Geometrical transformations and equivalent materials in computational electromagnetism

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Abstract

Purpose – This paper aims to review various techniques used in computational electromagnetism such as the treatment of open problems, helicoidal geometries and the design of arbitrarily shaped invisibility cloaks. This seemingly heterogeneous list is unified by the concept of geometrical transformation that leads to equivalent materials. The practical set-up is conveniently effected via the finite element method.

Design/methodology/approach – The change of coordinates is completely encapsulated in the material properties.

Findings – The most significant examples are the simple 2D treatment of helicoidal geometries and the design of arbitrarily shaped invisibility cloaks.

Originality/value – The paper provides a unifying point of view, bridging several techniques in electromagnetism.

Keywords Computational electromagnetism, Helicoidal geometry, Photonic Crystal Fibre, Metamaterial, Invisibility cloaking

Paper type Research paper

1. Geometrical transformations and equivalent materials

Beside Cartesian coordinates, cylindrical and spherical coordinates, and even the other orthogonal systems (Stratton, 1941), have been commonly used to set up electromagnetic problems. In this paper, much more general coordinate systems are discussed since they do not need to be orthogonal (and not even real valued). A modern approach is to write the equations of electromagnetism in the language of exterior calculus that is covariant, i.e. independent of the choice of the coordinate system (Bossavit, 1991). In this way, the Maxwell equations involve only the exterior derivative and are purely topological and differential while all the metric information is contained in the material properties via a Hodge star operator. This looks rather abstract but can nevertheless be encapsulated in a very simple and practical equivalence rule (Milton *et al.*, 2006; Zolla *et al.*, 2005):

> When you change your coordinate system, all you have to do is to replace your initial material (electric permittivity tensor $\underline{\underline{\varepsilon}}$ and magnetic permeability tensor $\underline{\underline{\mu}}$) properties by equivalent material properties given by the following rule:
>
> $$\underline{\underline{\varepsilon}}' = \mathbf{J}^{-1}\underline{\underline{\varepsilon}}\mathbf{J}^{-\mathrm{T}}\det(\mathbf{J}) \text{ and } \underline{\underline{\mu}}' = \mathbf{J}^{-1}\underline{\underline{\mu}}\mathbf{J}^{-\mathrm{T}}\det(\mathbf{J}), \tag{1}$$
>
> where $\mathbf{J}$ is the Jacobian matrix of the coordinate transformation consisting of the partial derivatives of the new coordinates with respect to the original ones ($\mathbf{J}^{-\mathrm{T}}$ is the transposed of its inverse).

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In equation (1), the right hand sides involve matrix products where the matrix associated with a second rank tensor containing the coefficients of its representation in the initial Cartesian coordinate system. The obtained matrix provides the new coefficients of the tensor corresponding to the equivalent material.

Explicitly, a map from a coordinate system $\{u, v, w\}$ to the coordinate system $\{x, y, z\}$ is given by the transformation characterized by $x(u, v, w), y(u, v, w)$ and $z(u, v, w)$. As we start with a given set of equations in a given coordinate system, it seems at first sight that we have to map these coordinates on the new ones. Nevertheless, it is the opposite that has to be done: the new coordinate system is mapped on the initial one (i.e. the new coordinates are defined as explicit functions of the initial coordinates) and the equations are then pulled back, according to differential geometry (Bossavit, 1991), on the new coordinates. This provides us directly with the functions whose derivatives are involved in the computation of the Jacobian matrix. The Jacobian is directly given by:

$$\mathbf{J}_{xu} \equiv \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}. \tag{2}$$

The equivalence rule (1) can be extended to more general material properties such as local Ohm's law and bianisotropic materials (Milton *et al.*, 2006). Moreover, the rule given by equation (1) may be easily applied to a composition of transformations. Let us consider three coordinate systems $\{u, v, w\}$, $\{X, Y, Z\}$ and $\{x, y, z\}$. The two successive changes of coordinates are given by the sets of functions $\{X(u, v, w), Y(u, v, w), Z(u, v, w)\}$ and $\{x(X, Y, Z), y(X, Y, Z), z(X, Y, Z)\}$. They lead to the Jacobians $\mathbf{J}_{Xu}$ and $\mathbf{J}_{xX}$ so that the global Jacobian $\mathbf{J}_{xu} = \mathbf{J}_{xX}\mathbf{J}_{Xu}$. The compound transformation can therefore be considered either as involving this global Jacobian or as successive applications of equation (1). This rule naturally applies for an arbitrary number of coordinate systems. Note that the maps are defined from the final u, v, w to the original x, y, z coordinate system and that the product of the Jacobians, corresponding to the composition of the pull back maps, is in the opposite order.

When the initial material properties ε and μ are isotropic and described by a scalar, they generally lead to anisotropic properties and are given via a transformation matrix $\mathbf{T} = \mathbf{J}^{\mathrm{T}}\mathbf{J}/\det(\mathbf{J})$ related to the metric expressed in the new coordinates so that the equivalence rule (1) becomes:

$$\underline{\underline{\varepsilon}}' = \varepsilon\mathbf{T}^{-1}, \text{ and } \underline{\underline{\mu}}' = \mu\mathbf{T}^{-1}. \tag{3}$$

We note that there is no change in the impedance of the media since the permittivity and permeability suffer the same transformation.

As for the vector analysis operators and products, everything works as if we were in Cartesian coordinates. It means that once the material properties have been set to their equivalent values, all the computations are performed as if the coordinates were Cartesian. Once the solution has been obtained in the new coordinate system, e.g. the electric field $\mathbf{E}'$, its components in the original Cartesian coordinate system, $\mathbf{E}$, are given by (in the rest of this section, the vectors are represented by 3×1 column matrices):

$$\mathbf{E} = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \mathbf{J}^{-\mathrm{T}} \begin{pmatrix} E'_u \\ E'_v \\ E'_w \end{pmatrix} = \mathbf{J}^{-\mathrm{T}}\mathbf{E}'. \tag{4}$$

It must be emphasized here that $\mathbf{E}$ and $\mathbf{E}'$ are the same field expressed in two different coordinate systems. The direct interpretation of $\mathbf{E}'$ is difficult since it is expressed in a possibly non-orthogonal and not normed basis. Other vector fields corresponding to 1-forms such as $\mathbf{H}$ or $\mathbf{A}$ are transformed in the same way while the vector fields corresponding to 2-forms (field densities) such as $\mathbf{D}$, $\mathbf{B}$, and $\mathbf{J}$ are transformed according to:

$$\mathbf{D} = \frac{\mathbf{J}\mathbf{D}'}{\det(\mathbf{J})}. \tag{5}$$

It may be checked that these transformations are compatible with the equivalence rule (1) assuming that $\mathbf{D} = \underline{\underline{\varepsilon}}\mathbf{E}$ is replaced by $\mathbf{D}' = \underline{\underline{\varepsilon}}'\mathbf{E}'$ in the equivalent formulation. They also preserve the form of energy densities since, for instance, $\int_\Omega \mathbf{E}^{\mathrm{T}}\mathbf{D}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{\Omega'} \mathbf{E}'^{\mathrm{T}}\mathbf{D}'\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$ where Ω' is the image of the domain Ω by the coordinate transformation and $\mathbf{E}^{\mathrm{T}}\mathbf{D}$ is the matrix notation for the dot product.

As inhomogeneous and anisotropic equivalent materials are obtained and as the theoretical framework is the exterior calculus, the (Whitney) finite element method is perfectly adapted to the numerical algorithm implementation (Bossavit, 1998; Dular *et al.*, 1994, 1995).

In fact, this goes beyond simple change of coordinates as we will also consider active transformations, i.e. changes of space (i.e. of manifold) where the equations are written.

It is very often useful to use radial transformations. In this case, the most simple way is to first perform a transformation to cylindrical or spherical coordinates and to perform the inverse transformation once the radial transformation has been made. First, the classical transformation from Cartesian coordinates (x, y, z) to polar coordinates (ρ, θ, z) is introduced via a map from ρ, θ to x, y:

$$\begin{cases} x(\rho, \theta) = \rho\cos\theta \\ y(\rho, \theta) = \rho\sin\theta. \end{cases} \tag{6}$$

The associated Jacobian is:

$$\mathbf{J}_{x\rho}(\rho, \theta) = \frac{\partial(x, y, z)}{\partial(\rho, \theta, z)} = \begin{pmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{R}(\theta)\mathrm{diag}(1, \rho, 1), \tag{7}$$

with:

$$\mathbf{R}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \mathrm{diag}(1, \rho, 1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$\mathbf{R}(\theta)$ has the well-known properties: $\mathbf{R}(\theta)^{-1} = \mathbf{R}(\theta)^{\mathrm{T}} = \mathbf{R}(-\theta)$.

Furthermore, the inverse transformation is given by the map:

$$\begin{cases} \rho(x,y) = \sqrt{x^2+y^2} \\ \theta(x,y) = 2\arctan\left(\frac{y}{x+\sqrt{x^2+y^2}}\right) \end{cases}, \tag{8}$$

and is associated with the Jacobian:

$$\mathbf{J}_{\rho x}(x,y) = \mathbf{J}_{x\rho}^{-1}(\rho(x,y), \theta(x,y)) = \operatorname{diag}\left(1, \frac{1}{\rho(x,y)}, 1\right)\mathbf{R}(-\theta(x,y)). \tag{9}$$

Similarly, the spherical coordinates are described via a map from ρ, θ, φ to x, y, z:

$$\begin{cases} x = \rho\cos\theta\sin\varphi \\ y = \rho\sin\theta\sin\varphi \\ z = \rho\cos\varphi \end{cases} \tag{10}$$

The spherical Jacobian:

$$\mathbf{J}_{x\rho}(\rho,\theta,\varphi) = \begin{pmatrix} \cos\theta\sin\varphi & -\rho\sin\theta\sin\varphi & \rho\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & \rho\cos\theta\sin\varphi & \rho\sin\theta\cos\varphi \\ \cos\varphi & 0 & -\rho\sin\varphi \end{pmatrix}, \tag{11}$$

can be written $\mathbf{J}_{x\rho} = \mathbf{R}(\theta)\mathbf{M}_2(\varphi)\operatorname{diag}(1, \rho\sin\varphi, \rho)$ still involving the $\mathbf{R}(\theta)$ matrix together with:

$$\mathbf{M}_2(\varphi) = \begin{pmatrix} \sin\varphi & 0 & \cos\varphi \\ 0 & 1 & 0 \\ \cos\varphi & 0 & -\sin\varphi \end{pmatrix}. \tag{12}$$

with the properties:

$$\mathbf{M}_2^{-1}(\varphi) = \mathbf{M}_2^{\mathrm{T}}(\varphi) = \mathbf{M}_2(\varphi).$$

2. Problems with open boundary conditions

One of the primary applications of non-orthogonal coordinates is the modelling of infinite domains (Lowther *et al.*, 1989). In the electrostatic or magnetostatic case as well as in the eddy current case, the solution decreases to zero at infinity. Several types of infinite elements have been introduced (when the problem was not brutally truncated at finite distance) but the most efficient ones correspond to a mapping of a finite domain on the exterior infinite domain (Imhoff *et al.*, 1990; Nicolet *et al.*, 1994).

In the case of propagation problems, a transformation of an infinite domain into a finite one as presented above would contract the wavelength to an infinitely small value as the outer boundary is approached so that a well adapted mesh would be difficult to obtain.

In this case, the solution is to introduce the perfectly matched layers (PML). Such regions have been introduced by Berenger (1994) and, nowadays, in the time harmonic case, the most natural way to introduce PML is to consider them as maps on a complex space (Lassas and Somersalo, 2001) so that the corresponding change of (complex) coordinates leads to equivalent ε and μ (that are complex, anisotropic, and inhomogeneous even if the original ones were real, isotropic, and homogeneous). This leads automatically to an equivalent medium with the same impedance as the one of the initial ambient medium since ε and μ are transformed in the same way and this insures that the interface with the layer is non-reflecting. Moreover, a correct choice of the complex map leads to an absorbing medium able to dissipate the outgoing waves (Ould Agha *et al.*, 2008). The problem can therefore be properly truncated under the condition that the artificial boundary is situated in a region where the field is damped to a negligible value.

For isotropic uniform media outside the region of interest, the cylindrical PML is an annulus whose characteristics are obtained by multiplying ε and μ by the following complex matrix:

$$\mathbf{T}_{\mathrm{PML}}^{-1} = \mathbf{J}_{\mathrm{PML}}^{-1}\mathbf{J}_{\mathrm{PML}}^{-\mathbf{T}}\det(\mathbf{J}_{\mathrm{PML}}) = \mathbf{R}(\theta)\,\mathrm{diag}\left(\frac{\tilde{\rho}}{s_\rho\rho}, \frac{s_\rho\rho}{\tilde{\rho}}, \frac{s_\rho\tilde{\rho}}{\rho}\right)\mathbf{R}(-\theta).$$

This latest expression is the metric tensor in Cartesian coordinates (x, y, z) for the cylindrical PML. θ, ρ, $\tilde{\rho}$, and $s_\rho(\rho)$ are explicit functions of the variables x and y, i.e.:

$$\theta = 2\arctan\left(\frac{y}{x+\sqrt{x^2+y^2}}\right), \quad \rho = \sqrt{x^2+y^2}, \quad s_\rho(\rho) = s_\rho\left(\sqrt{x^2+y^2}\right),$$

and:

$$\tilde{\rho} = \int_0^{\sqrt{x^2+y^2}} s_\rho(\rho')\mathrm{d}\rho',$$

where $s_\rho(\rho')$ is an arbitrary but well chosen complex valued function of a real variable that describes the radial stretch relating the initial radial distance ρ to the complex one $\tilde{\rho}$.

Another remarkable property of the PML is that they provide the correct extension to non-Hermitian operators (since $\mathbf{T}_{\mathrm{PML}}$ is complex and symmetric) that allows the computation of the leaky modes in waveguides (Nicolet *et al.*, 2007) and this may be obtained via a correct choice of the PML parameters, namely $\mathbf{R}^*$, $\mathbf{R}^{\mathrm{trunc}}$ such that $\mathbf{R}^* < \rho < \mathbf{R}^{\mathrm{trunc}}$ and $s_\rho(\rho)$ (Ould Agha *et al.*, 2008).

3. Helicoidal geometries and twisted optical fibres

The purpose of this section is to show how the equivalence rule (1) can be used to study the propagation of modes in twisted waveguides via a 2D model though the translational invariance of the geometry is lost (Figure 1).

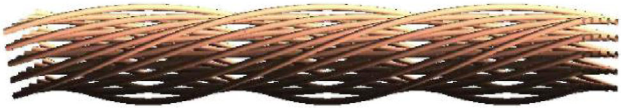

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Figure 1.
A twisted structure that may be described by the helicoidal coordinates

Let us introduce an helicoidal coordinate system (Lewin and Ruehle, 1980; Yabe and Mushiake, 1984; Igarashi and Honma, 1991) (ξ_1, ξ_2, ξ_3) deduced from rectangular Cartesian coordinates (x, y, z) in the following way:

$$\begin{cases} x = \xi_1 \cos(\alpha\xi_3) + \xi_2 \sin(\alpha\xi_3) \\ y = -\xi_1 \sin(\alpha\xi_3) + \xi_2 \cos(\alpha\xi_3) \\ z = \xi_3 \end{cases}, \quad (13)$$

where α is a parameter which characterizes the torsion of the structure. A twisted structure is a structure for which both geometrical and physical characteristics (here the permittivity ε and the permeability μ) together with the boundary conditions only depend on ξ_1 and ξ_2. Note that such a structure is invariant along ξ_3 but $(2\pi/\alpha)$-periodic along z (the period may be shorter depending on the symmetry of the cross section).

This general coordinate system is characterized by the Jacobian of the transformation equation (13):

$$\mathbf{J}_{\mathrm{hel}}(\xi_1, \xi_2, \xi_3) = \begin{pmatrix} \cos(\alpha\xi_3) & \sin(\alpha\xi_3) & \alpha\xi_2\cos(\alpha\xi_3) - \alpha\xi_1\sin(\alpha\xi_3) \\ -\sin(\alpha\xi_3) & \cos(\alpha\xi_3) & -\alpha\xi_1\cos(\alpha\xi_3) - \alpha\xi_2\sin(\alpha\xi_3) \\ 0 & 0 & 1 \end{pmatrix}, \quad (14)$$

which does depend on the three variables ξ_1, ξ_2 and ξ_3. On the contrary, the transformation matrix $\mathbf{T}_{\mathrm{hel}}$:

$$\mathbf{T}_{\mathrm{hel}}(\xi_1, \xi_2) = \frac{\mathbf{J}_{\mathrm{hel}}^{\mathrm{T}}\mathbf{J}_{\mathrm{hel}}}{\det(\mathbf{J}_{\mathrm{hel}})} = \begin{pmatrix} 1 & 0 & \alpha\xi_2 \\ 0 & 1 & -\alpha\xi_1 \\ \alpha\xi_2 & -\alpha\xi_1 & 1 + \alpha^2\left(\xi_1^2 + \xi_2^2\right) \end{pmatrix}, \quad (15)$$

which describes the change in the material properties, only depends on the first two variables ξ_1 and ξ_2 (Nicolet *et al.*, 2004, 2006, 2007). This matrix may also conveniently be expressed in terms of twisted cylindrical coordinates:

$$\mathbf{R}(\varphi)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\rho\alpha \\ 0 & -\rho\alpha & 1 + \rho^2\alpha^2 \end{pmatrix}\mathbf{R}(-\varphi) = \begin{pmatrix} 1 & 0 & \alpha\rho\sin(\varphi) \\ 0 & 1 & -\alpha\rho\cos(\varphi) \\ \alpha\rho\sin(\varphi) & -\alpha\rho\cos(\varphi) & 1 + \rho^2\alpha^2 \end{pmatrix},$$

with:

$$\varphi = 2\arctan\left(\frac{\xi_2}{\xi_1 + \sqrt{\xi_1^2 + \xi_2^2}}\right), \quad \rho = \sqrt{\xi_1^2 + \xi_2^2}.$$

Helicoidal coordinates have also been combined with PML to compute the leaky modes in twisted microstructured optical fibres (MOF) (Nicolet *et al.*, 2007):

$$\mathbf{T}_{\mathrm{hPML}} = \mathbf{R}(\varphi)\begin{pmatrix} \frac{\rho s_\rho}{\tilde{\rho}} & 0 & 0 \\ 0 & \frac{\tilde{\rho}}{\rho s_\rho} & -\alpha\frac{\tilde{\rho}}{s_\rho} \\ 0 & -\alpha\frac{\tilde{\rho}}{s_\rho} & \frac{\rho(1+\alpha^2\tilde{\rho}^2)}{\tilde{\rho} s_\rho} \end{pmatrix}\mathbf{R}(-\varphi). \tag{16}$$

This is the expression of the "twisted cylindrical PML tensor" in "helicoidal Cartesian modelling coordinates" ξ_1, ξ_2 and all the quantities involved in the previous expression can be given as explicit functions of these two variables, joining:

$$s_\rho(\rho) = s_\rho\left(\sqrt{\xi_2^1 + \xi_2^2}\right),$$

and:

$$\tilde{\rho} = \int_0^{\sqrt{\xi_1^2+\xi_2^2}} s_\rho(\rho')\mathrm{d}\rho',$$

to the expressions for ρ and φ given here above.

The fact that the equivalent materials are independent from the longitudinal coordinate ξ_3 allows a 2D model for the determination of the propagation modes and of the leaky modes via a classical model provided it allows completely anisotropic and inhomogeneous media. Luckily, the finite element method allows such a numerical computation.

Figure 2 shows a MOF (Guenneau *et al.*, 2001, 2002a, b, 2003; Zolla *et al.*, 2005). It is a dielectric waveguide whose structure consists of a bulk of silica (supposed to be unbounded) drilled by six air holes with a center-to-center spacing $\Lambda = 6.75\ \mu$m. Each hole is circular with a radius equal to $r_s = 2.5\ \mu$m. A given wavelength $\lambda_0 = 1.55\ \mu$m is considered for which the index of silica is about $\sqrt{\varepsilon_{\mathrm{Si}}} = n_{\mathrm{Si}} = 1.444024$.

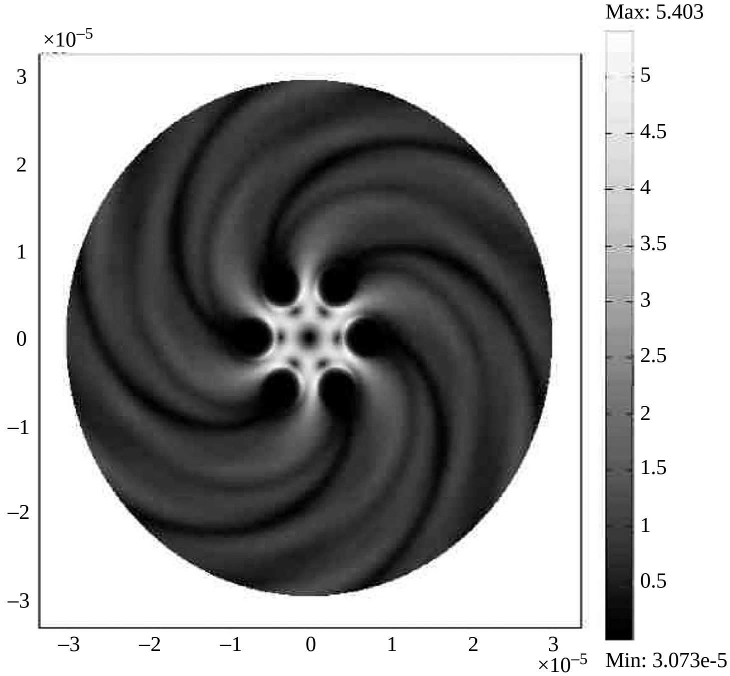
Note that for this structure no propagating mode can be found and the fundamental mode is a leaky mode. The figure shows the norm of the longitudinal component of the electric field (reconstructed via equation (4): $\mathbf{E}_z = \alpha\xi_2\mathbf{E}_{\xi 1} - \alpha\xi_1\mathbf{E}_{\xi 2} + \mathbf{E}_{\xi 3}$, Nicolet and Zolla, 2007) for the "fundamental mode" in the case of a very strong twist ($\alpha = 50{,}000\ \mathrm{m}^{-1}$) for which the losses are very large.

4. Invisibility cloaking

The geometrical transformations can also be used in the reverse sense to design new materials. In this case, a geometrical transformation is applied to free space to guess interesting material properties given by the equivalence rule. A new device can be built if the new material properties may be approximated, e.g. using electromagnetical metamaterials (Ramakrishna, 2005). For instance, a convex domain is mapped on a holey domain with the same exterior boundary. The structure made of the transformed equivalent material is an invisibility cloak and any object can be perfectly hidden in the central hole (Pendry *et al.*, 2006; Zolla *et al.*, 2007).

To compute the transformation matrix $\mathbf{T}$ associated with the cloak, we first map Cartesian coordinates onto polar co-ordinates (ρ, θ, z). The associated Jacobian matrix is given by equation (7).

Let us now consider a 2D object we want to cloak located within a disk of radius R_1. As proposed in (Pendry *et al.*, 2006), we consider a geometric transformation which maps the field within the disk $\rho < R_2$ onto the annulus $R_1 \leq \rho \leq R_2$:



Notes: The map is truncated at the inner boundary of the PML, the field unit on the gray scale is arbitrary, and dimensions on the axes are in *m*. Reproduced from the only available original

Figure 2. Map of the norm of the longitudinal component of the electric field $\mathbf{E}_z = \alpha\xi_2\mathbf{E}_{\xi 1} - \alpha\xi_1\mathbf{E}_{\xi 2} + \mathbf{E}_{\xi 3}$ for a strongly twisted fibre ($\alpha = 50{,}000\,\mathrm{m}^{-1}$) with very large losses

$$\begin{cases} \rho' = R_1 + \rho(R_2 - R_1)/R_2, \quad 0 \leq \rho \leq R_2 \\ \theta' = \theta, \ \ 0 < \theta \leq 2\pi, \\ z' = z, \ \ z \in \mathbf{R} \end{cases} \tag{17}$$

where ρ', θ' and z' are "radially contracted cylindrical coordinates". Moreover, this transformation maps the field for $\rho \geq R_2$ onto itself through the identity transformation. This leads to:

$$\mathbf{J}_{\rho\rho'} = \frac{\partial(\rho, \theta, z)}{\partial(\rho', \theta', z)} = \mathrm{diag}(c_{11}, 1, 1) \tag{18}$$

where $c_{11} = R_2/(R_2 - R_1)$ for $0 \leq \rho \leq R_2$ and $c_{11} = 1$ for $\rho > R_2$.

Last, we need to go to Cartesian coordinates x', y', z', which are "radially contracted Cartesian coordinates" where the modelling takes place to obtain a representation of the metric tensor in the suitable coordinate system. The associated Jacobian matrix is given by equation (9):

$$\mathbf{J}_{\rho' x'}(x', y') = \frac{\partial(\rho', \theta', z')}{\partial(x', y', z')} = \mathbf{J}^{\mathrm{T}}_{\rho x}\left(\frac{1}{\rho'}, \theta'\right) = \mathrm{diag}\left(1, \frac{1}{\rho'}, 1\right)\mathbf{R}(-\theta'). \tag{19}$$

Applying the composition rule twice, $\mathbf{J}_{xx'} = \mathbf{J}_{x\rho}\mathbf{J}_{\rho\rho'}\mathbf{J}_{\rho' x'}$, hence the material properties of the invisibility cloak are described by the transformation matrix $\mathbf{T} = \mathbf{J}_{xx'}^{\mathrm{T}}\mathbf{J}_{xx'}/\det(\mathbf{J}_{xx'})$. We will also need its inverse that we give explicitly, taking into account that $\rho(\rho') = c_{11}(\rho' - R_1)$:

$$\mathbf{T}^{-1} = \mathbf{R}(\theta')\mathrm{diag}\left(\frac{\rho' - R_1}{\rho'}, \frac{\rho'}{\rho' - R_1}, \frac{c_{11}^2(\rho' - R_1)}{\rho'}\right)\mathbf{R}(-\theta'). \tag{20}$$

4.1 Cylindrical cloaks of arbitrary cross section

A quite general situation is now considered, where the shape of the cloak is described by two arbitrary functions $R_1(\theta)$ and $R_2(\theta)$ giving an angle dependent distance from the origin corresponding, respectively, to the interior and exterior boundary of the cloak.

The geometric transformation which maps the field within the full domain $r \leq R_2(\theta)$ onto the hollow domain $R_1(\theta) \leq \rho \leq R_2(\theta)$ can be expressed as:

$$\begin{cases} \rho'(\rho, \theta) = R_1(\theta) + \rho(R_2(\theta) - R_1(\theta))/R_2(\theta), & 0 \leq \rho \leq R_2(\theta) \\ \theta' = \theta, & 0 < \theta \leq 2\pi \\ z' = z. & z \in \mathbb{R} \end{cases}$$

Note that the transformation maps the field for $\rho > R_2(\theta)$ onto itself through the identity transformation. This leads to:

$$\mathbf{J}_{\rho\rho'}(\rho', \theta') = \frac{\partial(\rho(\rho', \theta'), \theta, z)}{\partial(\rho', \theta', z')} = \begin{pmatrix} c_{11}(\theta') & c_{12}(\rho', \theta') & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $c_{11}(\theta') = R_2(\theta')/(R_2(\theta') - R_1(\theta'))$ for $0 \leq \rho' \leq R_2(\theta')$ and $c_{11} = 1$ for $\rho' > R_2(\theta')$ and:

$$c_{12}(\rho', \theta') = (\rho' - R_2(\theta'))R_2(\theta')\frac{\mathrm{d}R_1(\theta')}{\mathrm{d}\theta'} - \frac{(\rho' - R_1(\theta'))R_1(\theta')\frac{\mathrm{d}R_2(\theta')}{\mathrm{d}\theta'}}{(R_2(\theta') - R_1(\theta'))^2},$$

for $0 \leq \rho' \leq R_2(\theta')$, and $c_{12} = 0$ for $\rho' > R_2(\theta')$.

Finally, the properties of the cloak are given by:

$$\mathbf{T}^{-1} = \mathbf{R}(\theta')\begin{pmatrix} \frac{c_{12}^2 + f_\rho^2}{c_{11} f_\rho \rho'} & -\frac{c_{12}}{f_\rho} & 0 \\ -\frac{c_{12}}{f_\rho} & \frac{c_{11}\rho'}{f_\rho} & 0 \\ 0 & 0 & \frac{c_{11} f_\rho}{\rho'} \end{pmatrix}\mathbf{R}(\theta')^{\mathrm{T}},$$

with:

$$f_\rho = \frac{(\rho' - R_1(\theta'))R_2(\theta')}{(R_2(\theta') - R_1(\theta'))} = (\rho' - R_1(\theta'))c_{11}.$$

The parametric representation of the ellipse $\rho(\theta) = \mathrm{ab}/\sqrt{\mathrm{a}^2\cos(\theta)^2 + \mathrm{b}^2\sin(\theta)^2}$ corresponds to cloaks of elliptical cross section and it has been checked that it provides exactly the same result as in Nicolet *et al.* (2008), where similar results have been obtained by a space dilatation.

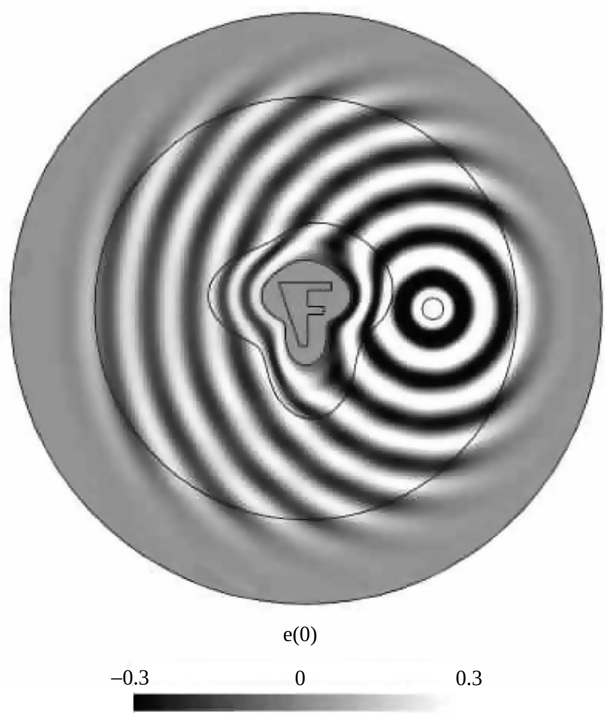
To obtain general shapes, Fourier series:

$$\rho(\theta) = a_0 + \sum_{k=1}^{n} (a_k \cos(k\theta) + b_k \sin(k\theta)),$$

may be used. An example of such a general cloak is shown on Figure 3: a source made of a wire of circular cross section (radius $=$ 0.25) centered at point $r_s = (2.5, 0)$ with a constant E_z imposed on its boundary, radiating in a vacuum with wavelength $\lambda = 1$ (note that all lengths are given in arbitrary units, μm for instance for near infrared). The electric field E_z is therefore a cylindical wave (note that the electric field is given in arbitrary units, V/m for instance, and $E_z = 1$ on the boundary of the source wire) and is not perturbed at all by a F-shaped scattering (lossy) obstacle of relative permittivity $1 + 4i$ placed near the origin (0,0) and surrounded by the cloak. Note also that the unbounded space is simulated via a circular PML in the annulus $4 \leq \rho \leq 5$.

4.2 3D cloaks

The 3D cloaks may be determined following the same guidelines but using the spherical coordinates.



Notes: $R_1(\theta)$ is with a_0 = 1, b_1 = 0.1, a_2 = –0.15, b_3 = 0.2, a_4 = 0.1, $R_2(\theta)$ is with a_0 = 2, a_2 = –0.1, a_3 = –0.15, b_3 = 0.3, a_4 = 0.2, all the other coefficients = 0. Reproduced from the only available original

Figure 3.
Cloak with a general shape given by Fourier series

The Jacobian of the radial contraction $\mathbf{J}_{\rho\rho'} = \mathrm{diag}(c_{11}, 1,1)$ is still the same (ρ is now the radius of a sphere). The total Jacobian is therefore:

$$\mathbf{R}(\theta)\mathbf{M}_2(\varphi)\ \mathrm{diag}\,(1, \rho\sin\varphi, \rho)\ \mathrm{diag}\,(c_{11}, 1, 1)\ \mathrm{diag}(1, 1/(\rho'\sin\varphi'), 1/\rho'\mathbf{M}_2(\varphi')\mathbf{R}^{\mathrm{T}}(\theta')$$
$$= \mathbf{R}(\theta)\mathbf{M}_2(\varphi)\ \mathrm{diag}(c_{11}, \rho/\rho', \rho/\rho')\,\mathbf{M}_2(\varphi)\mathbf{R}^{\mathrm{T}}(\theta),$$

where we used the fact that $\varphi = \varphi'$ and $\theta = \theta'$. The transformation matrix for the equivalent media is finally:

$$\mathbf{T}^{-1} = \mathbf{R}(\theta)\mathbf{M}_2(\varphi)\ \mathrm{diag}\left(\frac{\rho^2}{c_{11}\rho'^2}, c_{11}, c_{11}\right)\mathbf{M}_2(\varphi)\mathbf{R}^{\mathrm{T}}(\theta). \quad (21)$$

3D arbitrary cloaks can be found by varying their interior and exterior radii with respect to the angular coordinates: $R_1(\theta, \varphi)$, $R_2(\theta, \varphi)$.

5. Conclusion

The geometrical transformations may be viewed as a unifying point of view bridging several techniques in electromagnetism: treatment of unbounded domains and of twisted structures, design of invisibility cloaks… The cornerstone of the method is to remark that the Maxwell equations can be written in a covariant form such that all the metric properties are only involved in the material properties. The change of coordinates may therefore be encapsulated in exotic equivalent material properties, via the equivalence rule (1), and the rest of the computation is dealt with just as if rectangular Cartesian coordinates were used. Though this technique is completely general, the fact that the obtained material are usually anisotropic and inhomogeneous makes it of particular interest in the context of the finite element method where it provides very interesting models that do not require a modification of the existing code (if this one is general enough). It also provides a tool to design new electromagnetic devices such as the invisibility cloaks. It gives also an interpretation of negative refractive index materials together with a pictural view of the perfect lens that corresponds to a "folding" of the space (Pendry and Smith, 2004; Leonhardt and Philbin, 2006; Schurig *et al.*, 2007). Nevertheless, it should be emphasized that the space transformations that do not correspond to a diffeomorphism lead to material properties that are, if not impossible to obtain, at least challenging for the optical metamaterial science (even in a small frequency range). Thus, far, experimental verification of invisibility cloaks was chiefly achieved for microwaves (Schurig *et al.*, 2006).

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