Finite-Element Analysis of Cylindrical Invisibility Cloaks of Elliptical Cross Section

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We present a finite-element analysis of a diffraction problem involving a coated cylinder enabling the electromagnetic cloaking of a finite conducting object with sharp wedges located within its core. The coating consists of a hollow cylinder with a circular cross section made of heterogeneous anisotropic material deduced from a geometric transformation as first proposed by Pendry *et al.* The shape of the cloak is then generalized to elliptic cross sections.

***Index Terms*—Cloaking, electromagnetic scattering, finite elements, geometric transformation, invisibility.**

I. Introduction

Recently, it was suggested by Pendry *et al.* that an object surrounded by a coating consisting of an exotic material becomes invisible for electromagnetic waves [1]. The theoretical idea based on geometric transformations was supplied with numerics performed with a software based on geometrical optics. In the present paper, we provide a full electromagnetic wave picture which supports the results of [1] in the case of a 2-D cylindrical object first with a cloak of circular shape [2], and then we extend these results to the case of elliptical cross sections. In electromagnetism, a change of coordinates amounts to replacing the different materials (often homogeneous and isotropic, which corresponds to the case of scalar piecewise constant permittivities and permeabilities) by equivalent inhomogeneous anisotropic materials described by a transformation matrix $\mathbf{T}$ (metric tensor) [3]. The idea underpinning electromagnetic invisibility [1] is that a suitable geometric transformation provides the material characteristics of a cloak described by a tensor field $\mathbf{T}$ and then that newly discovered metamaterials enable control of the optical properties by mimicking the heterogeneous anisotropic nature of $\mathbf{T}$ [4].

II. Geometric Transformations

The basic principle of the methods presented in this paper is to transform a geometrical domain or a coordinate system into another one and to search how the equations to be solved have to be changed. As we start with a given set of equations in a given coordinate system, it seems at first sight that we have to map these coordinates on the new ones. Nevertheless it is the opposite that has to be done: the new coordinate system is mapped on the initial one (i.e., the new coordinates are defined as explicit functions of the initial coordinates) and the equations are then pulled back, according to differential geometry [5], on the new coordinates. This requires only the computation of the Jacobian (matrix) $\mathbf{J}$ made of the partial derivatives of the new coordinates with respect to the original ones.

In electromagnetism, such a change of coordinates amounts to replacing the different materials (often homogeneous and isotropic, which corresponds to the case of scalar piecewise constant permittivities and permeabilities) by equivalent inhomogeneous anisotropic materials described by a transformation matrix $\mathbf{T} = \mathbf{J}^T\mathbf{J}/det(\mathbf{J})$ [3].

From a geometric point of view, the matrix $\mathbf{T}$ is a representation of the metric tensor. The only thing to do in the transformed coordinates is to replace the materials (homogeneous and isotropic) by equivalent ones whose properties are given by the permittivity and permeability tensors

$$\underline{\underline{\varepsilon}}' = \varepsilon\mathbf{T}^{-1} \quad \underline{\underline{\mu}}' = \mu\mathbf{T}^{-1}. \tag{1}$$

We note that there is no change in the impedance of the media since the permittivity and permeability suffer the same transformation. As for the vector analysis operators and products, everything works as if we were in Cartesian coordinates.

In the more general case where the initial $\underline{\underline{\varepsilon}}$ and $\underline{\underline{\mu}}$ are tensors corresponding to anisotropic properties, the equivalent properties become [6]

$$\underline{\underline{\varepsilon}}' = \mathbf{J}^{-1}\underline{\underline{\varepsilon}}\mathbf{J}^{-T}det(\mathbf{J}) \quad \underline{\underline{\mu}}' = \mathbf{J}^{-1}\underline{\underline{\mu}}\mathbf{J}^{-T}det(\mathbf{J}) \tag{2}$$

where $\mathbf{J}^{-1}$ is the inverse of the Jacobian matrix and $\mathbf{J}^{-T} \equiv (\mathbf{J}^{-1})^T$ the transpose of this inverse.

We will also need to consider compound transformations. Let us consider three coordinate systems $\{u, v, w\}$, $\{X, Y, Z\}$, and (x, y, z). The two successive changes of coordinates are given by $\{X(u, v, w), Y(u, v, w), Z(u, v, w)\}$ and $\{x(X, Y, Z), y(X, Y, Z), z(X, Y, Z)\}$. They lead to the Jacobians $\mathbf{J}_{Xu}$ and $\mathbf{J}_{xX}$ so that

$$\mathbf{J}_{xu} = \mathbf{J}_{xX}\mathbf{J}_{Xu}. \tag{3}$$

This rule naturally applies for an arbitrary number of coordinate systems. The total transformation can therefore be considered either as involving a total Jacobian built according to (3), or as successive applications of (2). Note that the maps are defined from the final $\{u, v, w\}$ to the original $\{x, y, z\}$ coordinate system and that the product of the Jacobian matrices, corresponding to the composition of the pull back maps, is in the opposite order.

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III. Circular Cloak

To compute the transformation matrix $\mathbf{T}$ associated with the cloak, we first map Cartesian coordinates onto polar coordinates (r, θ, z). The associated Jacobian matrix is

$$\mathbf{J}_{xr}(r,\theta) = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{pmatrix} \cos(\theta) & -r\sin(\theta) & 0 \\ \sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{R}(\theta)\mathbf{diag}(1,r,1) \quad (4)$$

where $\mathbf{R}(\theta)$ is a rotation matrix and $\mathbf{diag}$ a diagonal matrix.

Let us now consider a 2-D object we want to cloak located within a disk of radius R_1. As proposed in [1], we consider a geometric transformation which maps the field within the disk $r \leq R_2$ onto the annulus $R_1 \leq r \leq R_2$

$$\begin{cases} r' = R_1 + r(R_2 - R_1)/R_2, 0 \leq r \leq R_2 \\ \theta' = \theta, 0 < \theta \leq 2\pi \\ z' = z, z \in \mathbb{R} \end{cases} \quad (5)$$

where r', θ', and z' are "radially contracted cylindrical coordinates." Moreover, this transformation maps the field for $r \geq R_2$ onto itself by the identity transformation. This leads to

$$\mathbf{J}_{rr'} = \frac{\partial(r,\theta,z)}{\partial(r',\theta',z')} = \mathbf{diag}\left(\frac{1}{\alpha},1,1\right) \quad (6)$$

where $\alpha = (R_2 - R_1)/R_2$ for $0 \leq r \leq R_2$ and $\alpha = 1$ for $r > R_2$.

Last, we need to go to Cartesian coordinates x', y', z', which are "contracted Cartesian coordinates" where the modeling takes place to obtain a representation of the metric tensor in the suitable coordinate system. The associated Jacobian matrix is

$$\mathbf{J}_{r'x'}(r',\theta') = \frac{\partial(r',\theta',z')}{\partial(x',y',z')} = \mathbf{J}_{rx}^{T}\left(\frac{1}{r'},\theta'\right) = \mathbf{diag}\left(1,\frac{1}{r'},1\right)\mathbf{R}(-\theta'). \quad (7)$$

The material in the cloak is obtained by mapping these coordinates on the initial classical Cartesian coordinates and pulling back the equation to obtain the $\mathbf{T}$ matrix. Applying the composition rule twice, $\mathbf{J}_{xx'} = \mathbf{J}_{xr}\mathbf{J}_{rr'}\mathbf{J}_{r'x'}$. Hence the material properties of the invisibility cloak are described by the transformation matrix $\mathbf{T} = \mathbf{J}_{xx'}^{T}\mathbf{J}_{xx'}/det(\mathbf{J}_{xx'})$. We will also need its inverse that we give explicitly, taking into account that $r(r') = (r' - R_1/\alpha)$

$$\mathbf{T}^{-1} = \mathbf{R}(\theta')\mathbf{diag}\left(\frac{r'-R_1}{r'},\frac{r'}{r'-R_1},\frac{r'-R_1}{\alpha^2 r'}\right)\mathbf{R}(-\theta'). \quad (8)$$

IV. Elliptical Cloak

A simple generalization is obtained by considering cylinders with an elliptical cross section. This may be deduced from the circular case by scaling the Cartesian coordinates. The global transformation is a mapping of a holey elliptic domain (the inner and outer boundaries are concentric ellipses with the same eccentricity) on a simply connected elliptic domain bounded by the outer ellipse of the cloak. The detailed computation of the equivalent material properties is given by the following sequence of transformations. The starting point is the ellipse that bounds the exterior limit of our cloak with its principal axes chosen conveniently parallel to the coordinates axes. The first step is aimed at restoring the previous situation namely a circular cloak. For this purpose, the plane is scaled by a factor s_y along the (arbitrary chosen) y-axis so that the initial ellipse becomes a circle (*in the scaled coordinates*) defined by the transformation $y = s_y y_s$ (and simply $x = x_s$) characterized by the Jacobian matrix $J_{xx_s} = \mathbf{diag}(1, s_y, 1)$. The next three transformations are then the ones used to build the circular cloak: transformation to cylindrical coordinates, radial contraction (the active part), and transformation to rectangular coordinates. It is important to note that it is the scaled variable y_s that is involved in these various operations. The last step to be performed is an inverse scaling along y-axis: $y' = (1/s_y)y'_s$ to recuperate the initial elliptical shape of the cloak and an identity for the transformation of the outside of the cloak. At the end, a cloak is obtained whose inner and outer boundaries are ellipses with $R_1 = x$-axis of the hole, $s_y\, R_1 = y$-axis of the hole, $R_2 = x$-axis of the external boundary, $s_y\, R_2 = y$-axis of the external boundary. The total Jacobian of this sequence of transformations is $\mathbf{J}_{xx'} = \mathbf{J}_{xx_s}\mathbf{J}_{x_s r}\mathbf{J}_{rr'}\mathbf{J}_{r'x'_s}\mathbf{J}_{x'_s x'}$.

The inverse of the $\mathbf{T}$ matrix is given explicitly by

$$\begin{aligned}\mathbf{T}^{-1} = &\,\mathbf{diag}(1,s_y,1)\mathbf{R}(\theta'_s)\,\mathbf{diag}\left(\alpha,\frac{r'_s}{r},1\right)\mathbf{R}(-\theta_s) \\ &\times \mathbf{diag}(1,1/s_y^2,1)\mathbf{R}(\theta_s)\mathbf{diag}(\alpha,\frac{r'_s}{r},1) \\ &\times \mathbf{R}(-\theta'_s)\mathbf{diag}(1,s_y,1)\frac{r}{\alpha r'_s}\end{aligned} \quad (9)$$

with $r'_s = \sqrt{x'^2 + (y'/s_y)^2}$, $\theta_s = \theta'_s = 2\arctan((y'/s_y)/(x' + \sqrt{x'^2 + (y'/s_y)^2}))$, and $r = (r'_s - R_1)/\alpha$.

Note the angles and distances are computed in the scaled coordinate systems (x_s, y_s) where the ellipses are mapped to circles.

V. Numerical Models

Although the previous computations are rigorous, we would like to make some numerical modelling of the cloaking process. Indeed, the material properties of the cloak involves an unavoidable singular behavior on the inner boundary of the cloak [7], and it is important to check that the invisibility will resist to an approximate numerical computation as this will be a clue for the feasibility of a real-life invisibility cloak made of metamaterials that themselves approximate the ideal behavior of the theoretical materials of the cloak.

The numerical modelling is based on the finite-element method that is perfectly adapted to inhomogeneous and anisotropic media and note, by the way, that the geometric transformations are a useful tool associated with this method [8]. Due to the singular behavior of the material properties and to the strong inhomogeneity and anisotropy, the use of second-order elements together with a very fine mesh in the vicinity of the inner boundary of the cloak are necessary (see Fig. 1). To model the outgoing waves, we use perfectly matched layers [9] that may also be interpreted via geometric transformations [10], [11]. The finite-element model is implemented in the freeware GetDP [12].

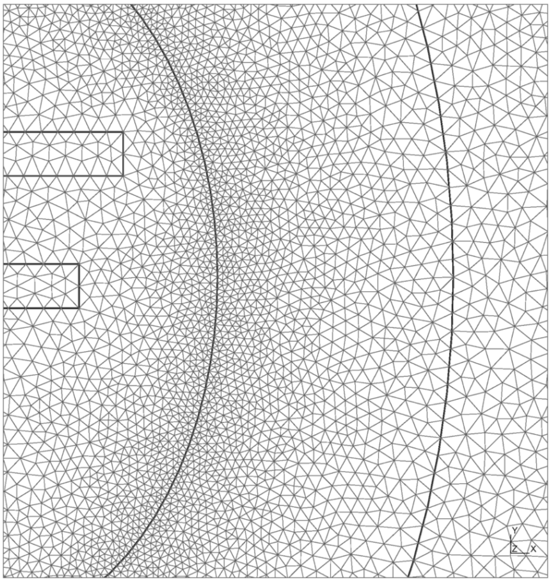

Fig. 1. Magnified picture of a part of the triangular mesh of the cloak. The use of a very fine mesh on the inner boundary of the cloak is necessary. A total of 56 148 second-order triangular elements have been used in this model.

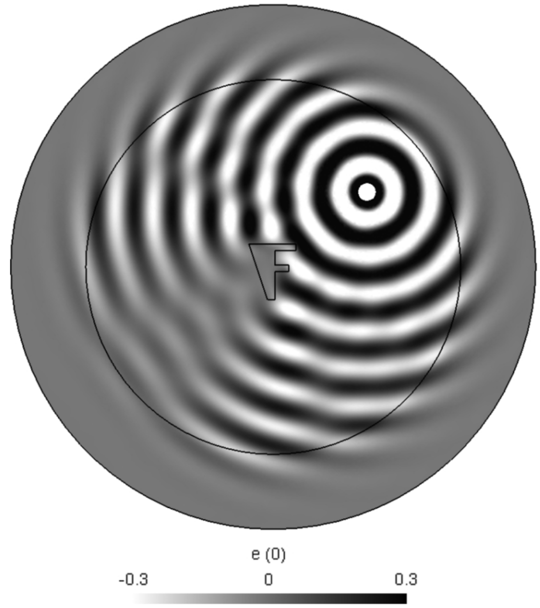

Fig. 3. Real part of the longitudinal electric field of the same line source which radiates in the presence of a F-shaped metallic obstacle.

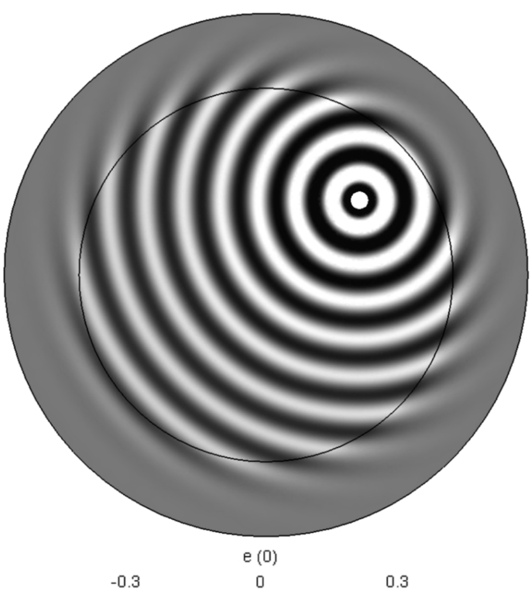

Fig. 2 Real part of the longitudinal electric field of a source which radiates in a vacuum. The source is a wire of circular cross section (the electric field is given on the boundary) and radiates cylindrical waves.

Fig. 4. Real part of the longitudinal electric field for the same line source which radiates in the presence of a F-shaped metallic obstacle surrounded by the invisibility cloak which is a cylinder of circular cross section (interior radius $= 1$, exterior radius $= 2$).

As a test problem, we look at the cylindrical waves radiated by a wire source of circular cross section in presence of a finite conducting object arbitrarily shaped as a letter "F" when it is surrounded by a cloak. We consider here p polarization

$$\nabla \times \left(\underline{\underline{\mu}}'^{-1} \nabla \times \mathbf{E}_l \right) - \mu_0 \varepsilon_0 \omega^2 \underline{\underline{\varepsilon}}' \mathbf{E}_l = 0 \qquad (10)$$

where $\mathbf{E}_l = E_z(x,y)\mathbf{e}_z$, $\underline{\underline{\varepsilon}}'$, and $\underline{\underline{\mu}}'$ are defined by (1).

Let us first consider a source made of a wire of circular cross section (radius $= 0.25$) centered at point $\mathbf{r}_s = (2.5, 2)$ with a constant E_z imposed on its boundary, radiating in a vacuum with wavelength $\lambda = 1$ (*note that all lengths are given in arbitrary units* (microns (μm), *e.g., for visible light*). The electric field E_z is therefore a cylindical wave (Fig. 2) (*note that the electric field is given in arbitrary units*, volts per meter (V/m) for instance, and $E_z = 1$ *on the boundary of the source wire*). In a second experiment, a F-shaped obstacle is placed near the origin (0, 0) beside the aforementioned line source as shown in Fig. 3. This obstacle is made up of an arbitrary homogeneous nonmagnetic lossy material characterized by its permittivity $\varepsilon_{r,F} = 1 + 4i$. Then, the letter "F" is surrounded by an annulus-shaped coating (cloak of invisibility) geometrically characterized by two circles centered on the origin $R_2 = 2$ and $R_1 = 1$ (Fig. 4) and optically characterized by $\underline{\underline{\varepsilon}}'$ and $\underline{\underline{\mu}}'$ given by (1). Finally, the circular cloak is replaced by an elliptical cloak

Fig. 5. Real part of the longitudinal electric field for the same line source which radiates in the presence of a F-shaped metallic obstacle surrounded by the invisibility cloak which is a cylinder of elliptical cross section.

Fig. 6. Magnified picture of the real part of the longitudinal electric field inside the elliptical cloak.

that is a dilation of the circular cloak by a factor 1.5 along the y direction (Fig. 5). Fig. 6 shows in a closer view the distribution of the electric field inside the elliptical cloak.

VI. Conclusion

In this paper, we used the well-known equivalence between a geometric transformation and a change of characteristics of material (permittivity and permeability) not to obtain a more tractable model but to design a new optical device. Previous results have been extended to the case of elliptic shapes. It must be noted that the characteristics of the material properties lead to equations that are theoretically outside the domain of application of the finite-element method (some coefficients become null or infinite and this destroys both the coercivity and the continuity of the operator). This is in fact fundamental for invisibility as this way the inside of the cloak is insulated from its electromagnetic environment. Nevertheless, the numerical computations based on the finite-element method have shown that the invisibility should resist to some approximation of the material properties.

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