Finite element analysis of helicoidal waveguides

A. Nicolet and F. Zolla

Abstract: The purpose of the paper is to propose an efficient method to compute propagation modes in helicoidal waveguides. An helicoidal system of co-ordinates is introduced to define the structure and to set up the problem. These co-ordinates, albeit non-orthogonal, preserve the translational invariance in a way that allows a two-dimensional finite element model similar to that of classical straight waveguides.

1 Introduction

The modelling of ideal electromagnetic waveguides relies on their translational invariance to reduce it to a two-dimensional computation. In the case of helicoidal waveguides, this invariance is lost. The helicoidal geometry sometimes arises in an unwanted way, for example, during the production process of microstructured optical fibres, an uncontrolled twist can appear. The proposed method presents a technique to preserve the two-dimensional character of the computation of the modes of such waveguides. The method is an exact (in the framework of classical electromagnetism) vector formulation together with a general numerical finite element analysis.

2 Helicoidal waveguides

Let us introduce a twisted co-ordinate system (ξ_1, ξ_2, ξ_3) [1–5] deduced from rectangular Cartesian co-ordinates (x_1, x_2, x_3) in the following way

$$\begin{cases} x_1 = \xi_1 \cos(\alpha\xi_3) + \xi_2 \sin(\alpha\xi_3) \\ x_2 = -\xi_1 \sin(\alpha\xi_3) + \xi_2 \cos(\alpha\xi_3) \\ x_3 = \xi_3 \end{cases} \tag{1}$$

where α is a parameter that characterises the torsion of the structure. A twisted structure is a structure for which both geometrical and physical characteristics (here, the permittivity ε, the permeability μ and the geometry of the perfect conductors) only depend on ξ_1 and ξ_2.

Note that such a structure is invariant along ξ_3 but $(2\pi/\alpha)$-periodic along x_3.

This general co-ordinate system is characterised by the Jacobian of the transformation (1)

$$J(\xi_1, \xi_2, \xi_3) = \frac{\partial(x_1, x_2, x_3)}{\partial(\xi_1, \xi_2, \xi_3)}$$

$$= \begin{pmatrix} \cos(\alpha\xi_3) & \sin(\alpha\xi_3) & \begin{matrix}\alpha\xi_2\cos(\alpha\xi_3) \\ -\alpha\xi_1\sin(\alpha\xi_3)\end{matrix} \\ -\sin(\alpha\xi_3) & \cos(\alpha\xi_3) & \begin{matrix}-\alpha\xi_1\cos(\alpha\xi_3) \\ -\alpha\xi_2\sin(\alpha\xi_3)\end{matrix} \\ 0 & 0 & 1 \end{pmatrix}$$

which does depend on the three variables ξ_1, ξ_2 and ξ_3. In contrast, the transformation matrix $\boldsymbol{T}$ [4]

$$\boldsymbol{T}(\xi_1, \xi_2) = \frac{\boldsymbol{J}^{\mathrm{T}}\boldsymbol{J}}{\det(\boldsymbol{J})} = \begin{pmatrix} 1 & 0 & \alpha\xi_2 \\ 0 & 1 & -\alpha\xi_1 \\ \alpha\xi_2 & -\alpha\xi_1 & 1+\alpha^2(\xi_1^2+\xi_2^2) \end{pmatrix} \tag{2}$$

which describes the change in the material properties, only depends on the first two variables ξ_1 and ξ_2. The inverse matrix is

$$\boldsymbol{T}^{-1}(\xi_1, \xi_2) = \begin{pmatrix} 1+\alpha^2\xi_2^2 & -\alpha^2\xi_1\xi_2 & -\alpha\xi_2 \\ -\alpha^2\xi_1\xi_2 & 1+\alpha^2\xi_1^2 & \alpha\xi_1 \\ -\alpha\xi_2 & \alpha\xi_1 & 1 \end{pmatrix} \tag{3}$$

A more compact way to write these matrices is

$$\boldsymbol{T} = \begin{pmatrix} \boldsymbol{I} & -\alpha\boldsymbol{R} \\ -\alpha\boldsymbol{R}^{\mathrm{T}} & 1+\alpha^2\boldsymbol{R}^{\mathrm{T}}\boldsymbol{R} \end{pmatrix} \tag{4}$$

and

$$\boldsymbol{T}^{-1} = \begin{pmatrix} \boldsymbol{I}+\alpha^2\boldsymbol{R}\boldsymbol{R}^{\mathrm{T}} & \alpha\boldsymbol{R} \\ \alpha\boldsymbol{R}^{\mathrm{T}} & 1 \end{pmatrix} \tag{5}$$

where $\boldsymbol{I}$ is the 2×2 unit matrix, and

$$\boldsymbol{R} = \begin{pmatrix} -\xi_2 \\ \xi_1 \end{pmatrix}$$

is the $\pi/2$ counterclockwise rotation of $\boldsymbol{\xi}$, the position vector in the cross-section with respect to the axis of rotation.

From a geometric point of view, the matrix $\boldsymbol{T}$ plays the role of the metric tensor [3]. Another way to see it, is that the change of co-ordinates amounts to replacement of the different materials (often homogeneous and isotropic) by equivalent, inhomogeneous anisotropic materials. The twisted problem is set up by the actual material characteristics ε and μ being replaced by new tensorial ones given by $\underline{\underline{\varepsilon}}' = \varepsilon\boldsymbol{T}^{-1}$ and $\underline{\underline{\mu}}' = \mu\boldsymbol{T}^{-1}$. Note that the permittivity and permeability undergo the same

doi:10.1049/iet-smt:20060042

Paper first received 15th March and in revised form 15th June 2006

A. Nicolet is with the Institut Fresnel, Université Paul Cézanne, UMR CNRS, Marseille 6133, France

F. Zolla is with the Institut Fresnel, Université de Provence, UMR CNRS, Marseille 6133, France

E-mail: andre.nicolet@fresnel.fr

Techset Composition Ltd, Salisbury
Doc: {IEE}smt/SMT51840.3d

transformation, so that the impedances of the media remain unchanged. It allows for a simple setting of the finite element method for twisted structures [4–6].

Our goal is to obtain numerically the propagation modes in an electromagnetic waveguide twisted along the ξ_3-axis and therefore described by its cross-section in the $\xi_1\xi_2$-plane. We choose to formulate the problem in terms of the electric field $\mathcal{E}$ with homegeneous boundary conditions, that is for a guide with perfectly conducting metallic walls. (A formulation of the problem in terms of the magnetic field could be handled in the same way.)

Choosing a time dependence in $e^{-i\omega t}$, and taking into account the invariance of the structure in the local co-ordinates (ξ_1, ξ_2, ξ_3) along the ξ_3-axis, we define the time-harmonic two-dimensional electric field $\boldsymbol{E}$

$$\mathcal{E}(\xi_1, \xi_2, \xi_3, t) = \Re e\left(\boldsymbol{E}(\xi_1, \xi_2)\mathrm{e}^{-i(\omega t-\beta\xi_3)}\right) \tag{6}$$

where $\omega = k_0 c = k_0/\sqrt{(\mu_0\varepsilon_0)}$ is the angular frequency, and β is the propagating constant of the guided mode. Note that $\boldsymbol{E}$ is a complex-valued field depending on two variables (co-ordinates ξ_1 and ξ_2), but still with three components (along the three axes $\boldsymbol{e}^{\xi_1}$, $\boldsymbol{e}^{\xi_2}$ and $\boldsymbol{e}^{\xi_3}$). The two-dimensional electric field is separated into a transverse component $\boldsymbol{E}_t$ in the $\xi_1\xi_2$-plane and a longitudinal field E_ℓ along the ξ_3-axis (unit vector $\boldsymbol{e}^{\xi_3}$) of invariance, so that $\boldsymbol{E} = \boldsymbol{E}_t + E_\ell \boldsymbol{e}^{\xi_3}$, with $\boldsymbol{E}_t \cdot \boldsymbol{e}^{\xi_3} = 0$. Note that the fact that the twisted co-ordinate system is not orthogonal does not lead to any ambiguity about this scalar product, as $\boldsymbol{e}^{\xi_3}$ is clearly always orthogonal to the $\xi_1\xi_2$-plane.

Writing Maxwell's equations in terms of $\boldsymbol{E}$, we obtain

$$\mathbf{curl}_\beta(\mu_r^{-1}\mathbf{curl}_\beta \boldsymbol{E}) = k_0^2 \epsilon_r \boldsymbol{E} \tag{7}$$

where the operator $\mathbf{curl}_\beta$ is defined as

$$\mathbf{curl}_\beta \boldsymbol{U}(\xi_1, \xi_2) = \mathbf{curl}(\boldsymbol{U}(\xi_1, \xi_2)\mathrm{e}^{i\beta\xi_3})\mathrm{e}^{-i\beta\xi_3}$$

with **curl** having the same formal definition in terms of partial derivatives as in the case of rectangular Cartesian co-ordinates. The following transverse operators are defined for a scalar function $\varphi(\xi_1, \xi_2)$ and a transverse field $\boldsymbol{v} = v_{\xi_1}(\xi_1, \xi_2)\boldsymbol{e}^{\xi_1} + v_{\xi_2}(\xi_1, \xi_2)\boldsymbol{e}^{\xi_2}$

$$\mathbf{grad}_t \varphi = \frac{\partial\varphi}{\partial\xi_1}\boldsymbol{e}^{\xi_1} + \frac{\partial\varphi}{\partial\xi_2}\boldsymbol{e}^{\xi_2}$$

$$\mathbf{curl}_t \boldsymbol{v} = \left(\frac{\partial v_{\xi_1}}{\partial\xi_2} - \frac{\partial v_{\xi_2}}{\partial\xi_1}\right)\boldsymbol{e}^{\xi_3}$$

and are used to separate $\mathbf{curl}_\beta$ into its transverse and longitudinal components

$$\mathbf{curl}_\beta(\boldsymbol{v} + \varphi\boldsymbol{e}^{\xi_3}) = \mathbf{curl}_t \boldsymbol{v} + (\mathbf{grad}_t \varphi - i\beta \boldsymbol{v}) \times \boldsymbol{e}^{\xi_3}$$

To summarise, the setting of the equations is exactly the same as for the case of straight guides in rectangular Cartesian co-ordinates, but for the introduction of equivalent inhomogeneous material properties involving the $\boldsymbol{T}$ matrix.

3 Finite element modelling

The discretisation of Maxwell's equations (together with the equivalent material properties $\epsilon_r'(\xi_1, \xi_2)$, $\mu_r'(\xi_1, \xi_2)$ is obtained with finite elements [4]. The cross-section of the guide is meshed with triangles, and Whitney finite elements are used, i.e. edge elements for the transverse field and nodal elements for the longitudinal field

$$\boldsymbol{E}_t = \sum_{j=1}^{\text{edges}} \mathrm{e}_j^t \boldsymbol{w}_e^j(\xi_1, \xi_2) \quad \text{and } E_\ell = \sum_{j=1}^{\text{nodes}} \mathrm{e}_j^\ell w_n^j(\xi_1, \xi_2)$$

where e_j^t denotes the line integral of the transverse component $\boldsymbol{E}_t$ on the edges, e_j^ℓ denotes the line integral of the longitudinal component E_ℓ along one unit of length of the ξ_3-axis (which is equivalent to the nodal value), and $\boldsymbol{w}_e^j$ and w_n^j are, respectively, the basis functions of Whitney 1-forms and 0-forms on triangles. On a triangle, if λ_i denotes the barycentric co-ordinate associated with the node i, $\boldsymbol{w}_e = \lambda_i \,\mathrm{grad}\, \lambda_j - \lambda_j \,\mathrm{grad}\, \lambda_i$ for the edge going from node i to node j, and $w_n = \lambda_i$ for the node i.

As the electric field satisfies a homogeneous Dirichlet boundary condition ($\boldsymbol{n} \times \boldsymbol{E} = 0$) on the boundary of the guide, the weak formulation of (7) gives

$$\mathcal{R}(\boldsymbol{E}, \boldsymbol{E}') = \int_\Omega (\mu_r'^{-1}\mathbf{curl}_\beta \boldsymbol{E}) \cdot \overline{\mathbf{curl}_\beta \boldsymbol{E}'}\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 - k_0^2 \times \int_\Omega (\varepsilon_r' \boldsymbol{E}) \cdot \overline{\boldsymbol{E}}' \mathrm{d}\xi_1 \mathrm{d}\xi_2 = 0 \quad \forall \boldsymbol{E}' \in H(\mathbf{curl}_\beta, \Omega) \tag{8}$$

where $\mu_r'^{-1} = \mu_r^{-1}\boldsymbol{T}$ and $\varepsilon_r' = \varepsilon_r \boldsymbol{T}^{-1}$ are the equivalent (anisotropic and inhomogeneous) material properties. The μ_r and ε_r are the original material properties that are commonly (but not necessarily) isotropic piecewise homogeneous. The space $H(\mathbf{curl}_\beta, \Omega)$ of curl-conforming fields is defined as $H(\mathbf{curl}_\beta, \Omega) = \{\boldsymbol{v} \in [L^2(\Omega)]^3, \mathbf{curl}_\beta \boldsymbol{v} \in [L^2(\Omega)]^3\}$.

In terms of transverse and longitudinal components, this weighted residual corresponding to the propagative mode problem for helicoidal waveguides is [4]

$$\begin{aligned}
&\mathcal{R}(\boldsymbol{E}, \boldsymbol{E}') \\
&= \int_\Omega \mu_r'^{-1}(\mathbf{curl}_t \boldsymbol{E}_t + (\mathbf{grad}_t E_\ell - i\beta \boldsymbol{E}_t) \times \boldsymbol{e}^{\xi_3}) \\
&\quad \times \overline{(\mathbf{curl}_t \boldsymbol{E}_t' + (\mathbf{grad}_t E_\ell' - i\beta \boldsymbol{E}_t') \times \boldsymbol{e}^{\xi_3})}\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \\
&\quad - k_0^2 \int_\Omega \varepsilon_r'(\boldsymbol{E}_t + E_\ell \boldsymbol{e}^{\xi_3}) \cdot \overline{(\boldsymbol{E}_t' + E_\ell' \boldsymbol{e}^{\xi_3})}\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \\
&= \int_\Omega \Big((\mu_r'^{-1}\mathbf{curl}_t \boldsymbol{E}_t) \cdot \mathbf{curl}_t \overline{\boldsymbol{E}_t'} \\
&\quad + (\mu_r'^{-1}\mathbf{curl}_t \boldsymbol{E}_t) \cdot (\mathbf{grad}_t \overline{E_\ell'} \times \boldsymbol{e}^{\xi_3}) \\
&\quad + i\beta(\mu_r'^{-1}\mathbf{curl}_t \boldsymbol{E}_t) \cdot (\overline{\boldsymbol{E}_t'} \times \boldsymbol{e}^{\xi_3}) \\
&\quad + (\mu_r'^{-1}(\mathbf{grad}_t E_\ell \times \boldsymbol{e}^{\xi_3})) \cdot \mathbf{curl}_t \overline{\boldsymbol{E}_t'} \\
&\quad + (\mu_r'^{-1}(\mathbf{grad}_t E_\ell \times \boldsymbol{e}^{\xi_3})) \cdot (\mathbf{grad}_t \overline{E_\ell'} \times \boldsymbol{e}^{\xi_3}) \\
&\quad + i\beta(\mu_r'^{-1}(\mathbf{grad}_t E_\ell \times \boldsymbol{e}^{\xi_3})) \cdot (\overline{\boldsymbol{E}_t'} \times \boldsymbol{e}^{\xi_3}) \\
&\quad - i\beta(\mu_r'^{-1}(\boldsymbol{E}_t \times \boldsymbol{e}^{\xi_3})) \cdot \mathbf{curl}_t \overline{\boldsymbol{E}_t'} \\
&\quad - i\beta(\mu_r'^{-1}(\boldsymbol{E}_t \times \boldsymbol{e}^{\xi_3})) \cdot (\mathbf{grad}_t \overline{E_\ell'} \times \boldsymbol{e}^{\xi_3}) \\
&\quad + \beta^2(\mu_r'^{-1}(\boldsymbol{E}_t \times \boldsymbol{e}^{\xi_3})) \cdot (\overline{\boldsymbol{E}_t'} \times \boldsymbol{e}^{\xi_3}) \Big)\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \\
&\quad - k_0^2 \int_\Omega \Big((\varepsilon_r' \boldsymbol{E}_t) \cdot \overline{\boldsymbol{E}_t'} + (\varepsilon_r' \boldsymbol{E}_t) \cdot \overline{E_\ell'} \boldsymbol{e}^{\xi_3} \\
&\quad + (\varepsilon_r' E_\ell \boldsymbol{e}^{\xi_3}) \cdot \overline{\boldsymbol{E}_t'} + (\varepsilon_r' E_\ell \boldsymbol{e}^{\xi_3}) \cdot \overline{E_\ell'} \boldsymbol{e}^{\xi_3} \Big)\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 = 0
\end{aligned}$$

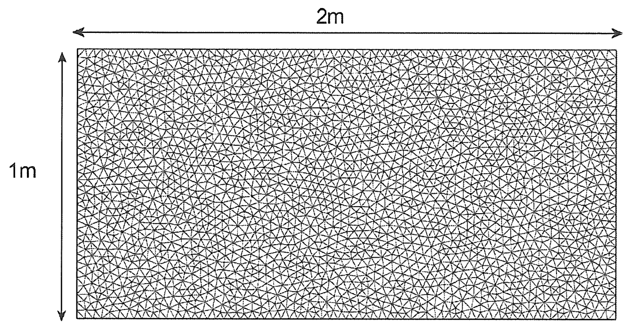

Fig. 1 *Geometry of rectangular waveguide with triangular finite element mesh*

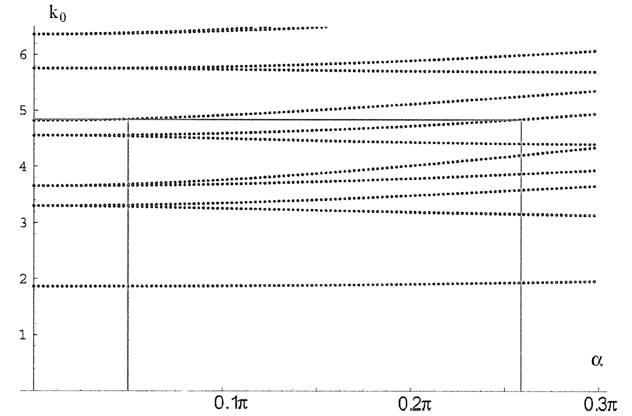

Fig. 2 *Variation of k_0 (m^{-1}) with respect to torsion $\alpha(m^{-1})$ of guide with $\beta = 1.0\ m^{-1}$*

If we introduce the finite element approximation in the weighted residual and take the basis of the discrete spaces as weights E'_ℓ and $\boldsymbol{E}'_t$ (Galerkin method), the problem reduces to a matrix system.

For fixed β, this system is a generalised eigenvalue problem

$$-k_0^2 \boldsymbol{M}_\beta \boldsymbol{u} + \boldsymbol{K}_\beta \boldsymbol{u} = 0$$

where $\boldsymbol{M}_\beta$ and $\boldsymbol{K}_\beta$ are $N \times N$ matrices (where N is the number of unknowns) depending on β, and k_0^2 and $\boldsymbol{u}$ are the eigenvalue and eigenvector, respectively.

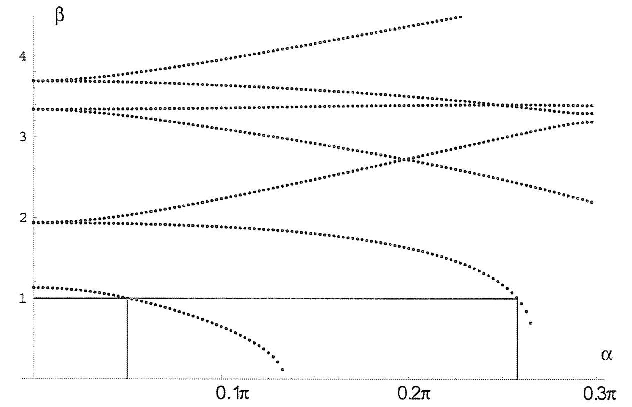

Fig. 3 *Variation of β (m^{-1}) with respect to torsion α (m^{-1}) of guide with $k_0 = 4.84596\ m^{-1}$*

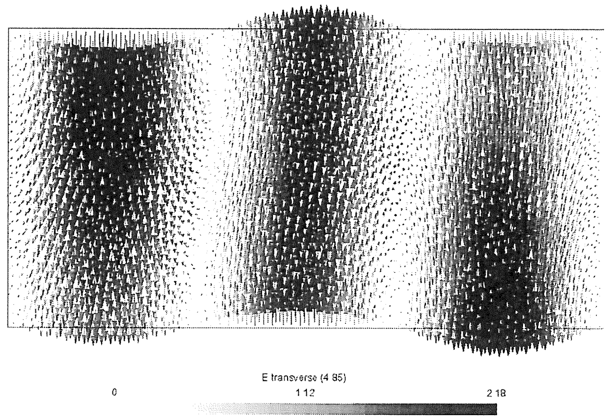

Fig. 4 *Transverse electric field (real part) of mode corresponding to $\alpha = 0.05\pi\ m^{-1}$, $\beta = 1.0\ m^{-1}$ and $k_0 = 4.84596\ m^{-1}$*

For fixed k_0, this system is a generalised quadratic eigenvalue problem of the form

$$-\beta^2 \boldsymbol{M}_{k_0} \boldsymbol{u} + i\beta \boldsymbol{L}_{k_0} \boldsymbol{u} + \boldsymbol{K}_{k_0} \boldsymbol{u} = 0$$

where $\boldsymbol{M}_{k_0}$, $\boldsymbol{L}_{k_0}$ and $\boldsymbol{K}_{k_0}$ are $N \times N$ matrices depending on k_0, and β and $\boldsymbol{u}$ are the eigenvalue and eigenvector, respectively. This quadratic problem is then transformed to a linear problem in β involving $2N \times 2N$ matrices, but where all the operations can be performed on $N \times N$ submatrices [7].

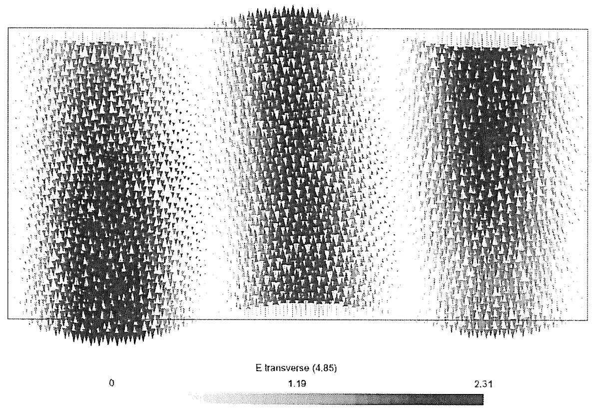

Fig. 5 *Transverse electric field (imaginary part) of mode corresponding to $\alpha = 0.05\ \pi\ m^{-1}$, $\beta = 1.0\ m^{-1}$ and $k_0 = 4.84596\ m^{-1}$*

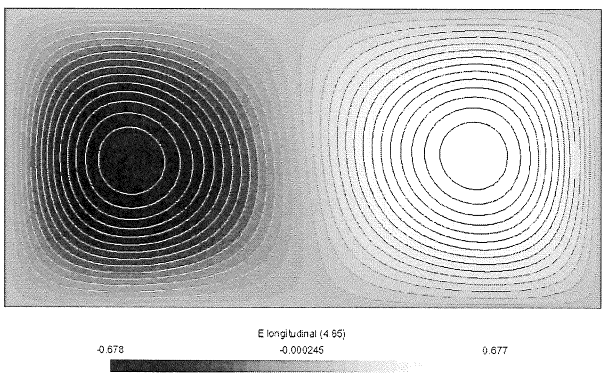

Fig. 6 *Longitudinal electric field (real part) of mode corresponding to $\alpha = 0.05\pi\ m^{-1}$, $\beta = 1.0\ m^{-1}$, and $k_0 = 4.84596\ m^{-1}$*

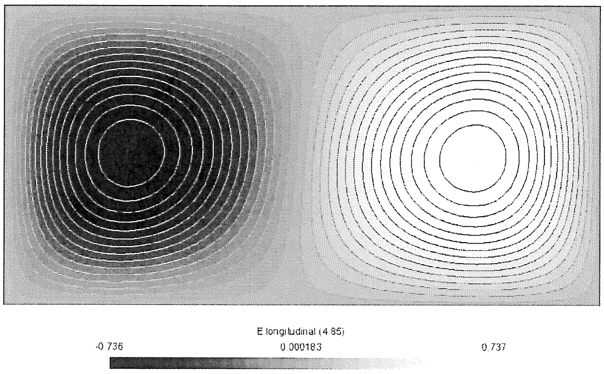

Fig. 7 *Longitudinal electric field (imaginary part) of mode corresponding to* $\alpha = 0.05\pi\,m^{-1}$, $\beta = 1.0\,m^{-1}$ *and* $k_0 = 4.84596\,m^{-1}$

4 Numerical example

As a numerical example, a rectangular waveguide with a width of 2 m and a height of 1 m is considered (the results can, of course, be easily scaled to more realistic dimensions), and Fig. 1 shows the triangular meshing used for the numerical computation. The two approaches, β fixed or k_0 fixed, are used to find the propagation modes.

Fig. 2 shows the evolution of the frequency of the modes for a given value of $\beta = 1.0\,\mathrm{m}^{-1}$. The 2 : 1 ratio of the width on the height gives a twofold degeneracy of some modes in the straight guide ($\alpha = 0$) that is destroyed by the twist.

Fig. 3 shows the evolution of the frequency of the modes for a given value of $k_0 = 4.84596\,\mathrm{m}^{-1}$. The shape of the curves is more dramatic, as some modes seem to disappear for large values of α.

As a verification, for $\alpha = 0.05\pi\,\mathrm{m}^{-1}$, there is a mode corresponding to $\beta = 1.0\,\mathrm{m}^{-1}$ and $k_0 = 4.84596\,\mathrm{m}^{-1}$. Another mode with the same values of β and k_0 is found for $\alpha = 0.256\pi\,\mathrm{m}^{-1}$. As indicated on Figs. 2 and 3, these two modes are found by the two approaches (β fixed or k_0 fixed).

Figs. 4–7 show the real and imaginary parts of the transverse and longitudinal electric fields of this mode. The twist of the waveguide appears clearly in the loss of symmetry of the field patterns with respect to the corresponding case of the straight rectangular waveguide. Moreover, the presence of the longitudinal electric field together with the transverse one shows that the mode is not transverse electric. The concept of transverse electric and transverse magnetic modes only makes sense in non-twisted guides.

Moreover, the longitudinal component represented here is $E_\ell = E_{\xi_3}$, but, in the $z = \xi_3 = 0$ plane (for which $\xi_j = x_j$, for $j = 1, 2, 3$), the relationship between electric field components in the rectangular and twisted systems is given by $E_x = E_{\xi_1}$, $E_y = E_{\xi_2}$, $E_z = -\alpha\xi_2 E_{\xi_1} + \alpha\xi_1 E_{\xi_2} + E_{\xi_3} = \alpha \boldsymbol{R} \cdot \boldsymbol{E}_t + E_\ell$.

5 Conclusions

Helicoidal waveguides provide a good example to apply the principle 'change of co-ordinate system can be represented by equivalent material properties' [4]. In terms of differential geometry, it means that the Maxwell equations involve only topological and differential concepts that are insensitive to smooth deformations of the space, whereas all the metric aspects (the Hodge operator) are concentrated in material properties, that is the dielectric and magnetic relationships.

A possible application of the model is the study of the effect of parasitic torsion in microstructured optical fibres. Future work will include the estimation of the losses of leaky modes in the twisted case.

6 Acknowledgment

The authors would like to thank Professor C. Geuzaine from Case Western University for the programming of the quadratic eigenvalue solution in GetDP [8].

7 References

1 Lewin, L., and Ruehle, T.: 'Propagation in twisted square waveguide', *IEEE Trans. Microw. Theory Tech.*, 1980, **28**, (1), pp. 44–48

2 Yabe, H., and Mushiake, Y.: 'An analysis of a hybrid-mode in a twisted rectangular waveguide', *IEEE Trans. Microw. Theory Tech.*, 1984, **32**, (1), pp. 65–71

3 Igarashi, H., and Honma, T.: 'A finite element analysis of TE modes in twisted waveguides', *IEEE Trans. Magn.*, 1991, **27**, (5), pp. 4052–4055

4 Zolla, F., Renversez, G., Nicolet, A., Khulmey, B., Guenneau, S., and Felbacq, D.: 'Foundations of photonic crystal fibres' (Imperial College Press, London, 2005)

5 Nicolet, A., Zolla, F., and Guenneau, S.: 'Modelling of twisted optical waveguides with edge elements', *Eur. Phys. J. Appl. Phys.*, 2004, **28**, pp. 153–157

6 Nicolet, A., Movchan, A.B., Guenneau, S., and Zolla, F.: 'Asymptotic modelling of weakly twisted electrostatic problems', *C. R. Mecanique*, 2006, **334**, (2), pp. 91–97

7 Nicolet, A., and Geuzaine, C.: 'Waveguide propagation modes and quadratic eigenvalue problems', *IEE Proc., Sci. Meas. Technol.*, this issue Q1

8 Dular, P., Geuzaine, C., Henrotte, F., and Legros, W.: 'A general environment for the treatment of discrete problems and its application to the finite element method', *IEEE Trans. Magn.*, 1998, **34**, (5), pp. 3395–3398 and available online at: http://www.geuz.org