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High order asymptotic analysis of twisted electrostatic problems

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Abstract

In this paper, we present a high order asymptotic approximation of the operator arising in the two-dimensional setting of a scalar helicoidal (electrostatic) problem.
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1. Introduction

We investigate the asymptotic analysis of twisted structures. Our aim is to understand the effect of a weak twist on waveguides such as microstructured optical fibers. At this early stage of the study, a simple scalar problem is considered.

2. Twisted electrostatics

The electrostatic problem of interest consists in the computation of the scalar potential u satisfying $\operatorname{div} \varepsilon \, \mathbf{grad}\, u = 0$ for a given space distribution of dielectric permittivity ε and known values of u on given metallic surfaces. The problem is *twisted* if ε and the metallic surfaces are independent of ξ_3 in the following helicoidal co-ordinate system [1–3]:

$$\begin{aligned}\xi_1 &= x_1 \cos(\alpha x_3) - x_2 \sin(\alpha x_3),\\ \xi_2 &= x_1 \sin(\alpha x_3) + x_2 \cos(\alpha x_3),\\ \xi_3 &= x_3,\end{aligned} \tag{1}$$

where α is a parameter which characterizes the torsion. This co-ordinate system is characterized by the Jacobian of the transformation:

$$\mathbf{J} = \frac{\partial(x_1, x_2, x_3)}{\partial(\xi_1, \xi_2, \xi_3)}.$$

When the electrostatic problem is expressed in this co-ordinate system, only the matrix

$$\mathbf{T}(\xi_1, \xi_2) = \frac{\mathbf{J}^{\mathrm{T}}\mathbf{J}}{\det(\mathbf{J})}$$

is involved in the equation with the remarkable property that it is independent of ξ_3, which allows the setting of the twisted problems as two-dimensional problems. The equation to be solved is then:

$$\operatorname{div}(\varepsilon \mathbf{T}^{-1}\, \mathbf{grad}\, u) = \operatorname{div}(\varepsilon\, \mathbf{grad}\, u) + \alpha^2 \mathscr{L}(\varepsilon \mathscr{L}(u)) = 0, \tag{2}$$

where div and **grad** involve now partial derivatives with respect to ξ_1 and ξ_2 *as if they were rectangular co-ordinates*, the third component of the gradient being equal to zero *in the helicoidal co-ordinate system*. We have

$$\mathbf{T}^{-1} = \begin{pmatrix} \mathbf{I} + \alpha^2 \mathbf{R}\mathbf{R}^{\mathrm{T}} & \alpha \mathbf{R} \\ \alpha \mathbf{R}^{\mathrm{T}} & 1 \end{pmatrix}, \tag{3}$$

with $\mathbf{R} = \begin{pmatrix} -\xi_2 \\ \xi_1 \end{pmatrix}$, and

$$\mathscr{L}(.) = \left(\xi_1 \frac{\partial .}{\partial \xi_2} - \xi_2 \frac{\partial .}{\partial \xi_1}\right), \tag{4}$$

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the "twisted" derivation operator. This operator can be expressed as:

$$\mathscr{L}(u) = \mathbf{R} \cdot \mathbf{grad}\, u = \mathrm{div}(\mathbf{R}u). \tag{5}$$

This problem is easily solved numerically via a finite element analysis but we would like to obtain a simple asymptotic formula in order to understand the effect of very weak twists (small values of α).

3. Asymptotic analysis

Given the twisted electrostatic problem on a domain of cross-section Ω (with prescribed non-homogeneous Dirichlet conditions on its boundary $\partial\Omega$), the rigorous setting of the asymptotic analysis requires the introduction of scaled variables in the cross-section

$$(y_1, y_2, x_3) = \left(\frac{x_1}{\eta}, \frac{x_2}{\eta}, x_3\right),$$

where η is a *small parameter* representing a ratio between the diameter of the cross-section Ω of the structure to the period $2\pi/\alpha$ of the twist along the x_3-axis.

The field u is then approximated by the asymptotic development $\tilde{u}_\eta$ given by the following *Ansatz* [4]:

$$\tilde{u}_\eta = \sum_{i=0}^{\infty} \eta^{2i} \tilde{u}_i \left(\frac{x_1}{\eta}, \frac{x_2}{\eta}, x_3\right). \tag{6}$$

Nevertheless, a convenient way to deal with the numerical computation of the various terms of the asymptotic development is to work in unscaled two-dimensional co-ordinates. Let us introduce the auxiliary functions u_i independent of η:

$$u_i(x_1, x_2) = \eta^{2i} \tilde{u}_i(x_1/\eta, x_2/\eta, 0) = \eta^{2i} \tilde{u}_i(y_1, y_2, 0),$$

for $i \geqslant 0$.

In this case, the asymptotic model reduces to a sum of the auxiliary functions u_i [4]:

$$\tilde{u}_\eta|_{x_3=0} = u(x_1, x_2) = u_0(x_1, x_2) + \sum_{i=1}^{\infty} u_i(x_1, x_2) \tag{7}$$

with $u_0|_{\partial\Omega} = u|_{\partial\Omega}$ (the prescribed boundary conditions) and $u_i|_{\partial\Omega} = 0$, for $i > 0$. The u_0 term is given by the solution of the untwisted problem $\mathrm{div}(\varepsilon\, \mathbf{grad}\, u_0) = 0$ with $u_0|_{\partial\Omega} = u|_{\partial\Omega}$ while the higher order terms u_i, for $i > 0$, are given by Poisson problems where the charge density depends on the previous term:

$$\mathrm{div}(\varepsilon\, \mathbf{grad}\, u_i) = -\alpha^2 \mathscr{L}(\varepsilon \mathscr{L}(u_{i-1})) \tag{8}$$

with $u_i|_{\partial\Omega} = 0$. The corresponding weak formulation necessary for the finite element model is

$$\int_\Omega \varepsilon\, \mathbf{grad}\, u_i \cdot \mathbf{grad}\, w \, \mathrm{d}S = -\alpha^2 \int_\Omega \varepsilon\, \mathscr{L}(u_{i-1}) \mathscr{L}(w) \, \mathrm{d}S, \tag{9}$$

for any admissible weight function w with $u_i|_{\partial\Omega} = 0$. It is noteworthy that the problem is linear and that u_i is proportional to $\alpha^2 u_{i-1}$. It is then sufficient to solve the problem for $\alpha = 1$ to find U_i such that $u_i = \alpha^{2i} U_i$ and therefore the practical computation of the asymptotic model reduces to the introduction of a development in powers of α:

$$u = u_0 + \sum_{i=1}^{\infty} \alpha^{2i} U_i. \tag{10}$$

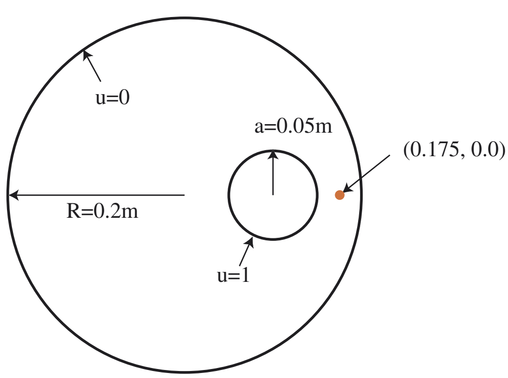

Fig. 1. Geometry of the simple test problem.

4. A numerical example

We consider the structure depicted in Fig. 1, whose cross-section is a hollow disk. The outer circle (of radius $R = 0.2\,\mathrm{m}$) is centered on the rotation axis of the twisted structure ($\xi_1 = 0, \xi_2 = 0$) and the inner circle (of radius $a = 0.05\,\mathrm{m}$) is centered at $0.1\,\mathrm{m}$ from the rotation axis. Table 1 presents the results of comparison of asymptotic approximations and finite element numerical data; it gives the variation of the error at a given point within the domain for the asymptotic prediction as a function of the

Table 1
Comparison of the values of the potential at point ($\xi_1 = 0.175, \xi_2 = 0$) (the origin is on the rotation axis) where the error is the most important

α	u exact	u asymptotic	abs. error	$\alpha^2 U_1$ correction
0	0.4323075525521577			
0.1	0.4323010126383421	0.43230101219128	4.4706e − 010	−6.5404e − 006
0.5	0.4321443225126496	0.43214404353032	2.7898e − 007	−1.6351e − 004
2	0.4297611380445051	0.42969140820282	6.9730e − 005	−2.6161e − 003
8	0.403178325558716	0.39044924296276	1.2729e − 002	−4.1858e − 002

On this point $U_1 = -6.54036087\mathrm{e} - 004$ and $U_0 = 0.4323075525521577$.

Table 2
Second order: comparison of the values of the potential at point ($\xi_1 = 0.175, \xi_2 = 0$) (the origin is on the rotation axis) where the error is the most important

α	u exact	u asymptotic	abs. error	$\alpha^4 U_2$ correction
0.1	0.4323010126383421	0.43230101263837	$-2.7978e-014$	$4.4709e-010$
0.5	0.4321443225126496	0.43214432295890	$-4.4625e-010$	$2.7943e-007$
2	0.4297611380445051	0.42976294191724	$-1.8039e-006$	$7.1534e-005$
8	0.403178325558716	0.40876187385424	$-5.5835e-003$	$1.8313e-002$

On this point $U_2 = 4.470857151240261e-006$.

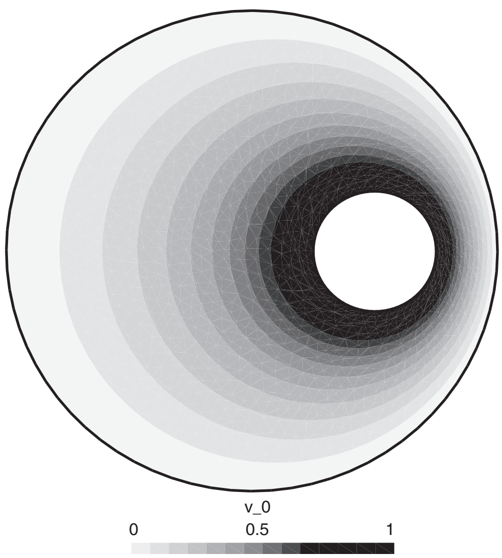

Fig. 2. The untwisted term u_0.

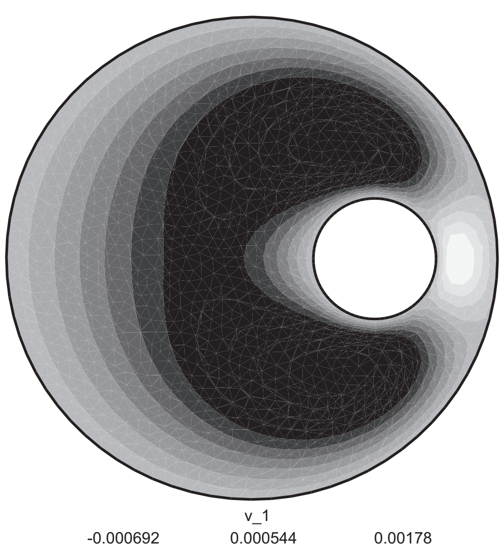

Fig. 3. The α^2 corrector U_1.

twist α. The first column is the value of the twist parameter α. The second column corresponds to an "exact" model, i.e. finite element solution of Eq. (2). The third column gives our first-order asymptotic approximation $u_0 + \alpha^2 U_1$. The fourth column gives the discrepancy between the two previous columns. The fifth column gives the value of the asymptotic correction $\alpha^2 U_1$ to be added to the untwisted case u_0 to obtain the third column. Table 2 presents similar results for the second-order asymptotic approximation: the first two columns are the same. The third column gives our second-order asymptotic approximation $u_0 + \alpha^2 U_1 + \alpha^4 U_2$. The fourth column gives the discrepancy between the two previous columns. The fifth column gives the value of the asymptotic correction $\alpha^4 U_2$ to be added to the first-order asymptotic approximation $u_0 + \alpha^2 U_1$ to obtain the third column. This higher order correction leads clearly to more accurate results as it improves the estimate and is not negligible with respect to the residual error. Fig. 2 shows the map of u_0, Fig. 3 the one of the first corrector U_1, and Fig. 4 the one of the second corrector U_2. The process seems to be stable and the numerical solutions preserve correctly the symmetry of the problem. It seems also convergent since the values of the correctors decrease. The range of values is [0.0, 1.0] for u_0, $[-6.92 \times 10^{-4}, 1.78 \times 10^{-3}]$ for U_1, $[-1.08 \times 10^{-5}, 4.35 \times 10^{-6}]$ for U_2, $[-3.49 \times 10^{-8}, 9.5 \times 10^{-8}]$ for U_3, $[-1.04 \times 10^{-9}, 6.63 \times 10^{-10}]$ for U_4, and $[-1.53 \times 10^{-11}, 1.41 \times 10^{-11}]$ for U_5. Next steps of this study will be the asymptotic estimation of the propagation modes and the extension to the vector case with the

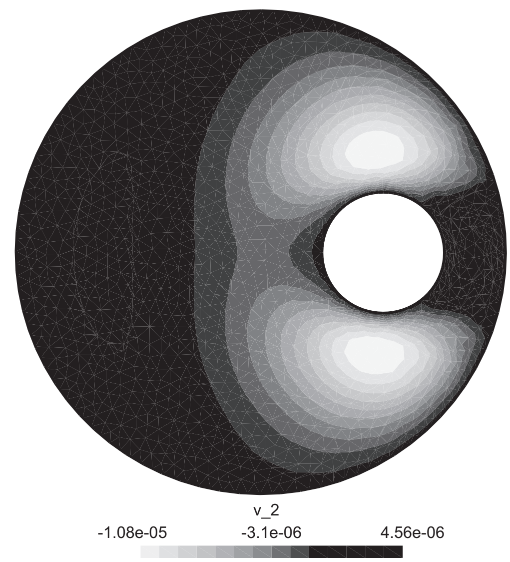

Fig. 4. The α^4 corrector U_2.

purpose of estimating the effect of a weak twist on the losses of leaky modes of microstructured optical fibers.

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