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Comparison of boundary elements and transformed finite elements for open magnetic problems

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ABSTRACT

The modelling of open problems, i.e., with an exterior domain of infinite extent, requires special techniques in order to keep the size of the problem limited. Among the most interesting ones are the boundary element (BE) method and the finite element (FE) transformation method. In this paper the two methods are compared as well from the computational efficiency point of view as from the accuracy (inside the problem and for the far field) point of view.

INTRODUCTION

Because of the propagation of electromagnetic fields in free space (or in the air), electromagnetic problems are often open i.e. characterised by the decay of the fields at infinity, the opposite to closed problems with boundary conditions at finite distance. As the numerical computations can only involve a finite number of degrees of freedom, various methods have been proposed to overcome this difficulty. Among the most interesting ones are the boundary element (BE) method and the finite element (FE) transformation method.

In a current free domain, the equation of the two-dimensional magnetostatics is the Laplace equation (1) :

$$\Delta A = 0 \qquad (1)$$

where A is the vector potential which has only one component. To solve open problems, it is therefore necessary to find a solution to (1) in an infinite region.

BOUNDARY ELEMENT METHOD

To solve the equation (1), the direct boundary element method (Brebbia[1]) is based on the following relation :

$$c\,A = \oint_{\Gamma} \left[A \frac{\partial G}{\partial n} - G \frac{\partial A}{\partial n} \right] d\Gamma \qquad (2)$$

where :
- G is the free space Green function of the two-dimensional Laplace operator;
- c = 0.5 on a smooth boundary;
- $\partial./\partial n$ is for the normal derivative.

Integrals are taken on the boundary Γ of the subdomains and the method involves only A and $\partial A/\partial n$ (tangential flux density) on the boundaries. Therefore neither meshing nor unknown need to be considered in the exterior domain. The boundary element method is based on the concept of Green's function which expresses in electromagnetism the remote action of sources. It takes into account naturally open problems because the way fields decrease at infinity is in a certain way contained in the Green's function.

FINITE ELEMENTS AND TRANSFORMATION METHOD

Some attempts have been made to use an a priori decrease of the field as shape functions in domain elements of infinite extent. Unfortunately the so called 'infinite elements' have never been successful in modelling open problems. Nevertheless the modelling of infinite regions by the finite element method is possible using the finite element transformation method (Imhoff[3]).

In order to set up the finite element method (Silvester[2]), a variational form is introduced. The magnetostatic Lagrangian is given by the integration of the magnetostatic Lagrangian density on the domain M (the coefficient ν is the magnetic reluctivity):

$$\int_M L(A)\, dM = \int_M \left(-1/2\, \nu \text{ grad } A \,.\, \text{grad } A \right) dM \qquad (3)$$

The finite element method consists of approximating A by $A = \Sigma A_i \alpha_i$ where A_i are parameters and α_i are shape functions obtained by assuming a simple behaviour on elements from a meshing of M. The equations for the parameters are found by expressing the extremum conditions for the discretised Lagrangian. The terms in the finite element equations are products of the unknowns A_j with the coefficients a_{ij} which are the following integrals on the elements E:

$$\int_E -\nu \text{ grad } \alpha_i \,.\, \text{grad } \alpha_j \; dE \qquad (4)$$

If a domain M* is mapped on the domain M, the differential operator and the integration undergo transformations involving the Jacobian matrix. The mapping of a domain M* with coordinates {X,Y} on the original domain M with coordinates {x,y} is given by two functions such that $\{X,Y\} \rightarrow \{x,y\} = \{f_1(X,Y), f_2(X,Y)\}$. The integrand of (4) transforms as:

$$\int_E -\nu\begin{pmatrix}\partial_x\alpha_i & \partial_y\alpha_i\end{pmatrix}\begin{pmatrix}\partial_x\alpha_j\\ \partial_y\alpha_j\end{pmatrix} dE =$$

$$\int_{E^*} -\nu\left[\begin{pmatrix}\partial_X\alpha_i(X,Y) & \partial_Y\alpha_i(X,Y)\end{pmatrix}\mathbf{J}^{-1}\right]\left[\mathbf{J}^{-T}\begin{pmatrix}\partial_X\alpha_j(X,Y)\\ \partial_Y\alpha_j(X,Y)\end{pmatrix}\right] \mathrm{dtm}(\mathbf{J})\ dE^* =$$

$$\int_{E^*} -\nu\begin{pmatrix}\partial_X\alpha_i & \partial_Y\alpha_i\end{pmatrix}\mathbf{T}\begin{pmatrix}\partial_X\alpha_j\\ \partial_Y\alpha_j\end{pmatrix} dE^* \qquad (5)$$

with the following Jacobian and transformation matrices:

$$\mathbf{J} = \begin{bmatrix}\partial_X f_1 & \partial_Y f_1\\ \partial_X f_2 & \partial_Y f_2\end{bmatrix} \qquad (6)$$

$$\mathbf{T} = \mathbf{J}^{-1}\mathbf{J}^{-T}\mathrm{dtm}(\mathbf{J}) = \begin{bmatrix}\dfrac{\partial_Y f_1{}^2 + \partial_Y f_2{}^2}{\partial_X f_1\,\partial_Y f_2 - \partial_X f_2\,\partial_Y f_1} & -\dfrac{\partial_X f_1\,\partial_Y f_1 + \partial_X f_2\,\partial_Y f_2}{\partial_X f_1\,\partial_Y f_2 - \partial_X f_2\,\partial_Y f_1}\\[2ex] -\dfrac{\partial_X f_1\,\partial_Y f_1 + \partial_X f_2\,\partial_Y f_2}{\partial_X f_1\,\partial_Y f_2 - \partial_X f_2\,\partial_Y f_1} & \dfrac{\partial_X f_1{}^2 + \partial_X f_2{}^2}{\partial_X f_1\,\partial_Y f_2 - \partial_X f_2\,\partial_Y f_1}\end{bmatrix} \qquad (7)$$

and where ∂_x, ∂_y, ∂_X, and ∂_Y indicate the partial derivative with respect to x, y, X, and Y respectively. E and E* are the physical and the transformed element respectively. $\alpha_i(X,Y)$, $\alpha_j(X,Y)$ are the shape functions on the transformed element.

This contribution of the transformed element is equal to the non transformed one up to the central matrix. If the transformation is trivial ($f_1(X,Y)=X$, $f_2(X,Y)=Y$) or corresponds to a conformal transformation ($f_1(X,Y) + i\, f_2(X,Y)$ is analytic i.e. $\partial_X f_1 = \partial_Y f_2$ and $\partial_X f_2 = -\partial_Y f_1$), this matrix reduces to the unit matrix.

The transformation method is thus the following one: map the 'transformed domain' M* on the original domain M and apply the finite element method with the elements obtained by meshing M* and using formula (5) . This is in fact very similar to the process of mapping a reference element on an actual element of the problem (figure 1). The direction of the mapping is noteworthy. It is different from the classical one adopted in finite element theory but it is much more coherent and is similar to the one used in algebraic topology (Bossavit[4]). For a justification of the relevance of the direction of the mappings in the finite element transformation method, see Nicolet[5] . This choice implies that only the direct functions f_1 and f_2 are involved and not their inverses. On figure 1, two transformations are chained. In fact, the number of chained transformations is arbitrary. In this case, the global Jacobian matrix is the product of the individual Jacobian matrices (the matrix of the transformation from the initial domain is the rightmost factor, each new transformation adds a left factor) and the transformation matrix is computed with this global transformation.

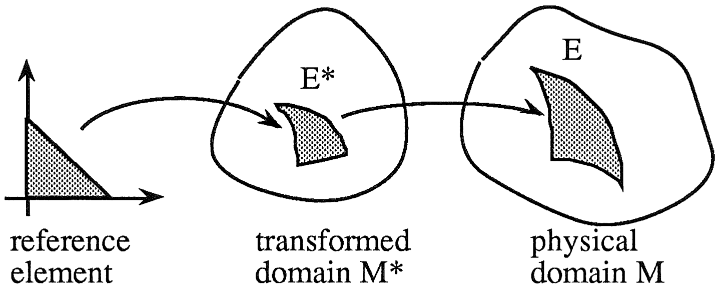

Figure 1: the operation of mapping the transformed domain M on the original domain M is similar to the mapping of the reference finite element on an element of the meshing.*

Common transformations

In the following transformations M is the physical domain with the Cartesian coordinates x and y and M* is the transformed domain with coordinates X and Y.

Kelvin transformation: A fictitious circular boundary of radius A is defined around the problem and the outside of this circle constitutes an infinite domain M. The method consists of mapping the inside M* of the circle on M. The transformation used is the Kelvin transformation given by the two functions (with $R = \sqrt{X^2 + Y^2}$):

$$x = f_1(X,Y) = A^2 X / R^2 \qquad (8)$$

$$y = f_2(X,Y) = A^2 Y / R^2 \qquad (9)$$

This transformation is known to map harmonic functions in M on harmonic functions in M*. Therefore the transformation matrix reduces to (minus) the unit matrix. No special treatment is necessary to use this method except to manage the fact that there are several elements at the same place because the transformed exterior elements are mapped inside the circle where the interior elements are. The minus sign introduced by the transformation matrix is taken into account by restoring the orientation of the elements changed by the geometric transformation.

Cylindrical shell: A fictitious circular boundary of radius A is defined around the problem and the outside of this circle constitutes an infinite domain M. The method consists of mapping a corona M* on M. The corona, a finite domain, has an inner radius A and an outer radius B, all the circles considered here having the same centre. The transformation is given by the two functions (with $R = \sqrt{X^2 + Y^2}$):

$$x = f_1(X,Y) = X\,[A\,(B-A)] / [R\,(B-R)] \qquad (10)$$

$$y = f_2(X,Y) = Y\,[A\,(B-A)] / [R\,(B-R)] \qquad (11)$$

There are no restrictions on B and for B = 0 one finds the Kelvin transformation. The interest of this method is to transform the open domain in a finite domain contiguous but separate from the interior domain. This leads to a more familiar situation for FE codes and B is generally chosen greater than A.

<u>Other methods</u>*:* ellipsoidal and rectangular shells are also used to match more efficiently geometries with a large aspect ratio.

NUMERICAL EXAMPLES

As an application, two problems are considered. The first one has an analytical solution: four circular wires placed symmetrically and fed by balanced currents (figures 2 and 3).

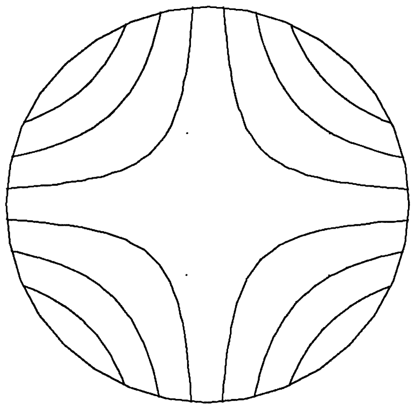

Figure 2: mapped domain with Kelvin transformation

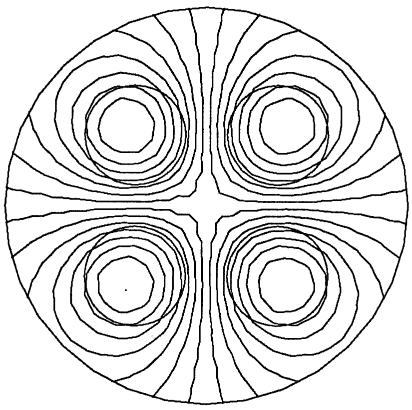

Figure 3 : the four wires.

Consider a wire of section a crossed by a current I. The potential A is calculated as follows :

$$A_z(r) = \begin{cases} \dfrac{\mu_0 I}{4\pi a^2}\left(a^2 - r^2\right) & \text{if } r < a \\ \dfrac{\mu_0 I}{2\pi}\ln\left(\dfrac{a}{r}\right) & \text{if } r \geq a \end{cases} \qquad (12)$$

where r is the radius calculated from the centre of a wire. The entire solution is the superposition of the four wires contributions. The magnetic flux density is derived from (12). The solution is calculated by all the methods described above. The next figures compare the magnetic flux density calculated along C (figure 3).

All the methods give much better results than only applying a boundary condition at a finite distance (figure 4). FE-BE coupling method gives the best results. The Kelvin transformation is quite equivalent but needs less computing time and does not implies any additional code to an existent F.E.M solver.

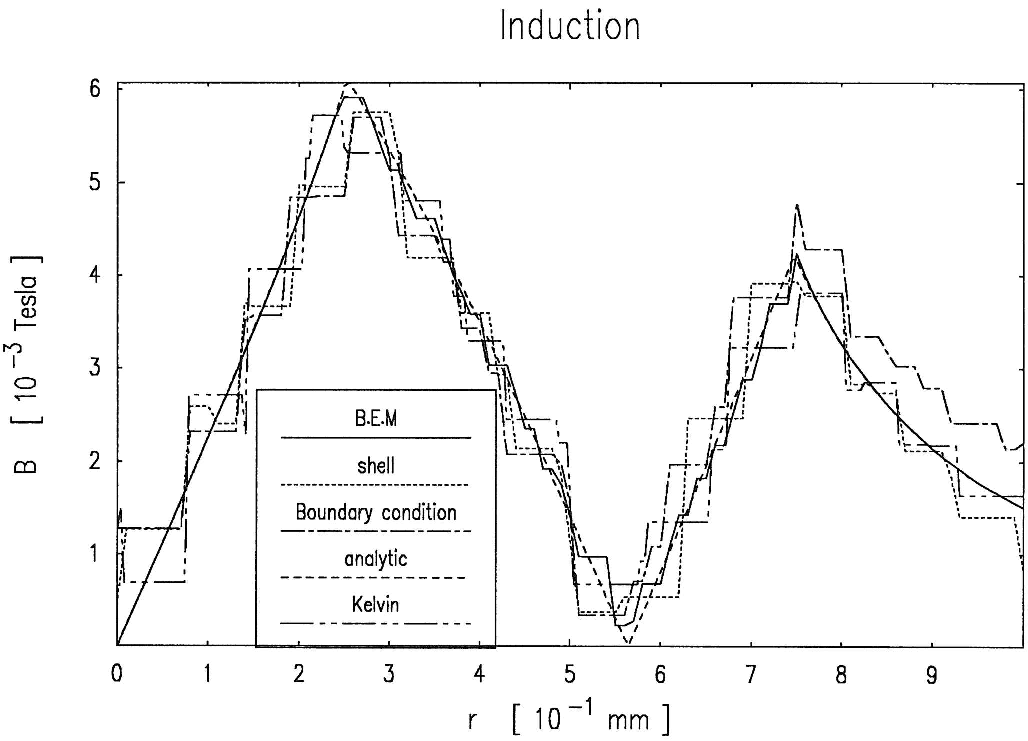

figure 4 : comparative results

The next table sums up the pros and the cons of each method :

Method	**Arguments for**	**Arguments against**
FE-BE.	High precision. No domain mesh.	High C.P.U time. High complexity.
Kelvin.	High precision. No additional code.	Only for 2-D.
Shell.	Easy to code. Problems with large aspect ratios.	Ill conditioned matrices especially for high order elements.
Boundary condition at finite distance.	Satisfactory if the field is enclosed.	Poor precision if the boundary is not taken far enough. Large number of d.o.f. if the boundary is far.

The second example is an industrial open problem : calculation of the magnetic field in an induction furnace. The furnace is composed of ten inductors that are heating a thin plate. It is obviously the kind of problem frequiring the computation of the exterior field.

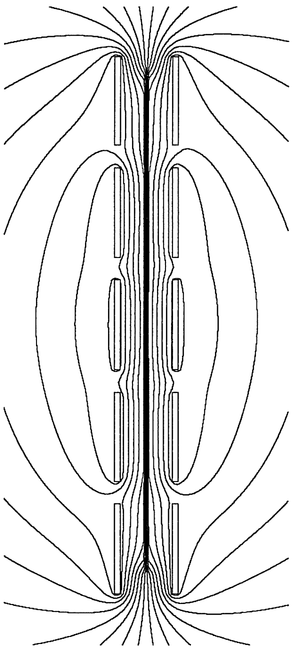

Figure 5: the furnace calculated by BEM method

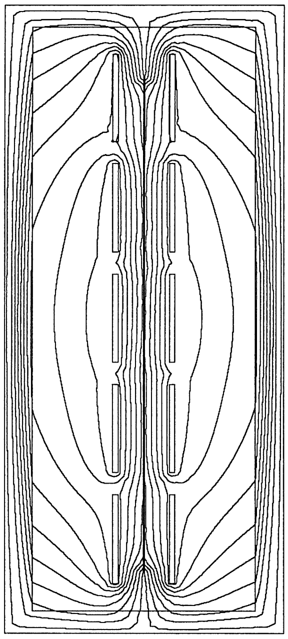

Figure 6 : the furnace calculated with a rectangular infinite shell

Using the rectangular shell allows to take into account the large aspect ratio of the problem and therefore limit the number of degrees of freedom. The BEM model contains 1080 degrees of freedom and 160 seconds are needed to invert the system by a direct method. The FEM model with the rectangular infinite shell contains 3556 degrees of freedom but the system is symmetric positive definite and can be solved by an iterative method in 17 seconds.

CONCLUSION

Both the FE-BE coupling and the finite element transformation method give excellent results. The boundary element method gives an excellent accuracy even for far field. With a fine meshing of the boundary, the exterior solution is in fact very close to exact solution. If only the interior solution is of interest a pure FE solution may be as accurate and faster than the FE-BE coupling.

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