Selective focusing for time dependent waves

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Abstract

We are concerned with focusing effects for time-dependent waves using an array of pointlike transducers. We consider a two-dimensional problem which models acoustic wave propagation in a medium which contains several unknown pointlike scatterers. Spatial focusing properties have been studied in the frequency domain in the context of the DORT method ("Decomposition of the Time Reversal Operator"). This method consists in doing a Singular Value Decomposition of the scattering operator, that is, the operator which maps the input signals sent to the transducers to the measure of the scattered wave. We show how to construct a wave that focuses in space and time near one of these scatterers, in the form of a superposition of time-harmonic waves related to the singular vectors of the scattering operator. Numerical results will be shown.

Introduction

We consider a reference medium, possibly inhomogeneous, filling the whole plane $\mathbb{R}^2$. We denote by G the time-dependent Green function of the acoustic wave equation, that is the causal solution to

$$\frac{1}{c^2(x)}\frac{\partial^2 G(x,y;t)}{\partial t^2} - \Delta_x G(x,y;t) = \delta(x-y)\otimes\delta(t)$$

where c is the wave speed function of the medium (e.g., $G(x,y;t) = -H(t-|x-y|)/(2\pi(t^2-|x-y|^2))^{\frac{1}{2}}$ for $c\equiv 1$, where H is the Heaviside function). We assume that the reference medium is perturbed by the presence of a family of P pointlike scatterers whose positions $s_1,\dots,s_P$ are unknown. Using an array of N point-like transducers located at x_n for $n=1,\dots,N$ (with $N\geq P$), our aim is to generate a wave that focuses in space and time on one of the scatterers. Such a wave is defined by

$$w(x,t) = \sum_{n=1}^{N}\left(G(x,x_n;\cdot)\overset{t}{\star} q_n\right)(t) \tag{1}$$

where $\mathbf{q}_{\mathrm{inp}}(t) := (q_1(t),\dots,q_N(t))^\top$ represents the input signals applied to the transducers and $\overset{t}{\star}$ denotes the time convolution. The question is to find signals $q_{\mathrm{inp}}(t)$ for which most part of the energy of the wave will be concentrated near one obstacle at a given time. In the present paper, we show how to deduce such signals from the only knowledge of the *scattering operator* $\mathbb{S}:\mathbf{q}_{\mathrm{inp}}\mapsto\mathbf{q}_{\mathrm{mes}}$ where q_{mes} represents the measures at points $x_1,\dots,x_N$ of the scattered wave associated with the incident wave (1), that is, the perturbation of this incident wave due to the presence of the unknown scatterers. The idea is to take advantage of the so-called DORT method (see, e.g., [2,4]) whose spatial focusing properties in the frequency domain are well known.

1. Space focusing in the frequency domain

Let $\widehat{G}$ denote the time-harmonic Green function of the reference medium which is related to the time-dependent Green function G by the Fourier transform:

$$G(x,y;t) = \frac{1}{\pi}\mathrm{Re}\left(\int_0^{+\infty} \widehat{G}(x,y;\omega)\, e^{-i\omega t}\, d\omega\right).$$

At a fixed frequency ω, the array of transducers emits a time-harmonic incident wave defined by

$$\hat{w}(x) = \sum_{n=1}^{N} \hat{q}_n\, \widehat{G}(x, x_n; \omega),$$

for a given $\hat{\mathbf{q}}_{\text{inp}} := (\hat{q}_1, \ldots, \hat{q}_N)^\top \in \mathbb{C}^N$ (complex amplitudes of the input signals at the N transducers). Then, the array measures the scattered wave $\hat{q}_{mes}$. This yields the time-harmonic scattering operator $\widehat{\mathbb{S}}_\omega : \hat{\mathbf{q}}_{\text{inp}} \mapsto \hat{\mathbf{q}}_{\text{mes}}$ which can be written here as a product of three matrices:

$$\widehat{\mathbb{S}}_\omega \quad = \quad \underbrace{\widehat{\mathbb{G}}_\omega^\top}_{\text{back propagation}} \quad \underbrace{\widehat{\Sigma}_\omega}_{\text{reflection}} \quad \underbrace{\widehat{\mathbb{G}}_\omega}_{\text{direct propagation}} \quad ,$$

where $\widehat{\mathbb{G}}_\omega$ is a $P \times N$ matrix defined by $(\widehat{\mathbb{G}}_\omega)_{pn} := \widehat{G}(x_n, s_p; \omega)$ and $\widehat{\Sigma}_\omega$ is a $P \times P$ symmetric matrix ($\widehat{\Sigma}_\omega^\top = \widehat{\Sigma}_\omega$) which represents the reflections on the scatterers. The latter matrix depends on the choice of an asymptotic model for the scatterers. In the simplest case (no interaction between the scatterers), this is a diagonal matrix composed of the reflection coefficients of the scatterers. The more elaborate Foldy–Lax model [1] takes into account isotropic interactions.

The DORT method consists in a Singular Value Decomposition (SVD) of $\widehat{\mathbb{S}}_\omega$:

$$\widehat{\mathbb{S}}_\omega = \widehat{\mathbb{P}}_\omega\, \widehat{\mathbb{D}}_\omega\, \overline{\widehat{\mathbb{Q}}}_\omega^\top, \tag{2}$$

where $\widehat{\mathbb{D}}_\omega$, $\widehat{\mathbb{P}}_\omega$, $\widehat{\mathbb{Q}}_\omega$ are respectively the diagonal matrix of singular values, the matrices of the left and right singular vectors. It is now well understood ([2, 4]) that in a homogeneous medium, for distant enough scatterers, the number of nonzero singular values of $\widehat{\mathbb{S}}_\omega$ coincide with the number of scatterers. Moreover if such a singular value $\lambda_p(\omega)$ is simple, the associated right singular vector $\widehat{\mathbf{q}}_p(\omega)$ (pth column of $\widehat{\mathbb{Q}}_\omega$) generates a wave which focuses selectively on one scatterer, say s_p.

2. Space-time focusing

Suppose that in a given frequency band $[\omega_1, \omega_2]$ (imposed by the physical properties of our array), we know a right singular vector $\widehat{\mathbf{q}}_p(\omega) \in \mathbb{C}^N$ associated with the pth obstacle and a simple singular value $\lambda_p(\omega)$. How can one choose a function $A(\omega)$ defined on the frequency band such that the superposition of the time-harmonic input signals:

$$\mathbf{q_p}(t) = \mathrm{Re}\int_{\omega_1}^{\omega_2} A(\omega)\, \widehat{\mathbf{q}}_p(\omega)\, e^{-\mathrm{i}\omega t}\, \mathrm{d}\omega \tag{3}$$

generates an incident wave which focuses not only in space near s_p, but also in time?

We look for a function A as a product $A(\omega) = \chi(\omega) e^{i\phi(\omega)}$ with χ a given real cutoff function and ϕ an unknown phase. This is a problem of frequency phase synchronization. The phase choice that we propose is based on a particular SVD of the scattering operator related to its symmetry. $\widehat{\mathbb{S}}_\omega$ is a symmetric operator, therefore up to a change of sign, there exists a unique $\phi_{sym} \in [-\pi, \pi[$ such that

$$\widehat{\mathbb{S}}_\omega \, e^{\mathrm{i}\phi_{\mathrm{sym}}(\omega)} \widehat{\mathbf{q}}_p(\omega) = \lambda_p(\omega) \, \overline{e^{\mathrm{i}\phi_{\mathrm{sym}}(\omega)} \widehat{\mathbf{q}}_p(\omega)}, \tag{4}$$

$e^{i\phi_{sym}(\omega)} \widehat{\mathbf{q}}_p(\omega)$ is then a right singular vector of a symmetric SVD of $\widehat{\mathbb{S}}_\omega$: $\overline{\mathbb{U}}_\omega \mathbb{D}_\omega \overline{\mathbb{U}}_\omega^\top$ (see [3] for more details). Does this signal yield an *optimal* focusing? We did not succeed in finding a mathematical functional representing the focusing quality which would be maximal for this particular choice. But several arguments are pointing in that direction.

The first one is heuristic. As the time reversal operation $\mathbb{J} : f(t) \mapsto f(-t)$ becomes a complex conjugation in the frequency domain, we see with (4) that at each frequency, the measure of the scattered field is (up to a positive real factor $\lambda_p(\omega)$) the time reversed emitted signal. This temporal symmetry synchronizes the spectral components of the emitted wave at the focusing time $t = 0$. The mathematical counterpart of this property lies in the following proposition. We denote for a function $\phi \in L^\infty([\omega_1, \omega_2])$,

$$\mathbf{q}_p[\phi] := \mathrm{Re} \left(\int_{\omega_1}^{\omega_2} \chi(\omega) \, e^{i\phi(\omega)} \, \hat{\mathbf{q}}_p(\omega) \, \mathrm{e}^{-\mathrm{i}\omega\cdot} \, \mathrm{d}\omega \right).$$

Proposition 2.1. *ϕ_{sym} satisfies the following optimization problem:*

$$\inf_{\nu \in \mathbb{R}^{+*}, \phi \in L^\infty([\omega_1, \omega_2])} \| (\mathbb{S} - \nu \mathbb{J}) \, \mathbf{q}_p[\phi] \|_{L^2(\mathbb{R}, \mathbb{C}^N)}$$

As $\mathbb{J}$ is an isometry, roughly speaking, this proposition says that the input signal $\mathbf{q}_p[\phi_{sym}]$ is close (for the L^2 norm) to an eigenfunction of the operator $\mathbb{J}\mathbb{S}$ associated to a positive eigenvalue.

The second one is related to the well-known time-reversal experiment: a time-reversed wave back-propagates towards the source. In this sense, the time-reversed Green function G emitted at s_p is some kind of optimal space-time focusing wave. We have checked that for high ω, the phases ϕ_{sym} given by (4) become close to those of the frequency components of the measures of the time-reversed Green function.

The last arguments are numerical experiments which confirm these focusing properties. In particular, we measured the focusing quality of (3) by means of an energy criterion. We compute the ratio of the local acoustic energy contained in a box which surrounds the obstacle s_p by the total energy sent by the transducers during the emission. In the two following figures, we compare this ratio for input signals $\mathbf{q}_p$ constructed with the time reversed Green function emitted at s_p (therefore, these signals require the position of the *pth* obstacle) with those constructed with ϕ_{sym}.

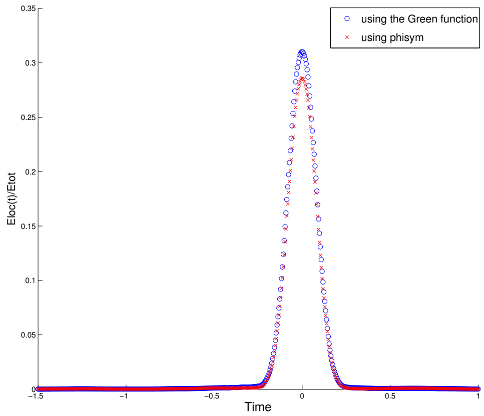

FIGURE 1. Case of two distant scatterers in a diffusive medium

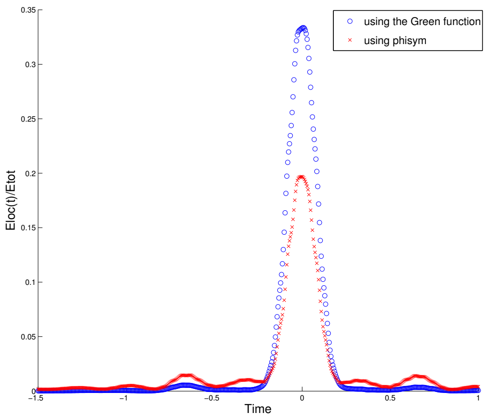

FIGURE 2. Case of two close scatterers in a diffusive medium

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