adjoint of the finite rank operator $\widetilde{\mathbb{K}}_{pq}^{\epsilon}$ is clearly the operator $\widetilde{\mathbb{L}}_{pq}^{\epsilon}$ of $\mathcal{L}(\ell^2(\mathbb{C}))$ defined by:

$$\widetilde{\mathbb{L}}_{pq}^{\epsilon} \,:\, c_q \longmapsto \left(\delta_{m,0} \frac{\overline{\psi_{q,0}(s_p)}}{H_0^{(1)}(\omega r_q^{\epsilon})} c_{q,0} \right)_{m\in\mathbb{Z}}$$

We have:

$$\left\|\left(\mathbb{L}_{pq}^{\epsilon} - \widetilde{\mathbb{L}}_{pq}^{\epsilon}\right) c_q\right\|_{\ell^2(\mathbb{C})}^2 \quad = \quad A(\epsilon) + B(\epsilon)$$

where:

$$A(\epsilon) = \sum_{m\in\mathbb{Z}^*} \frac{\left|\sum_{n\in\mathbb{Z}} \overline{\psi_{q,m-n}(s_p) J_n(\omega r_p^{\epsilon})} c_{q,n}\right|^2}{\left|H_m^{(1)}(\omega r_q^{\epsilon})\right|^2}$$

and

$$B(\epsilon) = \frac{\left|\sum_{n\in\mathbb{Z}} \overline{\psi_{q,-n}(s_p) J_n(\omega r_p^{\epsilon})} c_{q,n} - \overline{\psi_{q,0}(s_p)} c_{q,0}\right|^2}{\left|H_0^{(1)}(\omega r_q^{\epsilon})\right|^2}.$$

As in the proof above, we get:

$$A(\epsilon) \leq \left(\sum_{m\in\mathbb{Z}^*} \frac{\left\|\psi_{q,m}(r_p^{\epsilon},\cdot)\right\|_{L^2(0,2\pi)}^2}{\left|H_m^{(1)}(\omega r_q^{\epsilon})\right|^2} \right) \|c_q\|_{\ell^2(\mathbb{C})}^2$$

and show that there exists $C>0$ such that for ϵ small enough:

$$A(\epsilon) \leq C\,\epsilon^2 \,\|c_q\|_{\ell^2(\mathbb{C})}^2 .$$

For the term $B(\epsilon)$, using the asymptotic properties of the Bessel and the Hankel functions, we establish that there exists $C>0$ such that for ϵ small enough,

$$\sum_{n\in\mathbb{Z}^*} \left|\psi_{q,-n}(s_p) J_n(\omega r_p^{\epsilon})\right|^2 \;\leq\; C\,\epsilon^2 \quad (16)$$

$$\text{and } \left|J_0(\omega r_p^{\epsilon}) - 1\right|^2 \left|\psi_{q,0}(s_p)\right|^2 \;\leq\; C\,\epsilon^4. \quad (17)$$

Therefore, applying the Cauchy-Schwarz inequality, the asymptotic of $1/H_0^{(1)}$ for small arguments and the relations (16) and (17), there exists $C>0$ such that for ϵ small enough:

$$B(\epsilon) \leq C\,\left|\frac{\epsilon}{\log\epsilon}\right|^2 \|c_q\|_{\ell^2(\mathbb{C})}^2 .$$

Finally:

$$\left\|\mathbb{L}_{pq}^{\epsilon} - \widetilde{\mathbb{L}}_{pq}^{\epsilon}\right\|_{\mathcal{L}(\ell^2(\mathbb{C}))} = \left\|\mathbb{K}_{pq}^{\epsilon} - \widetilde{\mathbb{K}}_{pq}^{\epsilon}\right\|_{\mathcal{L}(\ell^2(\mathbb{C}))} \leq C\,\epsilon.$$

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