Bounds on Herglotz functions and physical limits to broadband passive cloaking in the quasitatic regime

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Introduction: Cloaks are specific structures placed near or around an object that render the electromagnetic response of cloak plus object equal or almost equal to that of free space. Ideally passive cloaks should work for waves in a broadband frequency range, giving rise to the challenging question: Is it possible to perform broadband passive cloaking over a finite frequency band? In the context of the quasistatic approximation of Maxwell's equations we prove that it is impossible and give quantitative limitations to cloaking over a finite frequency range. Our results, published in [2], hold for a cloak or object of any geometrical shape and do not depend on the cloaking methods: transformation optics, anomalous resonance, complementary media.

1) The passive cloaking problem

Let $\mathcal{O}$ be a bounded simply-connected dielectric inclusion with Lipschitz boundary that one wants to cloak. $\mathcal{O}$ is characterized by its permittivity $\boldsymbol{\varepsilon}(\boldsymbol{x},\omega)=\varepsilon\,\mathbf{I}$, where $\varepsilon>\varepsilon_0$ is constant on the frequency range of interest $[\omega_-,\omega_+]$ and strictly larger than the permittivity of the vacuum ε_0. The passive cloak is made of an anisotropic material of any shape characterized by its dielectric tensor $\boldsymbol{\varepsilon}(\boldsymbol{x},\omega)$ which depends both on the spatial variable $\boldsymbol{x}$ and the frequency ω. The whole device, the inclusion and the cloak, occupies an open bounded set $\Omega\subset B(\mathbf{0},R_0)$ of characteristic size R_0 and the remainder of space $\mathbb{R}^3\setminus\Omega$ is vacuum of permittivity $\boldsymbol{\varepsilon}(\boldsymbol{x},\omega)=\varepsilon_0\,\mathbf{I}$. The observer is assumed to be at a distance $R\gg R_0$.

We send a plane wave towards the device and assume that its wavelength is considerably larger than R in the frequency range of interest $\omega\in[\omega_-,\omega_+]\subset\mathbb{R}^{+,*}$ so that we can use the quasistatic approximation in this frequency band. In this approximation, the curl-free electrical field $\boldsymbol{E}(\boldsymbol{x},\omega)$ is given in terms of the gradient of some potential $V(\boldsymbol{x},\omega)$, i.e. $\boldsymbol{E}(\boldsymbol{x},\omega)=-\nabla V(\boldsymbol{x},\omega)$, the incident plane wave in the vicinity of a closed ball $B(\mathbf{0},R)$ corresponds to a uniform field $\boldsymbol{E}_0\in\mathbb{C}^3$ so that the potential $\nabla V(\boldsymbol{x},\omega)$ satisfies the following elliptic equation

$$\nabla\cdot\big(\boldsymbol{\varepsilon}(\boldsymbol{x},\omega)\nabla V(\boldsymbol{x},\omega)\big)=0 \text{ on } \mathbb{R}^3, \tag{1}$$

and admits the asymptotic expansion as $|\boldsymbol{x}|\to\infty$:

$$V(\boldsymbol{x},\omega)=-\boldsymbol{E}_0\cdot\boldsymbol{x}+\frac{\boldsymbol{p}(\omega)\cdot\boldsymbol{x}}{4\pi\varepsilon_0|\boldsymbol{x}|^3}+\mathcal{O}\Big(\frac{1}{|\boldsymbol{x}|^3}\Big), \text{ with } \boldsymbol{p}(\omega)=\boldsymbol{\alpha}(\omega)\boldsymbol{E}_0. \tag{2}$$

Thus, the main contribution of the scattered far field is a dipolar term $\boldsymbol{p}(\omega)\in\mathbb{C}^3$ which depends linearly on $\boldsymbol{E}_0$ via the polarizability tensor $\boldsymbol{\alpha}(\omega)\in M_3(\mathbb{C})$. Hence, to cloak the device Ω at a sufficient large distance R to any incident field $\boldsymbol{E}_0\in\mathbb{C}^3$ at a frequency $\omega\in[\omega_-,\omega_+]$, one needs that $\boldsymbol{\alpha}(\omega)$ vanishes at ω.

The electric induction $\boldsymbol{D}$ is given within the cloak $\Omega\setminus\mathcal{O}$ by the constitutive law (CL): $\boldsymbol{D}=\varepsilon_0\boldsymbol{E}+\varepsilon_0\boldsymbol{\chi}_E\star_t\boldsymbol{E}$, where $\star_t$ stands for the time convolution product between the real-valued susceptibility tensor $\boldsymbol{\chi}_E(\boldsymbol{x},t)$ and the electrical field $\boldsymbol{E}$. To define this convolution, one assumes for simplicity that $\boldsymbol{\chi}_E \in L^1(\mathbb{R}_t, L^\infty(\Omega \setminus \mathcal{O})^9)$ and that $\boldsymbol{E}, \partial_t \boldsymbol{E} \in L^2(\mathbb{R}_t, L^2(\Omega \setminus \mathcal{O})^3)$. The cloak is a passive material since **it is causal:** χ_E is supported in $(\Omega \setminus \mathcal{O}) \times \mathbb{R}_t^+$ and **passive**, i.e. for any real fields $(\boldsymbol{E}, \boldsymbol{D})$ satisfying the (CL) (and the regularity assumptions for $\boldsymbol{E}$):

$$\int_{-\infty}^{t} \int_{\Omega\setminus\mathcal{O}} \partial_t \boldsymbol{D}(\boldsymbol{x},t) \cdot \boldsymbol{E}(\boldsymbol{x},t) \mathrm{d}\boldsymbol{x} \mathrm{d}t \geq 0, \quad \forall t \in \mathbb{R}.$$

Let $\mathbb{C}^+ := \{\omega \in \mathbb{C} \mid \operatorname{Im} \omega > 0\}$ and $\operatorname{cl} \mathbb{C}^+ := \mathbb{C}^+ \cup \mathbb{R}$. For any causal $f \in L^1(\mathbb{R}_t)$, one defines the Fourier-Laplace transform as $\hat{f}(\omega) := \int_{\mathbb{R}^+} f(t) \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}t$, $\forall \omega \in \operatorname{cl} \mathbb{C}^+$ so that it coincides with the Fourier transform for real frequency. The (CL) in the frequency domain becomes: $\hat{\boldsymbol{D}}(\boldsymbol{x},\omega) = \boldsymbol{\varepsilon}(\boldsymbol{x},\omega)\ \hat{\boldsymbol{E}}(\boldsymbol{x},\omega)$ with $\boldsymbol{\varepsilon}(\boldsymbol{x},\omega) = \varepsilon_0(1 + \hat{\boldsymbol{\chi}}_E(\boldsymbol{x},\omega))$, $\forall \omega \in \mathbb{R}$. Thus, one shows that the passivity of the cloak is equivalent in the frequency domain to

$(\tilde{\mathrm{H}}_1)$: for a.e. $\boldsymbol{x} \in \Omega \setminus \mathcal{O}$, $\boldsymbol{\varepsilon}(\boldsymbol{x},\cdot)$ is analytic on $\mathbb{C}^+$ and continuous on $\operatorname{cl} \mathbb{C}^+$,
$(\tilde{\mathrm{H}}_2)$: for a.e. $\boldsymbol{x} \in \Omega \setminus \mathcal{O}$, $\forall \omega \in \operatorname{cl} \mathbb{C}^+$, $\boldsymbol{\varepsilon}(\boldsymbol{x}, -\overline{\omega}) = \overline{\boldsymbol{\varepsilon}(\boldsymbol{x},\omega)}$,
$(\tilde{\mathrm{H}}_3)$: for a.e. $\boldsymbol{x} \in \Omega \setminus \mathcal{O}$, $\forall \omega \in \mathbb{R}^+$, $\operatorname{Im} \boldsymbol{\varepsilon}(\boldsymbol{x},\omega) \geq 0$ (passivity),
$(\tilde{\mathrm{H}}_4)$: for a.e. $\boldsymbol{x} \in \Omega \setminus \mathcal{O}$, $\boldsymbol{\varepsilon}(\boldsymbol{x},\omega) \to \varepsilon_0 \mathbf{I}$ as $|\omega| \to \infty$ in $\operatorname{cl} \mathbb{C}^+$.

2) Bounds on Herglotz functions and passive systems

Herglotz functions are analytic functions of the upper-half plane with non-negative imaginary part. In [2], we derive bounds on Herglotz functions which apply to a wide class of linear passive systems and generalize those provided in [1]. To this aim, we consider a passive linear system characterized by a function $f : \operatorname{cl} \mathbb{C}^+ \to \mathbb{C}$ in the frequency domain which satisfies the assumptions

(H_1) f is analytic on $\mathbb{C}^+$, continuous on $\operatorname{cl} \mathbb{C}^+$, (H_2) $f(-\overline{\omega}) = \overline{f(\omega)}$, $\forall \omega \in \operatorname{cl} \mathbb{C}^+$,
(H_3) $\operatorname{Im} f(\omega) \geq 0$, $\forall \omega \in \mathbb{R}^+$, (H_4) $f(\omega) \to f_\infty > 0$, when $|\omega| \to \infty$ in $\operatorname{cl} \mathbb{C}^+$,

i.e. $(\tilde{\mathrm{H}}_1 - \tilde{\mathrm{H}}_4)$ but for a scalar function. We define the square root by $\sqrt{\omega} = |\omega|^{\frac{1}{2}} e^{\mathrm{i} \arg \omega / 2}$ if $\arg \omega \in (0, 2\pi)$ and by $\sqrt{x} = |x|^{\frac{1}{2}}$ for $x \in \mathbb{R}^+$. In [2], we show that v defined by $v(\omega) := \omega f(\sqrt{\omega})$, $\forall \omega \in \mathbb{C}$ is a Herglotz function that is analytic in $\mathbb{C} \setminus \mathbb{R}^+$, negative on $\mathbb{R}^{-,*}$ and satisfies by (H_2): $\overline{v(\overline{\omega})} = v(\omega), \forall \omega \in \mathbb{C}^+ \cup \mathbb{R}^{-,*}$ and by (H_4): $v(\omega) = f_\infty \omega + o(\omega)$ when $|\omega| \to \infty$ in $\mathbb{C}^+$.

Then, we introduce the Herglotz functions h_{m} and v_{m} defined by:

$$h_{\mathrm{m}}(\omega) = \int_{\mathbb{R}} \frac{\mathrm{dm}(\xi)}{\xi - \omega} \quad \text{and} \quad v_{\mathrm{m}}(\omega) = h_{\mathrm{m}}(v(\omega)),\ \forall \omega \in \mathbb{C}^+,$$

where $\mathrm{m} \in \mathcal{M}$ with $\mathcal{M}$ is the set of probability measures on $\mathbb{R}$. Using a sum rule derived in [1], we show (see [2]) the following theorem.

Theorem 1. *Let $[x_-, x_+]$ be a compact interval of $\mathbb{R}^{+*}$, then one has:*

$$\lim_{y \to 0^+} \frac{1}{\pi} \int_{x_-}^{x_+} \operatorname{Im} v_{\mathrm{m}}(x + \mathrm{i}y)\, \mathrm{d}x \leq \frac{1}{f_\infty} \quad \forall m \in \mathcal{M}, \tag{3}$$

and Dirac measures $(\delta_\xi)_{\xi \in \mathbb{R}}$ optimize the inequality (3) *since*

$$\sup_{\mathrm{m} \in \mathcal{M}} \frac{1}{\pi} \lim_{y \to 0^+} \int_{x_-}^{x_+} \operatorname{Im} v_{\mathrm{m}}(x + \mathrm{i}y)\, \mathrm{d}x = \sup_{\xi \in \mathbb{R}} \frac{1}{\pi} \lim_{y \to 0^+} \int_{x_-}^{x_+} \operatorname{Im} v_{\delta_\xi}(x + \mathrm{i}y)\, \mathrm{d}x.$$

We prove this result in [2] for compactly supported measures $\mathrm{m} \in \mathcal{M}$ but it can be shown exactly in the same way for any $\mathrm{m} \in \mathcal{M}$. If $\operatorname{Im} f(\omega) = 0$ for $\omega \in [\omega_-, \omega_+]$, we show by using Dirac measures $(\delta_\xi)_{\xi \in \mathbb{R}}$ in inequality (3) that

$$\omega_0^2(f(\omega_0) - f_\infty) \leq \omega^2(f(\omega) - f_\infty), \ \forall \omega, \omega_0 \in [\omega_-, \omega_+] \text{ such that } \ \omega_0 \leq \omega. \quad (4)$$

Without such assumption on $\operatorname{Im} f$, by using the uniform probability measure on $[-\Delta, \Delta]$ with $\Delta = \max_{[\omega_-^2, \omega_+^2]} |v(x)|$ in the bound (3) it follows that:

$$\frac{1}{4}(\omega_+^2 - \omega_-^2) f_\infty \leq \max_{\omega \in [\omega_-, \omega_+]} |\omega^2 f(\omega)|. \quad (5)$$

3) Fundamentals limits to broadband cloaking

We apply now the above bounds to cloaking. We give in [2] a functional framework to equations (1) and (2) (which are physically relevant in $[\omega_-, \omega_+]$ where the quasistatic approximation is valid but holds for any $\omega \in \operatorname{cl} \mathbb{C}^+$ by using the analytic extension of the permittivity in the inclusion and in the vacuum). Then, we show (with a coercivity assumption, see [2]) that for a passive cloak satisfying $(\tilde{\mathrm{H}}_1 - \tilde{\mathrm{H}}_4)$ and a reciprocity principle, the function $f_{\boldsymbol{E}_0}$ given by

$$f_{\boldsymbol{E}_0}(\omega) := \boldsymbol{\alpha}(\omega) \boldsymbol{E}_0 \cdot \overline{\boldsymbol{E}_0} = \int_\Omega (\boldsymbol{\varepsilon}(\boldsymbol{x}, \omega) - \varepsilon_0 \mathbf{I})\, \boldsymbol{E}(\boldsymbol{x}, \omega) \cdot \overline{\boldsymbol{E}_0}\, \mathrm{d}\boldsymbol{x}, \ \forall \omega \in \operatorname{cl} \mathbb{C}^+$$

is well-defined for $\boldsymbol{E}_0 \in \mathbb{C}^3$ and satisfies $(\tilde{\mathrm{H}}_1 - \tilde{\mathrm{H}}_4)$ with $f_{\boldsymbol{E}_0, \infty} := \boldsymbol{\alpha}(\infty) \boldsymbol{E}_0 \cdot \overline{\boldsymbol{E}_0}$ where $\boldsymbol{\alpha}(\infty) := \lim_{|\omega| \to +\infty} \boldsymbol{\alpha}(\omega)$ is the positive definite polarizability tensor of the inclusion $\mathcal{O}$ depending only on its geometry and contrast in permittivity. If the cloak is a lossless (i.e if $\operatorname{Im} \boldsymbol{\varepsilon}(\boldsymbol{x}, \omega) = 0$ inside the cloak on $[\omega_-, \omega_+]$), one shows that $\operatorname{Im} f_{\boldsymbol{E}_0}(\omega) = 0$ on $[\omega_-, \omega_+]$. Thus, using (4) on functions $f_{\boldsymbol{E}_0}$ gives

$$\omega_0^2 \left(\boldsymbol{\alpha}(\omega_0) - \boldsymbol{\alpha}(\infty)\right) \leq \omega^2 \left(\boldsymbol{\alpha}(\omega) - \boldsymbol{\alpha}(\infty)\right), \ \forall \omega, \omega_0 \in [\omega_-, \omega_+] \text{ such that } \ \omega_0 \leq \omega,$$

which turns to be an optimal bound (see [2]). Now assume that one can cloak at a frequency ω_0. Thus $\boldsymbol{\alpha}(\omega_0) = 0$ and it yields to

$$\boldsymbol{\alpha}(\omega) \leq -\boldsymbol{\alpha}(\infty) \frac{\omega_0^2 - \omega^2}{\omega^2}, \ \omega \in [\omega_-, \omega_0] \text{ and } \boldsymbol{\alpha}(\infty) \frac{\omega^2 - \omega_0^2}{\omega^2} \leq \boldsymbol{\alpha}(\omega), \ \omega \in [\omega_0, \omega_+]$$

which obviously forces $\boldsymbol{\alpha}(\omega)$ to be non-zero away from the frequency ω_0 on $[\omega_-, \omega_+]$ and makes cloaking impossible on $[\omega_-, \omega_+]$. If the cloak is not lossless, one applies the bound (5) on the functions $f_{\boldsymbol{E}_0}$ to get:

$$\tfrac{1}{4}(\omega_+^2 - \omega_-^2) \boldsymbol{\alpha}(\infty) \boldsymbol{E}_0 \cdot \overline{\boldsymbol{E}_0} \leq \max_{\omega \in [\omega_-, \omega_+]} |\omega^2 \boldsymbol{\alpha}(\omega) \boldsymbol{E}_0 \cdot \overline{\boldsymbol{E}_0}|, \quad \forall \boldsymbol{E}_0 \in \mathbb{C}^3.$$

This positive lower bound also gives a limitation to cloaking on $[\omega_-, \omega_+]$.

References

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