Electromagnetic analysis of cylindrical invisibility cloaks and the mirage effect

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We present a finite-element analysis of a diffraction problem involving a coated cylinder enabling the electromagnetic cloaking of a lossy object with sharp wedges located within its core. The coating consists of a heterogeneous anisotropic material deduced from a geometrical transformation as first proposed by Pendry *et al.* [Science **312**, 1780 (2006)]. We analyze the electromagnetic response of the cloak in the presence of an electric line source in p polarization and a loop of magnetic current in s polarization. We find that the electromagnetic field radiated by such a source located a fraction of a wavelength from the cloak is perturbed by less than 1%. When the source lies in the coating, it seems to radiate from a shifted location. © 2007 Optical Society of America

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Recently, it was suggested by Pendry *et al.* that an object surrounded by a coating consisting of an exotic material becomes invisible to electromagnetic waves.[1] The theoretical idea based on geometric transformations was supplied with numerics performed with software based on geometrical optics. Leonhardt independently analyzed conformal invisibility devices using the stationary Schrödinger equation.[2,3] An alternative route to invisibility was proposed by Alu and Engheta,[4] but it relies on a specific knowledge of the shape and material properties of the object being concealed. Lastly, Milton and Nicorovici proposed to cloak a countable set of line sources using anomalous resonance when it lies in the close neighborhood of a cylindrical coating filled with negative refractive index material.[5]

In the present Letter, we provide a full-electromagnetic wave picture that supports the results of Ref. 1 in the case of a harmonic electric line source or magnetic loop located in a close neighborhood of the invisibility cloak surrounding a two-dimensional lossy cylindrical object. The object and cloak lie well within the intense near field of the line source, thus confirming that cloaking holds not just for the far field but also for the near field, where a ray model of the field breaks down.

From now on, we adopt a covariant approach that uses Maxwell's equations, which was applied to transfer matrices in (Ref. 6) and to finite elements in Ref. 7. For this, let us consider a map from a coordinate system $\{u,v,w\}$ to the coordinate system $\{x,y,z\}$ given by the transformation characterized by $x(u,v,w)$, $y(u,v,w)$, and $z(u,v,w)$. We emphasize the fact that it is the transformed domain and coordinate system that are mapped onto the initial domain with Cartesian coordinates, and not the opposite. This change of coordinates is characterized by the transformation of the differentials through the Jacobian:

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \mathbf{J}_{xu} \begin{pmatrix} du \\ dv \\ dw \end{pmatrix}, \qquad \mathbf{J}_{xu} = \frac{\partial(x,y,z)}{\partial(u,v,w)}. \tag{1}$$

In electromagnetism, this change of coordinates amounts to replacing the different materials (often homogeneous and isotropic, which corresponds to the case of scalar piecewise constant permittivities and permeabilities) by equivalent inhomogeneous anisotropic materials described by a transformation matrix $\mathbf{T}$ (metric tensor).[3,8] The idea underpinning electromagnetic invisibility[1] is that newly discovered metamaterials enable control of the electromagnetic field by mimicking the heterogeneous anisotropic nature of $\mathbf{T}$, as recently demonstrated in the microwave regime.[9]

From a geometrical point of view, the matrix $\mathbf{T} = \mathbf{J}^T\mathbf{J}/\det(\mathbf{J})$ is a representation of the metric tensor. The only thing to do in the transformed coordinates is to replace the materials (often homogeneous and isotropic) by equivalent ones that are inhomogeneous (their characteristics are no longer piecewise constant but merely depend on u, v, w coordinates) and anisotropic ones (tensorial nature) whose properties are given by[3,8]

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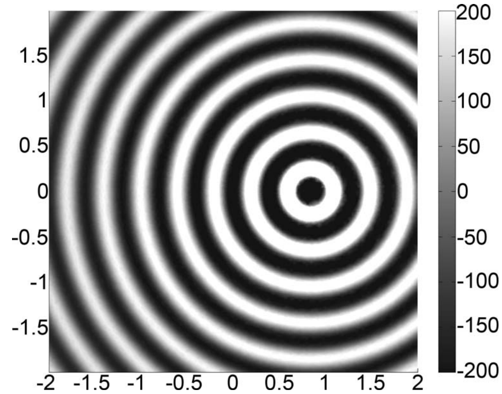

Fig. 1. Real part of the longitudinal electric field of a line source located at point $\mathbf{r}_s=(0.84,0)$ that radiates in a vacuum.

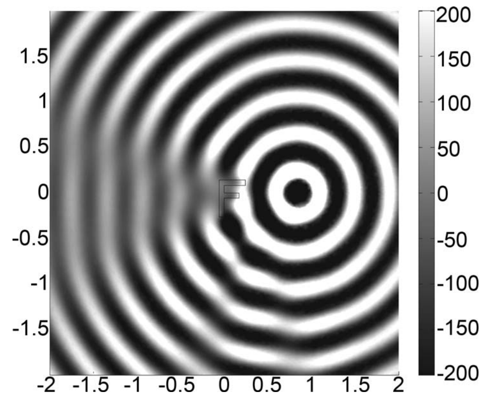

Fig. 2. Real part of the longitudinal electric field of the same line source that radiates in the presence of an F-shaped metallic obstacle.

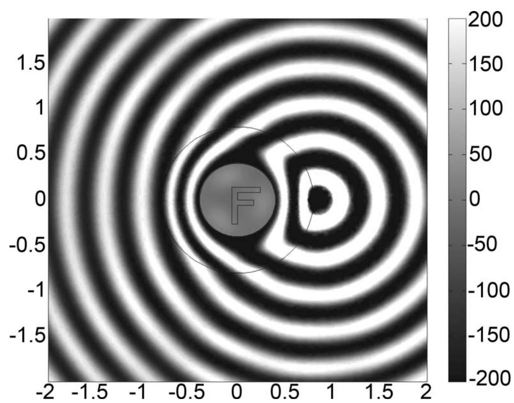

Fig. 3. Real part of the longitudinal electric field for the same line source that radiates in the presence of an F-shaped metallic obstacle surrounded by an invisibility cloak. Note that the cloak is of the same order as the wavelength: $R_2-R_1=0.8-0.4=\lambda$.

$$\underline{\underline{\varepsilon}}' = \varepsilon \mathbf{T}^{-1}, \qquad \underline{\underline{\mu}}' = \mu \mathbf{T}^{-1}. \tag{2}$$

We note that there is no change in the impedance of the media, since the permittivity and permeability suffer the same transformation. As for the vector analysis operator and product, everything works as if we were in Cartesian coordinates.

It remains to compute the transformation matrix $\mathbf{T}$ as expressed in the Cartesian coordinates (x',y',z) associated with the cylindrical cloak defined by the radii R_1 (interior radius) and R_2 (exterior radius). The Jacobian matrix $\mathbf{J}_{xx'}$ is therefore merely the product of three elementary Jacobians: $\mathbf{J}_{xx'}=\mathbf{J}_{xr}\mathbf{J}_{rr'}\mathbf{J}_{r'x'}$, where $\mathbf{J}_{xr}$ is the Jacobian associated with the change to cylindrical coordinates (r,θ,z), where $\mathbf{J}_{rr'}$ are the radially contracted cylindrical coordinates as proposed in Ref. 1 [$\mathbf{J}_{rr'}=\mathrm{diag}(1/\alpha,1,1,)$, where $\alpha=(R_2-R_1)/R_2$ for $0\leqslant r\leqslant R_2$ and $\alpha=1$ for $r>R_2$] and where the last Jacobian $\mathbf{J}_{r'x'}$ represents the Jacobian associated with the change to Cartesian coordinates (x',y',z). Hence the material properties of the invisibility cloak are described by the transformation matrix $\mathbf{T}=\mathbf{J}_{xx'}^{T}\mathbf{J}_{xx'}/\det(\mathbf{J}_{xx'})$.

We will also need its inverse, which we give explicitly as

$$\mathbf{T}^{-1} = \begin{pmatrix} (T^{-1})_{11} & (T^{-1})_{12} & 0 \\ (T^{-1})_{21} & (T^{-1})_{22} & 0 \\ 0 & 0 & \dfrac{r'-R_1}{\alpha^2 r'} \end{pmatrix}, \tag{3}$$

where

$$(T^{-1})_{11} = 1 - \frac{R_1 \sin^2(\theta')}{r'} + \frac{R_1 \cos^2(\theta')}{-R_1+r'},$$

$$(T^{-1})_{22} = 1 - \frac{R_1 \cos^2(\theta')}{r'} + \frac{R_1 \sin^2(\theta')}{-R_1+r'},$$

$$(T^{-1})_{12} = (T^{-1})_{21} = \frac{R_1 \cos(\theta')\sin(\theta')(R_1-2r')}{(R_1-r')r'}. \tag{4}$$

We now look at the electromagnetic field radiated by an electric line source located at point $\mathbf{r}_s$ and by a magnetic current dipole located at the same point in the presence of a finite conducting object arbitrarily shaped like the letter F when it is surrounded by a cloak. Thanks to the cylindrical geometry, the problem splits into p and s polarizations:

$$\nabla \times (\underline{\underline{\mu}}'^{-1} \nabla \times \mathbf{E}_l) - \mu_0\varepsilon_0\omega^2 \underline{\underline{\varepsilon}}' \mathbf{E}_l = -i\omega I_s \mu_0 \delta_{\mathbf{r}_s} \mathbf{e}_z, \tag{5}$$

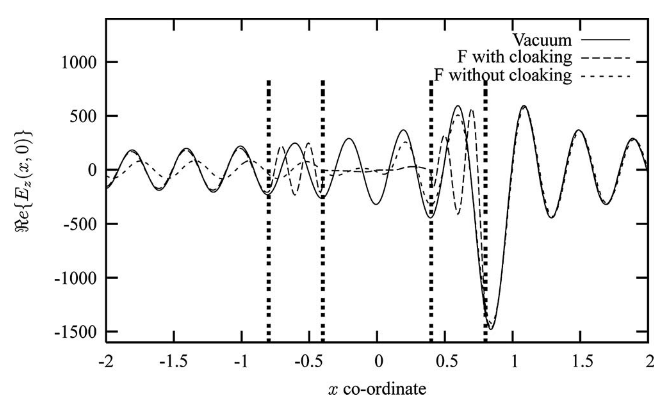

Fig. 4. Real part of E_z along the x-axis. The vertical dashed bold lines represent the boundary of the cloak.

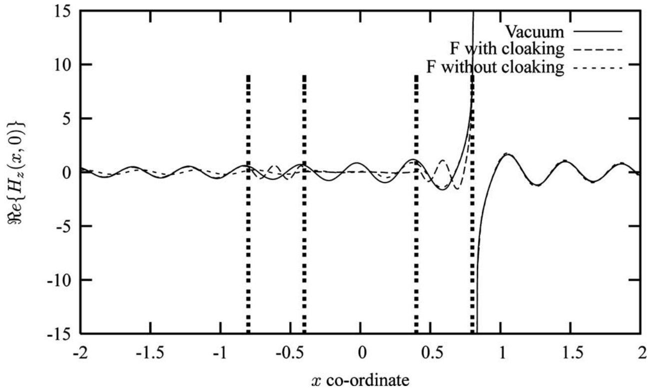

Fig. 5. Real part of H_z along the x-axis. The vertical dashed bold lines represent the boundary of the cloak. Note that $\Re e\{H_z\}$ is theoretically infinite at the source place.

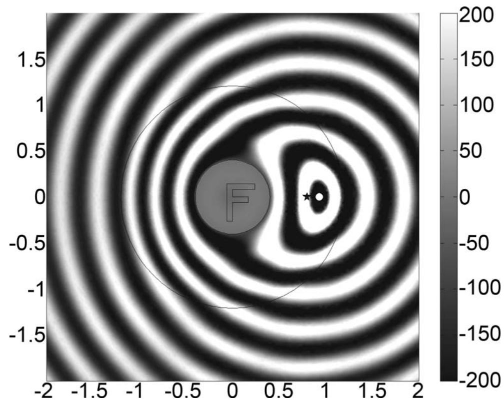

Fig. 6. Real part of the longitudinal electric field for a line source located at point $\mathbf{r}'_s=(0.96666,0)$ (○) that radiates within a broad cloak characterized by $R_1=0.4$ and $R_2=1.2$. For an exterior observer, the field seems to be emitted by a shifted source located at a point $\mathbf{r}_s=(0.84,0)$ (⋆): this is the *mirage effect*. Note that the shift, which is 0.12666, is nearly one third of the wavelength, $\lambda=0.4$. Compare the field outside the invisibility cloak with Fig. 1.

$$\nabla \times (\underline{\underline{\varepsilon}}'^{-1} \nabla \times \mathbf{H}_l) - \mu_0\varepsilon_0\omega^2 \underline{\underline{\mu}}' \mathbf{H}_l = \nabla \times (\underline{\underline{\varepsilon}}'^{-1}\mathbf{j}_T), \quad (6)$$

where $\mathbf{E}_l=E_z(x,y)\mathbf{e}_z$, $\mathbf{H}_l=H_z(x,y)\mathbf{e}_z$, and $\underline{\underline{\varepsilon}}'$ and $\underline{\underline{\mu}}'$ are defined by Eq. (2). Also, $\mathbf{j}_T$ denotes a current with a vanishing z-component. To model the outgoing waves, we use perfectly matched layers.[10] The finite-element method is implemented in the commercial software package COMSOL.

Let us first consider an electric line source located at point $\mathbf{r}_s=(0.84,0)$ radiating in a vacuum with wavelength $\lambda=0.4$ (Note that all lengths are given in arbitrary units, μm, for instance, for visible light) and carrying a current $I_s=1$. The electric field E_z is therefore given by $E_z=-\frac{1}{4}\omega\mu_0 I_s H_0^{(1)}(2\pi/\lambda|\mathbf{r}-\mathbf{r}_s|)$ (see Fig. 1). In a second experiment, an F-shaped obstacle is placed near the origin (0,0) beside the aforementioned line source as shown in Fig. 2. This obstacle is made up of an arbitrary homogeneous nonmagnetic lossy material characterized by its permittivity $\varepsilon_{r,F}=1+5i$. The real part of E_z, the solution of Eq. (5) is plotted in Fig. 2. Lastly, the letter F is surrounded by an annulus-shaped coating (cloak of invisibility) geometrically characterized by two circles centered on the origin $R_2=0.8$ and $R_1=0.4$ and optically characterized by $\underline{\underline{\varepsilon}}'$ and $\underline{\underline{\mu}}'$ given by Eq. (2). Figure 3 represents the real part of E_z in the vicinity of the invisibility cloak. Also, Fig. 4 sums up the last three configurations along the x-axis. Finally, we also tackle the problem in the s polarization. For this purpose, Fig. 5 gives the real part of the magnetic field H_z for a magnetic dipole located at point $\mathbf{r}_s=(0.84,0)$ and characterized by its dipole moment $\mathbf{m}=\mathbf{e}_y$.

Lastly, the electric line source is placed within the invisibility cloak at a point $\mathbf{r}'_s$. This point being the image of $\mathbf{r}_s$ through the radially contracted cylindrical transformation, we have $\mathbf{r}'_s=\alpha\mathbf{r}_s+R_1$, and therefore, for any observer in the free space, the radiated field seems to be emitted by a source located at $\mathbf{r}_s$ (see Fig. 6). This is the *mirage effect*.

In this paper, we used the equivalence between a geometrical transformation and a change of the characteristics of the material[3,6,7] (permittivity and permeability) to compute the electromagnetic response of the invisibility cloak proposed in Ref. 1. We proved that the finite-element method is numerically stable even if the source is very close to the invisibility cloak (a fraction of a wavelength away). We investigated both p and s polarizations (for an electric line current and a magnetic loop) when the working wavelength is of the same order as the radius of the cloak. This suggests that the experimental and numerical results obtained by Schurig *et al.* in the far field for an invisibility cloak consisting of concentric layers of split ring resonators[9] may still hold in our configuration. Finally, we emphasize that Eqs. (5) and (6) tackle electric and magnetic sources located in the cloak itself. In this last case, we observed some mirage effect whereby the source seems to radiate from a shifted location in accordance with the involved geometrical transformation.

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