Multiple scattering of acoustic waves by small sound-soft obstacles in two dimensions: mathematical justification of the Foldy–Lax model

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Abstract

We are concerned with a two-dimensional problem which models the scattering of a time-harmonic acoustic wave by an arbitrary number of sound-soft circular obstacles. Assuming that their radii are small compared to the wavelength, we propose a mathematical justification of different levels of asymptotic models available in the physical literature, including the so-called Foldy–Lax model.

Keywords: Asymptotics, Multiple scattering, Foldy–Lax model, Point scatterers

1. Introduction

We consider the scattering of an acoustic time-harmonic wave by an arbitrary number of sound-soft obstacles located in a homogeneous medium. When the size of the obstacles is small compared with the wavelength, the numerical simulation of such a problem by classical methods (e.g., integral equation techniques or methods based on a Dirichlet-to Neumann map) can become highly time-consuming, particularly when the number of scatterers is large. In this case, the use of an asymptotic model may reduce considerably the numerical cost. Such a model was introduced by Foldy [9] in the middle of the last century to study multiple isotropic scattering in a medium which contains randomly distributed small scatterers. Its use was extended by Lax in [14, 15] to anisotropic or inelastic scattering. Their asymptotic model is based on the fact that the scattered wave can be approximated by a wave emitted by point sources placed at the centers of the scatterers; the amplitudes of the sources are calculated by solving a linear system which represents the interactions between the scatterers (see [18] for an overview). The same system can be derived formally from the usual first-kind integral equation associated with the scattering problem [22].

Nowadays, the Foldy–Lax model is widely used in the community of physicists to approximate scattered waves in numerous physical and numerical applications. These applications concern on one hand deterministic media – see, e.g., [4] (evaluation of the scattering amplitude of a cluster of small obstacles), [10] (attenuation, dispersion and anisotropy caused by multiple scattering), [11] (multiple scattering between an extended scatterer and small obstacles), [17] (localization of targets using time-reversal techniques) – and on the other hand random media – see, e.g., [5] (generalization of well-established derivations of the radiative transfer equation from first principles), [16] (effective wavenumber in a dilute random array of identical scatterers), [23] (higher order Foldy-Lax models). Surprisingly, to our knowledge, there is no mathematical justification of the Foldy–Lax model. The case of one single small scatterer is well understood. As shown in [18] for sound-soft or sound-hard obstacles, using a simple dilation, it can be derived from the low-frequency behavior [12] of the wave scattered by a (non small) obstacle (see also [2] for the case of a small inhomogeneity). To a certain extent, the Foldy–Lax model provides us the path from single scattering to multiple scattering. Our purpose is to propose a rigorous justification of this model and to obtain local error estimates for the two-dimensional problem in the case of circular obstacles. This assumption allows us to represent the scattered wave by Fourier series, which yields explicit and simple proofs. Other more involved

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techniques (based on matched asymptotic expansions or multiple scale methods) have been developed in a slightly different context and could probably be adapted to our problem for non circular obstacles. For instance, the Laplace equation in a bounded domain with small inclusions (with Dirichlet boundary conditions) is dealt with in [19, 21] for the two-dimensional case and [20] for three-dimensional case. Note that in [20], the authors consider two-scale asymptotic expansions by assuming that the size of the obstacles as well as the distance between them are both small parameters. We also refer to [7] which studies the case of a Neumann boundary condition on two small and close obstacles for the Laplace equation.

The paper is organized as follows. In section 2, we show the equations of our scattering problem and follow a physical point of view to introduce different levels of asymptotic models including the Foldy–Lax model. We end this section with the statement of the main result of this paper, which concerns local error estimates between the exact solution and its different levels of approximation. In section 3, we adopt an asymptotic point of view: starting from a representation of the exact solution by means of Fourier series, we derive in a formal way the same asymptotic models. This approach is well adapted to the mathematical justification of the asymptotic models, which is the subject of section 4. For the sake of clarity, some general properties and technical lemmas used in the latter section are collected in section 5. We conclude in section 6 by some comments about the possible generalizations of the work proposed in the present paper.

2. A physical point of view

We are concerned with a two-dimensional problem which models the scattering of a given time-harmonic acoustic wave of circular frequency ω by a family of P disjoint small circular sound-soft obstacles $O_1^\varepsilon, \dots, O_P^\varepsilon$ located in a homogeneous medium filling the whole plane $\mathbb{R}^2$. We denote by $s_1, \dots, s_P$ their respective centers and $r_1^\varepsilon, \dots, r_P^\varepsilon$ their respective radii. For the sake of simplicity, we consider non-dimensional equations, which amounts to choosing a constant celerity $c = 1$ in the propagative medium. We suppose that the radii of the obstacles are small compared to the wavelength $2\pi/\omega$ and are all of the same order of magnitude represented by a small positive parameter ε, i.e.,

$$\omega r_p^\varepsilon = O(\varepsilon) \quad \text{for } p = 1, \dots, P. \tag{1}$$

Let w be a given incident field, solution to

$$\Delta w + \omega^2 w = 0 \quad \text{in } \mathbb{R}^2. \tag{2}$$

The associated scattered field u^ε is the solution to

$$\Delta u^\varepsilon + \omega^2 u^\varepsilon = 0 \quad \text{in } \mathbb{R}^2 \setminus \textstyle\bigcup_{p=1}^P \overline{O_p^\varepsilon}, \tag{3}$$

$$u^\varepsilon = -w \quad \text{on } \textstyle\bigcup_{p=1}^P \partial O_p^\varepsilon, \tag{4}$$

which satisfies the usual Sommerfeld radiation condition

$$\frac{\partial u^\varepsilon}{\partial |x|} - i\omega\, u^\varepsilon = O(|x|^{-3/2}) \quad \text{as } |x| \to \infty. \tag{5}$$

In order to approximate the solution to the above system of equations, we define below a family of asymptotic models which are based on the fact that in the case of one single small scatterer ($P = 1$), the scattered field is similar to the field emitted by a point source. More precisely, u^ε can be approximated (see [18]) by

$$u^\varepsilon(x) \approx \sigma_1^\varepsilon\, w(s_1)\, G(x - s_1),$$

where $G(x) = H_0^{(1)}(\omega |x|)/4i$ is the outgoing Green's function of the Helmholtz equation ($H_0^{(1)}$ is the Hankel function of the first kind and order 0, see [1]) and σ_1^ε is the reflection coefficient of the scatterer, which is given by $\sigma_1^\varepsilon := -4i/H_0^{(1)}(\omega r_1^\varepsilon)$ for a circular obstacle. Indeed it is readily seen that the above function satisfies (3) and (5), as well as the boundary condition $u^\varepsilon = -w(s_1)$ on ∂O_1^ε, which actually approximates (4) since $w(x) - w(s_1) = O(\varepsilon)$ for all $x \in \partial O_1^\varepsilon$ by virtue of our assumption (1).

For several obstacles ($P > 1$), we can consider different levels of approximation $u^{\varepsilon,0}, u^{\varepsilon,1}, \dots, u^{\varepsilon,\infty}$ of u^ε which consist in superpositions of the form

$$u^{\varepsilon,k}(x) := \sum_{p=1}^{P} \sigma_p^\varepsilon \, w_p^{\varepsilon,k} \, G(x - s_p) \quad \text{where } \sigma_p^\varepsilon := -4i/H_0^{(1)}(\omega r_p^\varepsilon) \tag{6}$$

and $w_p^{\varepsilon,k}$ represents different approximations of an "exciting field" on the p-th scatterer. In the simplest model ($k = 0$), we choose

$$w_p^{\varepsilon,0} := w(s_p) \quad \text{for } p = 1, \dots, P,$$

which amounts to the well-known Born approximation [17] where the interactions between the obstacles are neglected, since in this case, $u^{\varepsilon,0}$ is nothing but a superposition of single scattering approximations. The case $k = \infty$ corresponds to the so-called Foldy–Lax model [9, 14, 15], which takes into account these interactions. In this case, the exciting field for one given obstacle is the superposition of the incident field and the waves scattered by all the other obstacles, i.e.,

$$w_p^{\varepsilon,\infty} := w(s_p) + \sum_{q \neq p} \sigma_q^\varepsilon \, w_q^{\varepsilon,\infty} \, G(s_p - s_q) \quad \text{for } p = 1, \dots, P. \tag{7}$$

If we denote by $W^{\varepsilon,\infty}$ and W the vectors of $\mathbb{C}^P$ with components $w_p^{\varepsilon,\infty}$ and $w(s_p)$ respectively, this coupling between the exciting fields can be written equivalently as

$$(\mathbb{I} + \mathbb{M}^\varepsilon) \, W^{\varepsilon,\infty} = W, \tag{8}$$

where $\mathbb{M}^\varepsilon$ is the $P \times P$ matrix defined by

$$\mathbb{M}_{pq}^\varepsilon := -\sigma_q^\varepsilon \, G(s_p - s_q) \ \text{ if } q \neq p \quad \text{and} \quad \mathbb{M}_{pp}^\varepsilon := 0.$$

Between the cases $k = 0$ and $k = \infty$, one can consider intermediate models which take into account the successive reflections between the scatterers. Indeed, instead of (7), the exciting field is defined recursively by

$$w_p^{\varepsilon,k+1} := w(s_p) + \sum_{q \neq p} \sigma_q^\varepsilon w_q^{\varepsilon,k} G(s_p - s_q) \quad \text{for } p = 1, \dots, P.$$

It is readily seen that this relation amounts to approximating the inverse of operator $\mathbb{I} + \mathbb{M}^\varepsilon$ involved in (8) by a truncated Neumann series, so that we can summarize these different models by the formula

$$W^{\varepsilon,k} := \sum_{\ell=0}^{k} (-\mathbb{M}^\varepsilon)^\ell \, W \quad \text{for } k = 0, 1, \dots, \infty. \tag{9}$$

The aim of this paper is to prove the following error estimates between the solution u^ε of our initial problem (3)–(5) and the different levels of approximations $u^{\varepsilon,k}$ given by (6) and (9).

Theorem 1. *For every compact subset K of $\mathbb{R}^2 \setminus \bigcup_{p=1}^{P} \{s_p\}$ and every $s \geq 0$, there exists a constant $C_{K,s} > 0$ independent of ε such that for ε small enough,*

$$\left\| u^\varepsilon - u^{\varepsilon,k} \right\|_{H^s(K)} \leq \begin{cases} \left| \dfrac{C_{K,s}}{\log \varepsilon} \right|^{k+2} & \textit{if } k \in \mathbb{N}, \\[2ex] \dfrac{C_{K,s} \ \varepsilon}{\left| \log \varepsilon \right|} & \textit{if } k = \infty, \end{cases}$$

where $\| \cdot \|_{H^s(K)}$ denotes the norm of the usual Sobolev space $H^s(K)$.

This result tells us that for the Foldy–Lax model ($k = \infty$), the magnitude of the approximation error is the same as for the single scattering problem (see, e.g. [18]), whereas for the intermediate models ($k \in \mathbb{N}$), it corresponds to the first neglected reflections between the scatterers, that is, the reflections of order $k + 1$. For instance, if $k = 1$, the wave scattered by one of the scatterer is $O((\log\varepsilon)^{-1})$ so that the first reflections of this wave by the other scatterers are $O((\log\varepsilon)^{-2})$ and the magnitude of the approximation error is $O((\log\varepsilon)^{-3})$ which corresponds to second-order reflections. However it is important to notice that the constant $C_{K,s}$ involved in these estimates depends on the layout of the scatterers. In particular, $C_{K,s}$ is likely to be an increasing function of the scatterer density. For a high density, $C_{K,s}$ may be large, which means that the Foldy–Lax approximation will be valid for very small scatterers. To a certain extent, the above theorem only tells us that the Foldy–Lax model becomes efficient for small enough scatterers, but it does not give us a quantitative interpretation of the expression "small enough". Unfortunately, we did not find a quantitative estimate of the $C_{K,s}$ parameter with respect to the layout of the scatterers. This is a challenging but open question. Maybe the two-scale approach developed in [20] in a different context could be used for our problem.

3. An asymptotic point of view

In this section, we revisit the asymptotic models defined by (6) and (9) by a more rigorous approach which will allow us to prove Theorem 1 in section 4. The idea is to rewrite the initial problem (3)–(5) in an equivalent form using standard tools for multiple scattering and Fourier series.

3.1. Rewriting the scattering problem

In a first step, we consider a representation of u^ε which transforms our *multiple scattering* problem into a family of P coupled *single scattering* problems. This representation consists in splitting u^ε as follows (see [6]):

$$u^\varepsilon = \sum_{p=1}^{P} u_p^\varepsilon, \tag{10}$$

where $u_1^\varepsilon, u_2^\varepsilon, \dots, u_P^\varepsilon$ are the outgoing (in the sense of (5)) solutions to

$$\Delta u_p^\varepsilon + \omega^2 u_p^\varepsilon = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{O_p^\varepsilon}, \tag{11}$$

$$u_p^\varepsilon = -w - \sum_{q\neq p} u_q^\varepsilon \quad \text{on } \partial O_p^\varepsilon, \tag{12}$$

for $p = 1, \dots, P$. This representation of u^ε makes clear the notion of *exciting field* introduced in the previous section. Indeed each function u_p^ε represents the wave scattered by the p-th obstacle illuminated by the exciting field $w + \sum_{q\neq p} u_q^\varepsilon$.

The next step consists in taking advantage of the particular shape of the scatterers by using Fourier series expansions of the single scattering fields u_p^ε. We briefly recall here the main results; all details can be found in [18]. We equip $\mathbb{R}^2$ with a Cartesian coordinate system (O, e_1, e_2) and define for each $p = 1, \dots, P$ the local polar coordinates by (ρ_p, θ_p) where $\rho_p := |x - s_p|$ and $\theta_p \in [0, 2\pi)$ is the angle between e_1 and $x - s_p$. We denote by $x(\rho_p, \theta_p)$ or simply (ρ_p, θ_p) (if there is no ambiguity) a point of $\mathbb{R}^2$ defined by its local polar coordinates, so that a function of x can be considered equivalently as a function of (ρ_p, θ_p) without change of notation.

As u_p^ε is an outgoing solution to the homogeneous Helmholtz equation outside $\overline{O_p^\varepsilon}$, it has a modal decomposition on the Hankel functions $H_m^{(1)}$ (see [1]):

$$u_p^\varepsilon(x) = \sum_{m\in\mathbb{Z}} \frac{c_{p,m}^\varepsilon}{H_m^{(1)}(\omega r_p^\varepsilon)} \mathcal{H}_{p,m}(x) \ \text{ for all } x \in \mathbb{R}^2 \setminus \overline{O_p^\varepsilon} \quad \text{where} \quad c_{p,m}^\varepsilon := \frac{1}{2\pi}\int_0^{2\pi} u_p^\varepsilon(r_p^\varepsilon, \theta_p)\, e^{-im\theta_p}\, d\theta_p \tag{13}$$

is the m-th Fourier coefficient of $u_p^\varepsilon(r_p^\varepsilon, \cdot)$ and $\mathcal{H}_{p,m}(x)$ denotes the local outgoing cylindrical wave-functions associated with the p-th scatterer defined by

$$\mathcal{H}_{p,m}(x) := H_m^{(1)}(\omega\rho_p)\, e^{im\theta_p} \quad \text{for } x \neq s_p.$$

Similarly, w is a solution to the homogeneous Helmholtz equation (2), thus it has an analogous modal decomposition on the Bessel functions J_m [1]. We shall use such a decomposition by choosing $r_0 > 0$ such that ωr_0 is smaller than the smallest zero of J_0 (which ensures that $J_m(\omega r_0) \neq 0$ for all $m \in \mathbb{Z}$). Then we have

$$w(x) = \sum_{m\in\mathbb{Z}} \frac{d_{p,m}}{J_m(\omega r_0)} J_m(\omega\rho_p)\, e^{im\theta_p} \ \text{ for all } x \in \mathbb{R}^2 \quad \text{where} \quad d_{p,m} := \frac{1}{2\pi}\int_0^{2\pi} w(r_0, \theta_p)\, e^{-im\theta_p}\, d\theta_p \tag{14}$$

is the m-th Fourier coefficient of $w(r_0, \cdot)$.

The idea is to use the Fourier coefficients $c^{\varepsilon}_{p,m}$ as new unknowns, more precisely to express the coupling boundary condition (12) by means of the above decompositions (13) and (14), which yields the relation between $c^{\varepsilon}_{p,m}$ and the Fourier coefficients $d_{p,m}$ of the incident field. Thanks to the Graf's addition formula [1]

$$\mathcal{H}_{q,m}(x) = \sum_{n\in\mathbb{Z}} \mathcal{H}_{q,m-n}(s_p)\, J_n(\omega\rho_p) e^{in\theta_p} \quad \text{for } p \neq q \text{ and } \rho_p < \left|s_p - s_q\right|, \tag{15}$$

condition (12) becomes

$$\sum_{m\in\mathbb{Z}} c^{\varepsilon}_{p,m}\, e^{im\theta_p} + \sum_{q\neq p}\sum_{m\in\mathbb{Z}} \frac{c^{\varepsilon}_{q,m}}{H^{(1)}_m(\omega r^{\varepsilon}_q)} \sum_{n\in\mathbb{Z}} \mathcal{H}_{q,m-n}(s_p)\, J_n(\omega r^{\varepsilon}_p)\, e^{in\theta_p} = -\sum_{m\in\mathbb{Z}} \frac{d_{p,m}}{J_m(\omega r_0)}\, J_m(\omega r^{\varepsilon}_p)\, e^{im\theta_p}, \tag{16}$$

for all $\theta_p \in [0, 2\pi)$. Interchanging the order of summation in the second term of the left-hand side (which will be justified in the next section, see Remark 3) and using the fact that a Fourier series vanishes for all $\theta_p \in [0, 2\pi)$ if and only if all the Fourier coefficients vanish, this equation writes equivalently as

$$c^{\varepsilon}_{p,m} + J_m(\omega r^{\varepsilon}_p) \sum_{q\neq p}\sum_{n\in\mathbb{Z}} \frac{\mathcal{H}_{q,n-m}(s_p)}{H^{(1)}_n(\omega r^{\varepsilon}_q)}\, c^{\varepsilon}_{q,n} = -\frac{J_m(\omega r^{\varepsilon}_p)}{J_m(\omega r_0)}\, d_{p,m} \quad \text{for all } m \in \mathbb{Z},$$

which can be expressed in a more concise form as

$$(\mathbb{I} + \mathbb{K}^{\varepsilon})\, c^{\varepsilon} = f^{\varepsilon}, \tag{17}$$

where c^{ε} is the vector $(c^{\varepsilon}_1, \ldots, c^{\varepsilon}_P)^{\top}$ whose components c^{ε}_p are the sequences of Fourier coefficients $(c^{\varepsilon}_{p,m})_{m\in\mathbb{Z}}$. Moreover $\mathbb{K}^{\varepsilon}$ can be defined in matrix form by

$$\mathbb{K}^{\varepsilon} := \begin{pmatrix} 0 & \mathbb{K}^{\varepsilon}_{12} & \ldots & \mathbb{K}^{\varepsilon}_{1P} \\ \mathbb{K}^{\varepsilon}_{21} & 0 & \ldots & \mathbb{K}^{\varepsilon}_{2P} \\ \vdots & \ldots & \ddots & \vdots \\ \mathbb{K}^{\varepsilon}_{P1} & \mathbb{K}^{\varepsilon}_{P2} & \ldots & 0 \end{pmatrix}$$

where each term $\mathbb{K}^{\varepsilon}_{pq}$, for $p \neq q$, is an operator which represents the action of the q-th obstacle on the p-th obstacle, defined by

$$\mathbb{K}^{\varepsilon}_{pq}\, c_q := \left(\sum_{n\in\mathbb{Z}} \mathbb{K}^{\varepsilon}_{pq,mn}\, c_{q,n}\right)_{m\in\mathbb{Z}} \quad \text{where} \quad \mathbb{K}^{\varepsilon}_{pq,mn} := \frac{\mathcal{H}_{q,n-m}(s_p)\, J_m(\omega r^{\varepsilon}_p)}{H^{(1)}_n(\omega r^{\varepsilon}_q)}.$$

Finally the right-hand side f^{ε} of (17) is the vector $(f^{\varepsilon}_1, \ldots, f^{\varepsilon}_P)^{\top}$ whose components f^{ε}_p are the sequences

$$f^{\varepsilon}_p = \left(f^{\varepsilon}_{p,m}\right)_{m\in\mathbb{Z}} := -\left(\frac{J_m(\omega r^{\varepsilon}_p)}{J_m(\omega r_0)}\, d_{p,m}\right)_{m\in\mathbb{Z}}.$$

Note that f^{ε}_p is the sequence of Fourier coefficients of $-w(r^{\varepsilon}_p, \cdot)$. Indeed from (14), we have

$$-w(r^{\varepsilon}_p, \theta_p) = \sum_{m\in\mathbb{Z}} f^{\varepsilon}_{p,m}\, e^{im\theta_p}. \tag{18}$$

Thanks to (13) and (14), the linear equation (17) is clearly equivalent to the family of P coupled problems (11)–(12), hence also to our initial problem (3)–(5). It provides us a numerical strategy for multiple scattering (see [3] for a detailed analysis). Here we shall use this equivalent formulation to derive the asymptotic models exhibited in section 2 and to prove Theorem 1.

3.2. *Asymptotic models*

Using the asymptotic behavior (28) of Bessel and Hankel functions for small arguments and the fact that $w(s_p) = d_{p,0}/J_0(\omega r_0)$ (which follows from (14)), it is easy to see that for $p \neq q$,

$$\mathbb{K}^{\varepsilon}_{pq,mn} = \begin{cases} \dfrac{\mathcal{H}_{q,0}(s_p)}{H_0^{(1)}(\omega r_q^{\varepsilon})} + O\left(\dfrac{\varepsilon^2}{\log \varepsilon}\right) = O\left(\dfrac{1}{\log \varepsilon}\right) & \text{if } (m,m) = (0,0), \\ O(\varepsilon) & \text{else,} \end{cases}$$

and

$$f^{\varepsilon}_{p,m} = \begin{cases} -w(s_p) + O(\varepsilon^2) & \text{if } m = 0, \\ O(\varepsilon) & \text{else.} \end{cases}$$

So the dominant coefficients of $\mathbb{K}^{\varepsilon}_{pq}$ and f^{ε}_p are reached respectively for $(m,n) = (0,0)$ and $m = 0$. As a consequence, a formal approximation of order ε of (17) is given by

$$(\mathbb{I} + \widetilde{\mathbb{K}}^{\varepsilon})\, c^{\varepsilon} = f^0, \tag{19}$$

where $\widetilde{\mathbb{K}}^{\varepsilon}$ and f^0 are defined as $\mathbb{K}^{\varepsilon}$ and f^{ε} by replacing $\mathbb{K}^{\varepsilon}_{pq,mn}$ and $f^{\varepsilon}_{p,m}$ by

$$\widetilde{\mathbb{K}}^{\varepsilon}_{pq,mn} := \delta_{m,0}\, \delta_{n,0}\, \frac{\mathcal{H}_{q,0}(s_p)}{H_0^{(1)}(\omega r_q^{\varepsilon})} \quad \text{and} \quad f^0_{p,m} := -\delta_{m,0}\, w(s_p) = -\delta_{m,0}\, \frac{1}{J_0(\omega r_0)}\, d_{p,0}.$$

The above approximate system (19) is equivalent to the Foldy–Lax model (8). Indeed, following the notations introduced in section 2, let $c^{\varepsilon,k}$, for $k \in \mathbb{N}$, denote the approximation of the solution to (19) obtained by a truncated Neumann series of the inverse of $\mathbb{I} + \widetilde{\mathbb{K}}^{\varepsilon}$, and $c^{\varepsilon,\infty}$, the exact solution, i.e.,

$$c^{\varepsilon,k} := \sum_{\ell=0}^{k} (-\widetilde{\mathbb{K}}^{\varepsilon})^{\ell}\, f^0 \quad \text{for } k = 0, 1, \ldots, \infty,$$

which shows in particular that $c^{\varepsilon,k}_{p,m} = 0$ for all p and $m \neq 0$ in view of the above definition of $\widetilde{\mathbb{K}}^{\varepsilon}$ and f^0. The acoustic field associated with these Fourier coefficients by (13) is then given by

$$u^{\varepsilon,k}(x) := \sum_{p=1}^{P} \frac{c^{\varepsilon,k}_{p,0}}{H_0^{(1)}(\omega r_p^{\varepsilon})}\, H_0^{(1)}(\omega |x - s_p|) \quad \text{for } k = 0, 1, \ldots, \infty. \tag{20}$$

This expression is nothing but the field (6) obtained by the physical approach, which is readily verified by noticing that $\widetilde{\mathbb{K}}^{\varepsilon}_{pq,00} = \mathbb{M}^{\varepsilon}_{pq}$ and $f^0_{p,0} = -W_p$ (thanks to the definitions of the reflection coefficients σ^{ε}_p and of the Green's function), thus $c^{\varepsilon,k}_{p,0} = -W^{\varepsilon,k}_p$.

4. A mathematical point of view

4.1. *Functional framework*

As we shall see in the sequel, the natural function space for each component c^{ε}_p of the solution c^{ε} to (17) is $\ell^2(\mathbb{Z})$, that is, the Hilbert space composed of the sequences $c_p = (c_{p,m})_{m\in\mathbb{Z}}$ such that $\sum_{m\in\mathbb{Z}} |c_{p,m}|^2 < \infty$, equipped with the following inner product and associated norm:

$$(c_p, c'_p)_{\ell^2(\mathbb{Z})} := \sum_{m\in\mathbb{Z}} c_{p,m}\, \overline{c'_{p,m}} \quad \text{and} \quad \|c_p\|_{\ell^2(\mathbb{Z})} = \left\{ (c_p, c_p)_{\ell^2(\mathbb{Z})} \right\}^{1/2}.$$

Moreover, for $s > 0$, we denote by $h^s(\mathbb{Z})$ the subspace of $\ell^2(\mathbb{Z})$ composed of the sequences $c_p = (c_{p,m})_{m\in\mathbb{Z}}$ such that $\|c_p\|^2_{h^s(\mathbb{Z})} := \sum_{m\in\mathbb{Z}} (1+m^2)^s |c_{p,m}|^2$ is finite.

Proposition 2. *On the assumption that the scatterers are disjoint, $\mathbb{K}^\varepsilon$ is a compact operator in $\ell^2(\mathbb{Z})^P$.*

Proof. Let us first prove that $\mathbb{K}^\varepsilon \in \mathcal{L}(\ell^2(\mathbb{Z})^P)$, i.e., that $\mathbb{K}^\varepsilon$ is a bounded operator in $\ell^2(\mathbb{Z})^P$. It is enough to verify that each operator $\mathbb{K}^\varepsilon_{pq}$ is bounded in $\ell^2(\mathbb{Z})$. Instead of working with $\mathbb{K}^\varepsilon_{pq}$, we define its formal adjoint by

$$\mathbb{L}^\varepsilon_{qp}\, c_p := \left(\sum_{n\in\mathbb{Z}} \mathbb{L}^\varepsilon_{qp,mn}\, c_{p,n} \right)_{m\in\mathbb{Z}} \qquad \text{with} \qquad \mathbb{L}^\varepsilon_{qp,mn} := \overline{\mathbb{K}^\varepsilon_{pq,nm}} = \frac{\overline{\mathcal{H}_{q,m-n}(s_p)\, J_n(\omega r^\varepsilon_p)}}{\overline{H^{(1)}_m(\omega r^\varepsilon_q)}}.$$

Let $c_p \in \ell^2(\mathbb{Z})$ and $m \in \mathbb{Z}$. Using the Cauchy-Schwarz inequality, we have

$$\left| \sum_{n\in\mathbb{Z}} \mathbb{L}^\varepsilon_{qp,mn}\, c_{p,n} \right|^2 \leq \frac{\sum_{n\in\mathbb{Z}} \left| \mathcal{H}_{q,m-n}(s_p)\, J_n(\omega r^\varepsilon_p) \right|^2}{\left| H^{(1)}_m(\omega r^\varepsilon_q) \right|^2} \, \left\| c_p \right\|^2_{\ell^2(\mathbb{Z})}. \tag{21}$$

The main advantage of working with $\mathbb{L}^\varepsilon_{qp}$ instead of $\mathbb{K}^\varepsilon_{pq}$ lies in the observation that the sum in the right-hand side is nothing but the sum of the squared moduli of the Fourier coefficients which appear in the Graf's addition formula (15) for $x = (r^\varepsilon_p, \theta_p)$. Hence the Parseval identity yields

$$\sum_{n\in\mathbb{Z}} \left| \mathcal{H}_{q,m-n}(s_p)\, J_n(\omega r^\varepsilon_p) \right|^2 = \left\| \mathcal{H}_{q,m}(r^\varepsilon_p, \cdot) \right\|^2_{L^2(0,2\pi)} := \frac{1}{2\pi} \int_0^{2\pi} \left| \mathcal{H}_{q,m}(r^\varepsilon_p, \theta_p) \right|^2 d\theta_p.$$

Summing over $m \in \mathbb{Z}$ in (21), we obtain

$$\left\| \mathbb{L}^\varepsilon_{qp}\, c_p \right\|^2_{\ell^2(\mathbb{Z})} \leq \left(\sum_{m\in\mathbb{Z}} \frac{\left\| \mathcal{H}_{q,m}(r^\varepsilon_p, \cdot) \right\|^2_{L^2(0,2\pi)}}{\left| H^{(1)}_m(\omega r^\varepsilon_q) \right|^2} \right) \left\| c_p \right\|^2_{\ell^2(\mathbb{Z})}. \tag{22}$$

We prove in section 5 (see Lemma 8) that for disjoints obstacles, the series in the right-hand side is convergent, so $\mathbb{L}^\varepsilon_{qp} \in \mathcal{L}(\ell^2(\mathbb{Z}))$. This justifies the fact that $\mathbb{L}^\varepsilon_{qp}$ and $\mathbb{K}^\varepsilon_{pq}$ are adjoint to each other, so $\mathbb{K}^\varepsilon_{pq}$ is also bounded in $\ell^2(\mathbb{Z})$.

Using the same arguments, we can verify that for all $s > 0$, $\mathbb{L}^\varepsilon_{qp}$ is bounded from $\ell^2(\mathbb{Z})$ to the space $h^s(\mathbb{Z})$ defined above. Indeed, instead of (22), we obtain

$$\left\| \mathbb{L}^\varepsilon_{qp}\, c_p \right\|^2_{h^s(\mathbb{Z})} \leq \left(\sum_{m\in\mathbb{Z}} (1+m^2)^s\, \frac{\left\| \mathcal{H}_{q,m}(r^\varepsilon_p, \cdot) \right\|^2_{L^2(0,2\pi)}}{\left| H^{(1)}_m(\omega r^\varepsilon_q) \right|^2} \right) \left\| c_p \right\|^2_{\ell^2(\mathbb{Z})},$$

where Lemma 8 tells us that the series in the right-hand side is convergent. As $h^s(\mathbb{Z})$ is compactly embedded in $\ell^2(\mathbb{Z})$, the compactness of $\mathbb{L}^\varepsilon_{qp}$ follows. Hence its adjoint is also compact in $\ell^2(\mathbb{Z})$. □

Remark 3. It is easy to see that the convergence of the series which appears in (22) allows us to interchange the order of summation of the double series in (16).

Proposition 4. *On the assumption that the scatterers are disjoint, the multiple scattering problem* (17) *as well as the Foldy–Lax approximation* (19) *are well-posed in $\ell^2(\mathbb{Z})^P$.*

Proof. Proposition 2 tells us that (17) is a Fredholm equation in the space $\ell^2(\mathbb{Z})^P$ (note that the right-hand side f^ε belongs to $\ell^2(\mathbb{Z})^P$ since each f^ε_p is the sequence of Fourier coefficients of $-w(r^\varepsilon_p, \cdot)$, see (18)). By virtue of Fredholm's alternative, the well-posedness of this equation follows from the uniqueness of the solution to our initial problem (3)–(5). Indeed, it is well-known that this problem has a unique solution $u^\varepsilon \in H^1_{\mathrm{loc}}(\Omega^\varepsilon)$ where $\Omega^\varepsilon := \mathbb{R}^2 \setminus \bigcup_{p=1}^P \overline{\mathcal{O}^\varepsilon_p}$. Suppose then that $c^\varepsilon \in \ell^2(\mathbb{Z})^P$ is a solution to $(\mathbb{I} + \mathbb{K}^\varepsilon)\, c^\varepsilon = 0$. We have seen in the proof of Proposition 2 that $\mathbb{K}^\varepsilon$ is bounded from $\ell^2(\mathbb{Z})^P$ to $h^s(\mathbb{Z})^P$. Hence $c^\varepsilon = -\mathbb{K}^\varepsilon\, c^\varepsilon$ belongs to $h^s(\mathbb{Z})^P$ for all $s > 0$. This shows that the associated function u^ε defined by (10) and (13) is a smooth solution to (3)–(5) with $w = 0$. So $u^\varepsilon = 0$, which implies that $c^\varepsilon = 0$.

For the Foldy–Lax approximation (19), the proof is very different. As mentioned in section 3.2, this equation is equivalent to the finite dimensional system (8). Noticing that $\left|H_0^{(1)}(\cdot)\right|$ is decreasing on $(0, +\infty)$ (which can be easily deduced from formula (31)), we infer that

$$\left|\mathbb{M}^\varepsilon_{pq}\right| = \left|\sigma^\varepsilon_q\, G(s_p - s_q)\right| = \frac{\left|H_0^{(1)}(\omega|s_p - s_q|)\right|}{\left|H_0^{(1)}(\omega r^\varepsilon_q)\right|} < 1 \quad \text{for all } p, q \in \{1, \ldots, P\} \text{ with } p \neq q.$$

As a consequence, $\|\mathbb{M}^\varepsilon\|_\infty := \sup_{p,q\in\{1,\ldots,P\}} \left|\mathbb{M}^\varepsilon_{pq}\right| < 1$, thus $\mathbb{I} + \mathbb{M}^\varepsilon$ is invertible. □

4.2. Error analysis

The proof of Theorem 1 is mainly based on the following error estimates.

Proposition 5. *There exits a constant $C > 0$ such that for ε small enough,*

$$\|\mathbb{K}^\varepsilon\|_{\mathcal{L}(\ell^2(\mathbb{Z})^P)} \leq \frac{C}{\left|\log\varepsilon\right|}, \tag{23}$$

$$\left\|\mathbb{K}^\varepsilon - \widetilde{\mathbb{K}}^\varepsilon\right\|_{\mathcal{L}(\ell^2(\mathbb{Z})^P)} \leq C\,\varepsilon, \tag{24}$$

$$\left\|f^\varepsilon - f^0\right\|_{\ell^2(\mathbb{Z})^P} \leq C\,\varepsilon. \tag{25}$$

Proof. For inequality (23), we follow the same approach as in the proof of Proposition 2. The announced estimate simply results from (22) and Lemma 9.

To prove formula (24), we use similar arguments. First notice that the adjoint of the finite rank operator $\widetilde{\mathbb{K}}^\varepsilon_{pq}$ is clearly the operator $\widetilde{\mathbb{L}}^\varepsilon_{qp}$ of $\mathcal{L}(\ell^2(\mathbb{Z}))$ defined by

$$\widetilde{\mathbb{L}}^\varepsilon_{qp}\, c_p := \left(\delta_{m,0}\, \frac{\overline{\mathcal{H}_{q,0}(s_p)}}{H_0^{(1)}(\omega r^\varepsilon_q)}\, c_{p,0}\right)_{m\in\mathbb{Z}}.$$

Hence, for $c_p \in \ell^2(\mathbb{Z})$, we have

$$\left\|\left(\mathbb{L}^\varepsilon_{qp} - \widetilde{\mathbb{L}}^\varepsilon_{qp}\right) c_p\right\|^2_{\ell^2(\mathbb{Z})} = \sum_{|m|>0} \left|\left(\mathbb{L}^\varepsilon_{qp}\, c_p\right)_m\right|^2 + \left|\left((\mathbb{L}^\varepsilon_{qp} - \widetilde{\mathbb{L}}^\varepsilon_{qp})\, c_p\right)_0\right|^2.$$

On one hand, following the same lines as in the proof of Proposition 2 and using Lemma 9, we obtain

$$\sum_{|m|>0} \left|\left(\mathbb{L}^\varepsilon_{qp}\, c_p\right)_m\right|^2 \leq \left(\sum_{|m|>0} \frac{\left\|\mathcal{H}_{q,m}(r^\varepsilon_p, \cdot)\right\|^2_{L^2(0,2\pi)}}{\left|H_m^{(1)}(\omega r^\varepsilon_q)\right|^2}\right) \left\|c_p\right\|^2_{\ell^2(\mathbb{Z})} \leq C\,\varepsilon^2 \left\|c_p\right\|^2_{\ell^2(\mathbb{Z})},$$

for ε small enough. On the other hand, applying the Cauchy-Schwarz inequality, we get

$$\left|\left((\mathbb{L}^\varepsilon_{qp} - \widetilde{\mathbb{L}}^\varepsilon_{qp})\, c_p\right)_0\right|^2 \leq \frac{\sum_{|n|>0} \left|\mathcal{H}_{q,-n}(s_p)\, J_n(\omega r^\varepsilon_p)\right|^2 + \left|(J_0(\omega r^\varepsilon_p) - 1)\, \mathcal{H}_{q,0}(s_p))\right|^2}{\left|H_0^{(1)}(\omega r^\varepsilon_q)\right|^2} \left\|c_p\right\|^2_{\ell^2(\mathbb{Z})}.$$

Again thanks to the Graf's addition formula (15) and the Parseval identity, we see that the numerator of the fraction in the right-hand side is nothing but $\left\|\mathcal{H}_{q,0}(r^\varepsilon_p, \cdot) - \mathcal{H}_{q,0}(s_p)\right\|^2_{L^2(0,2\pi)}$. As $\mathcal{H}_{q,0}(x)$ is a smooth function near $x = s_p$, this quantity is $O(\varepsilon^2)$ for small ε. Noticing that $H_0^{(1)}(\omega r^\varepsilon_q) = O(|\log\varepsilon|)$ (see (28)), we conclude that

$$\left|\left((\mathbb{L}^\varepsilon_{qp} - \widetilde{\mathbb{L}}^\varepsilon_{qp})\, c_p\right)_0\right|^2 \leq C\,\varepsilon^2 \left|\log\varepsilon\right|^{-2} \left\|c_p\right\|^2_{\ell^2(\mathbb{Z})}.$$

which completes the proof of (24).

It remains to prove (25). From the respective definitions of f_p^ε and f_p^0, we have

$$\left\|f_p^\varepsilon - f_p^0\right\|^2_{\ell^2(\mathbb{Z})} = \sum_{m\in\mathbb{Z}} A^\varepsilon_{p,m} \left|d_{p,m}\right|^2 \quad \text{where} \quad A^\varepsilon_{p,0} := \left|\frac{J_0(\omega r_p^\varepsilon) - 1}{J_0(\omega r_0)}\right|^2 \quad \text{and} \quad A^\varepsilon_{p,m} := \left|\frac{J_m(\omega r_p^\varepsilon)}{J_m(\omega r_0)}\right|^2 \quad \text{if } m \neq 0.$$

Using the asymptotic behaviors of J_m for small arguments (see (28)) and for large orders (see (29)), we infer that there exists $C > 0$ independent of m such that

$$A^\varepsilon_{p,0} \leq C \left(\frac{r_p^\varepsilon}{r_0}\right)^4 \quad \text{and} \quad A^\varepsilon_{p,m} \leq C \left(\frac{r_p^\varepsilon}{r_0}\right)^{2m} \quad \text{if } m \neq 0.$$

This shows that the dominant coefficient is obtained for $m = 1$, so $\sup_{m\in\mathbb{Z}} A^\varepsilon_{p,m} \leq C\,\varepsilon^2$. As a consequence,

$$\left\|f_p^\varepsilon - f_p^0\right\|^2_{\ell^2(\mathbb{Z})} \leq C\,\varepsilon^2 \left\|d_p\right\|^2_{\ell^2(\mathbb{Z})},$$

where $\left\|d_p\right\|_{\ell^2(\mathbb{Z})}$ is finite since d_p is the sequence of Fourier coefficients of $w(r_0, \cdot)$ (see (14)). □

Corollary 6. *There exits a constant $C > 0$ such that for ε small enough,*

$$\left\|c^\varepsilon - c^{\varepsilon,k}\right\|_{\ell^2(\mathbb{Z})^P} \leq \begin{cases} \left|\dfrac{C}{\log\varepsilon}\right|^{k+1} & \text{if } k \in \mathbb{N}, \\ C\,\varepsilon & \text{if } k = \infty. \end{cases}$$

Proof. The case $k = \infty$ is a consequence of Proposition 5 using standard approximation results [13, Theorem 10.1]. The case $k \in \mathbb{N}$ then follows by noticing that

$$\left\|c^\varepsilon - c^{\varepsilon,k}\right\|_{\ell^2(\mathbb{Z})^P} \leq \|c^\varepsilon - c^{\varepsilon,\infty}\|_{\ell^2(\mathbb{Z})^P} + \left\|c^{\varepsilon,\infty} - c^{\varepsilon,k}\right\|_{\ell^2(\mathbb{Z})^P} \quad \text{where} \quad \|c^\varepsilon - c^{\varepsilon,\infty}\|_{\ell^2(\mathbb{Z})^P} \leq C\,\varepsilon$$

and

$$\left\|c^{\varepsilon,\infty} - c^{\varepsilon,k}\right\|_{\ell^2(\mathbb{Z})^P} = \left\|\sum_{\ell=k+1}^{\infty} (-\widetilde{\mathbb{K}}^\varepsilon)^\ell f^0\right\|_{\ell^2(\mathbb{Z})^P} \leq \sum_{\ell=k+1}^{\infty} \left\|\widetilde{\mathbb{K}}^\varepsilon\right\|^\ell_{\mathcal{L}(\ell^2(\mathbb{Z})^P)} \left\|f^0\right\|_{\ell^2(\mathbb{Z})^P} \leq C' \left|\frac{C}{\log\varepsilon}\right|^{k+1},$$

where the last inequality derives from (23) and (24). □

We are now able to prove Theorem 1. We first deal with the case $s = 0$, that is, local L^2 error estimates. Let K be a compact subset of $\mathbb{R}^2 \setminus \left(\bigcup_{p=1,P}\{s_p\}\right)$ and $k \in \mathbb{N} \cup \{\infty\}$. For a given $p \in \{1, \ldots, P\}$, there exists $\rho_* > 0$ and $\rho^* > \rho_*$ such that K is contained in the domain $\Xi_p := \left\{x \in \mathbb{R}^2;\ \rho_* \leq |x - s_p| \leq \rho^*\right\}$. Moreover for ε small enough, the obstacle $\mathcal{O}_p^\varepsilon$ is outside Ξ_p. Thanks to formulas (13) and (20) (recall that $c^{\varepsilon,k}_{p,m} = 0$ for all p and $m \neq 0$), Parseval identity yields

$$\frac{1}{2\pi}\int_0^{2\pi} \left|u_p^\varepsilon(\rho_p, \theta_p) - u_p^{\varepsilon,k}(\rho_p, \theta_p)\right|^2 d\theta_p = \sum_{m\in\mathbb{Z}} \left|c^\varepsilon_{p,m} - c^{\varepsilon,k}_{p,m}\right|^2 \left|\frac{H_m^{(1)}(\omega\rho_p)}{H_m^{(1)}(\omega r_p^\varepsilon)}\right|^2 \quad \text{for all } \rho_p \in (\rho_*, \rho^*).$$

Using the same arguments as in the proof of Lemma 9, it is easy to see that there exists some positive constant C such that for ε small enough, we have

$$\left|\frac{H_m^{(1)}(\omega\rho_p)}{H_m^{(1)}(\omega r_p^\varepsilon)}\right|^2 \leq C\,|\log\varepsilon|^{-2} \quad \text{for all } m \in \mathbb{Z} \text{ and } \rho_p \in (\rho_*, \rho^*).$$

Hence,

$$\left\|u_p^\varepsilon - u_p^{\varepsilon,k}\right\|^2_{L^2(K)} \leq \int_{\rho_*}^{\rho^*}\int_0^{2\pi} \left|u_p^\varepsilon(\rho_p, \theta_p) - u_p^{\varepsilon,k}(\rho_p, \theta_p)\right|^2 \rho_p\, d\theta_p d\rho_p \leq C\,|\log\varepsilon|^{-2} \left\|c_p^\varepsilon - c_p^{\varepsilon,k}\right\|^2_{\ell^2(\mathbb{Z})},$$

which shows finally that

$$\left\|u^\varepsilon - u^{\varepsilon,k}\right\|_{L^2(K)} \leq \sum_{p=1}^{P} \left\|u_p^\varepsilon - u_p^{\varepsilon,k}\right\|_{L^2(K)} \leq C \, |\log \varepsilon|^{-1} \, \left\|c^\varepsilon - c^{\varepsilon,k}\right\|_{\ell^2(\mathbb{Z})^P} .$$

The conclusion then follows from Corollary 6.

In order to deal with the case $s > 0$, we simply have to use some classical interior regularity results for second order elliptic equations (see, e.g., [8, Theorem 2, p. 314]). Indeed, noticing that

$$\Delta(u^\varepsilon - u^{\varepsilon,k}) + \omega^2(u^\varepsilon - u^{\varepsilon,k}) = 0 \quad \text{in } K,$$

we infer that for any compact set K' contained in the interior of K, $u^\varepsilon - u^{\varepsilon,k}$ belongs to $H^s(K)$ for all $s > 0$ and there exists $C_s > 0$ (depending only of s, K and K') such that:

$$\left\|u^\varepsilon - u^{\varepsilon,k}\right\|_{H^s(K')} \leq C_s \, \left\|u^\varepsilon - u^{\varepsilon,k}\right\|_{L^2(K)} .$$

This completes the proof of Theorem 1.

5. Technical lemnas

We collect in this section different results about Bessel functions which are used in the paper. First recall some well-known properties (see, e.g., [1]):

- Symmetry relations: for all $n \in \mathbb{N}$,

$$J_{-n}(x) = (-1)^n \, J_n(x) \;\text{ for } x \in \mathbb{R} \quad \text{and} \quad H_{-n}^{(1)}(x) = (-1)^n \, H_n^{(1)}(x) \;\text{ for } x \in (0, +\infty). \tag{26}$$

- Recurrence formulas: for all $n \in \mathbb{Z}$,

$$H_{n+1}^{(1)}(x) = \frac{2n}{x} H_n^{(1)}(x) - H_{n-1}^{(1)}(x) \;\text{ for } x \in (0, +\infty). \tag{27}$$

- Asymptotic behaviors for small arguments: when $n \in \mathbb{N}$ is fixed (see (26) if $n < 0$) and $x \searrow 0$,

$$J_n(x) = \begin{cases} 1 + O(x^2) & \text{if } n = 0, \\ O(x^n) & \text{else,} \end{cases} \quad \text{and} \quad H_n^{(1)}(x) = \begin{cases} \dfrac{2i}{\pi} \log\left(\dfrac{x}{2}\right) + O(1) & \text{if } n = 0, \\ \dfrac{(n-1)!}{i\pi} \left(\dfrac{x}{2}\right)^{-n} + O(x^{-n+1}) & \text{else.} \end{cases} \tag{28}$$

- Asymptotic behaviors for large orders: when $n \to +\infty$ (see (26) for $n \to -\infty$),

$$J_n(x) = \frac{x^n}{2^n \, n!} \left(1 + O\left(\frac{1}{n}\right)\right) \quad \text{uniformly on compact subsets of } [0, +\infty), \tag{29}$$

$$H_n^{(1)}(x) = \frac{2^n \, (n-1)!}{i\pi \, x^n} \left(1 + O\left(\frac{1}{n}\right)\right) \quad \text{uniformly on compact subsets of } (0, +\infty). \tag{30}$$

The following apparently non usual inequality plays an essential role in our estimates.

Lemma 7. *For all integer $n \geq 2$ and $x \in (0, 1)$, we have*

$$\left|\frac{1}{H_n^{(1)}(x)}\right| \leq \frac{2x^{n-2}}{n! \, \left|H_2^{(1)}(x)\right|}.$$

Proof. First notice that for all $n \in \mathbb{N}$ and $x \in (0, +\infty)$, we have

$$\left|H_n^{(1)}(x)\right| \leq \left|H_{n+1}^{(1)}(x)\right|,$$

which can be easily derived from the following formula (see [24, p. 444]):

$$\left|H_n^{(1)}(x)\right|^2 = \frac{8}{\pi^2}\int_0^\infty K_0(2x\sinh t)\cosh(2nt)\,dt \quad \text{where} \quad K_0(x) := \int_0^\infty e^{-x\cosh s}ds \tag{31}$$

is the modified Bessel function of order 0. Hence, from the recurrence relation (27), we deduce

$$\left|H_n^{(1)}(x)\right| \geq \left|\frac{2n-2}{x}H_{n-1}^{(1)}(x)\right| - \left|H_{n-2}^{(1)}(x)\right| \geq \frac{n}{x}\left|H_{n-1}^{(1)}(x)\right| \quad \text{for } n \geq 3 \text{ and } x \in (0,1).$$

Applying this inequality recursively, the conclusion follows. $\square$

The following lemmas concern the convergence and the order of magnitude of the series involved in (22).

Lemma 8. *For a fixed $\varepsilon > 0$, if the obstacles O_p^ε and O_q^ε are disjoint, then for all $s \geq 0$,*

$$\sum_{m\in\mathbb{Z}}(1+m^2)^s\frac{\left\|\mathcal{H}_{q,m}(r_p^\varepsilon,\cdot)\right\|_{L^2(0,2\pi)}^2}{\left|H_m^{(1)}(\omega r_q^\varepsilon)\right|^2} < \infty.$$

Proof. Thanks to the symmetry relation (26), we can consider the series for $m \in \mathbb{N}$. The asymptotic behavior (30) shows that for a fixed $\varepsilon > 0$, when $m \to +\infty$,

$$\frac{1}{\left|H_m^{(1)}(\omega r_q^\varepsilon)\right|^2} = O\left(\frac{\left\{\omega r_q^\varepsilon\right\}^{2m}}{2^{2m}(m-1)!^2}\right), \tag{32}$$

$$\left\|\mathcal{H}_{q,m}(r_p^\varepsilon,\cdot)\right\|_{L^2(0,2\pi)}^2 = \frac{1}{2\pi}\int_0^{2\pi}\left|H_m^{(1)}\left(\omega|x(r_p^\varepsilon,\theta_p)-s_q|\right)\right|^2 d\theta_p = O\left(\frac{2^{2m}(m-1)!^2}{\left\{\omega\left(|s_p-s_q|-r_p^\varepsilon\right)\right\}^{2m}}\right), \tag{33}$$

where the last equality follows from the fact that $|x(r_p^\varepsilon,\theta_p)-s_q| \geq |s_p-s_q| - r_p^\varepsilon$ for all $\theta_p \in (0,2\pi)$. As a consequence

$$(1+m^2)^s\frac{\left\|\mathcal{H}_{q,m}(r_p^\varepsilon,\cdot)\right\|_{L^2(0,2\pi)}^2}{\left|H_m^{(1)}(\omega r_q^\varepsilon)\right|^2} = O\left(m^{2s}\left(\frac{r_q^\varepsilon}{|s_p-s_q|-r_p^\varepsilon}\right)^{2m}\right) \quad \text{as } m \to +\infty.$$

Hence the series is convergent since $r_p^\varepsilon + r_q^\varepsilon < |s_p - s_q|$ (for the obstacles are disjoint). $\square$

Lemma 9. *Let $N \in \mathbb{N}$ and $p \neq q$. For $\varepsilon > 0$ small enough, we have*

$$\sum_{|m|\geq N}\frac{\left\|\mathcal{H}_{q,m}(r_p^\varepsilon,\cdot)\right\|_{L^2(0,2\pi)}^2}{\left|H_m^{(1)}(\omega r_q^\varepsilon)\right|^2} = \begin{cases} O\left(|\log\varepsilon|^{-2}\right) & \text{if } N = 0, \\ O\left(\varepsilon^{2N}\right) & \text{else.}\end{cases}$$

Proof. We proceed as in the proof of Lemma 8. We denote by $h_m(\varepsilon) = \left\|\mathcal{H}_{q,m}(r_p^\varepsilon,\cdot)\right\|_{L^2(0,2\pi)}^2 / \left|H_m^{(1)}(\omega r_q^\varepsilon)\right|^2$ the term of order m of the above series. On one hand, formula (32) is no more valid for small ε since (30) cannot be used for small arguments. We use instead Lemma 7 as well as the asymptotic behavior (28) of $H_2^{(1)}$ for small arguments, which shows that there exists $C > 0$ and $\varepsilon_0 > 0$ such that

$$\frac{1}{\left|H_m^{(1)}(\omega r_q^\varepsilon)\right|^2} \leq C\,\frac{\left\{\omega r_q^\varepsilon\right\}^{2m}}{m!^2} \quad \text{for all } m \geq 2 \text{ and } \varepsilon \leq \varepsilon_0.$$

On the other hand, formula (33) holds. Noticing that $|s_p - s_q| - r_p^\varepsilon \geq |s_p - s_q|/2$ for small enough ε, say $\varepsilon \leq \varepsilon_1$, we infer that there exists $C' > 0$ and $M \geq 2$ such that

$$\left\|\mathcal{H}_{q,m}(r_p^\varepsilon, \cdot)\right\|^2_{L^2(0,2\pi)} \leq C' \, \frac{4^{2m}\,(m-1)!^2}{\left\{\omega\,|s_p - s_q|\right\}^{2m}} \quad \text{for all } m \geq M \text{ and } \varepsilon \leq \varepsilon_1.$$

As a consequence,

$$h_m(\varepsilon) \leq \frac{C''}{m^2}\left(\frac{4r_q^\varepsilon}{|s_p - s_q|}\right)^{2m} \quad \text{for all } m \geq M \text{ and small enough } \varepsilon.$$

This shows that $\sum_{m=M}^{\infty} h_m(\varepsilon) = O\left(\varepsilon^{2M}\right)$. Moreover, from the asymptotic behavior (28) of $H_n^{(1)}$ for small arguments, we have

$$h_0(\varepsilon) \leq C_0\,|\log\varepsilon|^{-2} \quad \text{and} \quad h_m(\varepsilon) \leq C_m\,\varepsilon^{2m} \text{ for } 0 < m < M.$$

This completes the proof. □

6. Conclusion

We have proposed in this paper a justification of different levels of asymptotic models available in the physical literature, including the Foldy-Lax model, for the two-dimensional scattering of an acoustic wave by an arbitrary number of sound-soft circular obstacles. In the models considered here, each obstacle has an isotropic behavior in the sense that in the series (13) which describes the wave scattered by one obstacle, we only keep the dominant contribution, associated with $m = 0$, which does not depend on θ_p. As shown in [23], the method we have presented can be seen as the first order of a more general approach which allows higher order approximations that take into account the angular dependence of the field scattered by each obstacle. Instead of keeping only the radial component ($m = 0$) in the series (13), we simply have to truncate this series at $|m| = \ell$ for some $\ell \geq 1$: the new unknowns are the Fourier coefficients $c_{p,m}^\varepsilon$ for $p = 1, \ldots, P$ and $m = -\ell, \ldots, +\ell$. Instead of (19), we are then led to an approximation of order $\varepsilon^{\ell+1}$ of (17) which amounts to a linear system of size $P(2\ell + 1)$. Using the same arguments as in section 4, we will obtain an error estimate similar to that of Theorem 1 (for $k = \infty$) where $\varepsilon/|\log\varepsilon|$ will be replaced by $\varepsilon^{\ell+1}/|\log\varepsilon|$. Such an improved Foldy-Lax model may become useful when the scatterer density is high. Moreover, a numerical comparison between the original Foldy-Lax model and a high order approximation could help us to quantify the value of $C_{K,s}$ in Theorem 1.

What about the possible generalizations of our study? First notice that our approach should also apply for other boundary conditions on circular scatterers (Neumann or Robin) as well as for penetrable obstacles, since the scattered wave can still be represented by means of Fourier series. However, for such boundary conditions, the asymptotic behavior of each scatterer is no more isotropic: higher order terms in the multipole expansion of the scattered field must be kept. Hence the implementation of a Foldy–Lax model in such situations is similar to the higher order approximations mentioned above. In the three-dimensional case for spherical obstacles, the same ideas apply using spherical harmonics expansions instead of Fourier series. The main difference lies in the fact that the asymptotic expansions will only involve powers of ε whereas the expansions considered in the present paper also involve powers of $|\log\varepsilon|$. Let us finally mention that if we consider non-circular or non-spherical scatterers, the results we have obtained are likely to hold, but the method we have used does no longer apply, for the scattered wave can no longer be represented by Fourier series. Other techniques such as matched asymptotic expansions or multiple scale methods [19–21] should be used.

References

[1] M. Abramowitz, I. A. Stegun, Handbook of Mathematical Functions, Dover Publications, New-York, 1972.

[2] H. Ammari, H. Kang, Boundary layer techniques for solving the Helmholtz equation in the presence of small inhomogeneities, J. Math. Anal. Appl. 296 (2004) 190–208.

[3] X. Antoine, C. Chniti, K. Ramdani, On the numerical approximation of high-frequency acoustic multiple scattering problems by circular cylinders, J. Comput. Phys. 227 (2008) 1754–1771.

[4] I.I. Argatov, F.J. Sabina, Acoustic scattering by a cluster of small sound-soft obstacles, Wave Motion 47 (2010) 537–551.

[5] G. Bal and K. Ren, Transport-based imaging in random media, SIAM J. Appl. Math. 68 (2008) 1738–1762.
[6] M. Balabane, Boundary decomposition for Helmholtz and Maxwell equations 1: disjoint sub-scatterers, Asymptotic Analysis 38 (2004) 1–10.
[7] V. Bonnaillie-Noël, M. Dambrine, S. Tordeux, G. Vial, Interactions between moderately close inclusions for the Laplace equation, M3AS. 10 (2009) 1853–1882.
[8] L.C. Evans, Partial Differential Equations, American Mathematical Society, Providence, 2008.
[9] L.L. Foldy, The multiple scattering of waves. I. General theory of isotropic scattering by randomly distributed scatterers, Phys. Rev. 67 (1945) 107-119.
[10] J. Groenenboom, R. Snieder, Attenuation, dispersion, and anisotropy by multiple scattering of transmitted waves through distributions of scatterers, J. Acoust. Soc. Am. 98 (1995) 3482–3492.
[11] K. Huang, K. Solna , H. Zhao, Generalized Foldy–Lax formulation, J. Comput. Phys. 229 (2010) 4544–4553.
[12] R. Kleinman, B. Vainberg, Full low-frequency asymptotic expansion for second-order elliptic equations in two dimensions, Math. Methods Appl. Sci. 17 (1994) 989–1004.
[13] R. Kress, Linear Integral Equations, Springer-Verlag, Berlin, 1989.
[14] M. Lax, Multiple scattering of waves, Rev. Mod. Phys. 23 (1951), 287-310.
[15] M. Lax, Multiple scattering of waves. II. The effective field in dense systems, Phys. Rev. 85 (1952) 621–629.
[16] C.M. Linton, P.A. Martin, Multiple scattering by random configurations of circular cylinders: second-order corrections for the effective wavenumber, J. Acoust. Soc. Am. 117 (2005) 3413–3423.
[17] E.A. Marengo, F.K. Gruber, Subspace-based localization and inverse scattering of multiply scattering point targets, EURASIP Journal on Advances in Signal Processing (2007), Article ID 17342.
[18] P.A. Martin, Multiple Scattering. Interaction of Time-Harmonic Waves with *N* Obstacles, Cambridge University Press, 2006.
[19] V. Maz'ya, A. Movchan, M. Nieves, Uniform asymptotic formulae for Green's tensor in elastic singularly perturbed domains, Asymptot. Anal. 52 (2007) 173–206.
[20] V. Maz'ya, A. Movchan, Asymptotic treatment of perforated domains without homogenization, Math. Nachr. 283 (2010) 104–125.
[21] S.A. Nazarov, J. Sokolowski, Self-adjoint extensions for the Neumann Laplacian and applications, Acta Mathematica Sinica 22 (2006) 879–906.
[22] A.G. Ramm, Wave scattering by small bodies, Reports on Mathematical Physics 21 (1985) 69–77.
[23] K.K. Tse, L. Tsang, C.H. Chan, K.-H. Ding, Multiple scattering of waves by random distribution of particles for applications in light scattering by metal nanoparticles, in: A.A. Maradudin (Ed.), Light Scattering and Nanoscale Surface Roughness, Springer, 2007.
[24] G.N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press, 1944.