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# Finite elements-boundary elements coupling for the movement modeling in two-dimensional structures

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**Résumé.** — Cet article présente le couplage entre les éléments finis et les éléments frontières comme technique efficace pour résoudre le problème de la modélisation du mouvement dans des calculs de champs électromagnétiques bidimensionnels. D'un côté, les éléments finis sont utilisés à l'intérieur des matériaux magnétiques et conducteurs où ils permettent de prendre en compte des comportements complexes tels que la saturation des matériaux ferromagnétiques. D'un autre côté, les éléments frontières sont utilisés à l'extérieur des parties mobiles où ils évitent la déformation du maillage. La formulation du problème des courants induits conduit à un calcul implicite de la force électromotrice induite de mouvement.

**Abstract.** — The paper presents the coupling between the finite element and the boundary element methods as an efficient technique to solve the problem of the modelization of movement in two-dimensional electromagnetic field computations. On the one hand, the finite elements are used inside the magnetic and conducting parts where they permit to take into account the complex behaviour of materials such as saturable ferromagnetic materials. On the other hand, the boundary elements are used around the moving parts where they prevent the deformation of the meshes. The formulation of the eddy current problems leads to an automatic computation of the motional induced electromotive force.

## Introduction.

Movement is of crucial importance in electromechanical devices such as motors and relays. A correct treatment of the movement involves a coupling between the mechanical model and the electromagnetic model. On the one hand, electromagnetic phenomena produce forces that mechanically act on the devices producing, among others, movement.

On the other hand, mechanical movement acts on the electromagnetic model : it transforms the geometry of the problem and it produces phenomena such as the motional induced electromotive force.

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In this paper, we only consider rigid moving parts. In the two-dimensional model, the only degrees of freedom associated to such parts are 2 degrees of translation and 1 degree of rotation.

The mechanical model consists of the simple movement equation where electromagnetic force and torque are introduced. For the electromagnetic field modelization we use an existing program called LUCIE, which is based on the coupling of the finite element method and of the boundary element method.

This powerful method allows a rigorous treatment of open boundary problems with BEM together with a correct modelization of the saturable ferromagnetic materials. The modularity of the LUCIE software has allowed an easy and workable introduction of movement in the modelization.

The philosophy behind the use of the coupled method to model the movement is to use the finite element method inside the rigid moving parts that can be the seat of phenomena such as saturation and eddy currents.

The boundary elements are used for linear magnetic and non conducting media wherein rigid parts can move. The advantages of this formulation are that there is no mesh deformation in the finite element domains and that we need no explicit terms for the motional induced electromotive force, which is automatically taken into account by the variation of the boundary element method coefficients.

**Numerical methods : boundary element-finite element coupling.**

The equation for the two-dimensional magnetostatics is :

$$\Delta A = -\mu J \tag{1}$$

- $A$ is the vector potential which has only one component ;
- $J$ is the current density ;
- $\mu$ is the magnetic permeability ( $= 1/\nu$ inverse of the magnetic reluctivity).

In linear magnetic non conducting media, the direct boundary element method (Brebbia [1]) is based on the relation (2) :

$$cA = \int_\Gamma A \, \partial G/\partial n \, \mathrm{d}\Gamma - \int_\Gamma G \, \partial A/\partial n \, \mathrm{d}\Gamma \tag{2}$$

where :

- $G$ is the free space Green function of the two-dimensional Laplace operator ;
- $c = 0.5$ on a smooth boundary ;
- $\partial \cdot /\partial n$ is for the normal derivative.

Integrals are taken on the boundary $\Gamma$ of the subdomains and the method involves only $A$ and $\partial A/\partial n$ (tangential induction) on the boundaries.

The finite element formulation (Silvester [2]) is based on the Galerkin method (3) and for domain $\Omega$ of boundary $\Gamma$ we have :

$$\int_\Omega (\nu \operatorname{grad} A \operatorname{grad} w - Jw) \, \mathrm{d}\Omega - \int_\Gamma w\nu \, \partial A/\partial n \, \mathrm{d}\Gamma = 0 \tag{3}$$

where $\nu$ is the magnetic reluctivity, $J$ the current density and $w$ the weighting function.

The boundary term is usually used to apply a Neumann boundary condition.

Here it is used at the interface between finite element and boundary element domains to couple the methods. This can be extended to the boundary between two finite element subdomains. The tangential field is taken as an unknown on the boundary and the boundary term of equation (3) is introduced. It gives the same solution for vector potential $A$ as the classical formulation but also furnishes the value of the tangential field $\nu\ \partial A / \partial n = H_t$ on the boundary (Nicolet [3], Bamps [4]).

**Eddy currents.**

Using Ohm's law, the case of eddy currents can be treated by introducing (Bourmanne [5]) :

$$J = -\sigma (U + \partial A / \partial t) \tag{4}$$

as the expression for current density in equation (3). The conductor is characterized by its electrical conductivity $\sigma$. The time derivative of $A$ expresses the inductive effects since $U = \operatorname{grad} V$ is due to an externally imposed electromotive force and can be interpreted as the terminal voltage of the conductor (per unit of length). If $U$ is given for all the conductors of the problem, the discretization of equation (3) leads to a differential system that can be solved. Unfortunately, in most cases, it is the total current $I$ in a conductor that has to be imposed. As the finite element formulation involves the tangential field on the boundary, the total current $I$ can be expressed as the line integral of the magnetic field (Ampere's law) :

$$I = \int_{\Gamma} \nu\ \partial A / \partial n \ \mathrm{d}\Gamma . \tag{5}$$

The numerical method used to discretize the transient eddy current problem is a semi-discrete Galerkin method. The first step is a spatial discretization of the domains leaving a continuous time variation of the degrees of freedom. This spatial discretization is a classical finite element meshing. It leads to a system of ordinary differential equations. The second step is a time discretization of the degrees of freedom. A finite difference scheme is generally used. In our particular case, a backward Euler method is used. It leads to an algebraic system of equations. This non symmetric system is solved by a direct LU decomposition and the solution is improved by an iterative refinement.

**Motional induced electromotive force.**

One of the major difficulties related to the numerical modelization of the movement is to take into account the motional induced electromotive force.

The problem can be analyzed as follows.

We have related current density $\mathbf{J}$ in a conductor to the electric field $\mathbf{E}$ through Ohm's law :

$$\mathbf{J} = \sigma \mathbf{E} . \tag{6}$$

However this law is only valid for a conductor at rest. To determine the relation between current density and the electric field in a restless conductor, we have to consider a referential associated to the conductor and to use the non-relativistic field transformation formulae. We obtain (Landau [6]) :

$$\mathbf{J} = \sigma (\mathbf{E} + \mathbf{v} \times \mathbf{B}) . \tag{7}$$

This modified Ohm's law for restless conductors relates the current density in the conductor to the electric field $\mathbf{E}$ and to the induction $\mathbf{B}$ defined in a *referential at rest* with respect to which the conductors are moving.

If the electromotive force acting on a closed linear circuit is considered, it can be shown that the first term of the right hand member of equation (7) corresponds to a variation of the magnetic flux due to the time variation of the magnetic field itself when the position of the circuit does not change.

The second term of the right hand member of equation (7) corresponds to the magnetic flux variation due to the displacement of the conductor in a magnetic field constant in time.

Those results can be summed up by Faraday's induction law. The induced electromotive force in a circuit is equal to the total magnetic flux variation through the circuit whatever the origin of the variation : time variation of the field or movement of the circuit.

In the numerical treatment of the eddy currents the flux variation on a time step is computed as the difference of nodal values of the vector potential at two successive moments.

If the node belongs to a moving object, the flux variation is computed with two vector potential nodal values taken at different points in space but corresponding to the same « particle » of the moving body (Bouillault [7]). The time derivative of the potential defined in this way is the « substantial » or « material » derivative.

If objects are displaced during the computation without any other change to the numerical methods, flux variations will be correctly computed. This automatically takes into account movements of conductors or of magnetic parts because, according to Faraday's induction law, it involves the total flux variation at one point of the conductor.

The coupling between the boundary elements and the finite elements naturally solves the problems related to the modelization of the movement.

**Forces and torques computation.**

Two kinds of movement can be considered : forced movement under external forces and natural movement due to the electromagnetic forces. In this later case, the determination of the electromagnetic forces is an important point of the movement modelization. The knowledge of the vector potential $A$ and of the tangential field $H_t$ on the boundaries given by the coupling method suggests using the Maxwell stress tensor method (Aldefeld [8]).

This method permits to define, at the boundary of objects, an equivalent normal force density $\mathbf{F}_n$ and an equivalent tangential force density $\mathbf{F}_t$ as :

$$\mathbf{F}_n = \left( \frac{B_n^2}{2\,\mu_0} - \frac{\mu_0}{2} H_t^2 \right) \mathbf{n} \tag{8}$$

$$\mathbf{F}_t = B_n H_t \, \mathbf{t} \, . \tag{9}$$

The tangential field $H_t$ and the normal induction $B_n$ are well defined even on surfaces of discontinuity. $H_t$ is directly given by the coupling method since $B_n$ is computed from the vector potential by :

$$\begin{aligned} B_n &= \operatorname{curl} A \cdot \mathbf{n} = \frac{\partial A}{\partial y} \cdot n_x - \frac{\partial A}{\partial x} \cdot n_y \\ &= \left( - \frac{\partial A}{\partial y} \frac{\partial y}{\partial \xi} - \frac{\partial A}{\partial x} \frac{\partial x}{\partial \xi} \right) / J(\xi) \\ &= - \frac{\partial A}{\partial \xi} / J(\xi) \end{aligned} \tag{10}$$

where $\xi$ is the local coordinate of the shape functions used for the boundary element discretization and $J(\xi)$ the corresponding Jacobian.

The real total force acting on an object of boundary $\Gamma$ is given by :

$$\mathbf{F} = \int_\Gamma (\mathbf{F}_n + \mathbf{F}_t) \, \mathrm{d}\Gamma \, . \tag{11}$$

The real total torque is given by :

$$\mathbf{T} = \int_{\Gamma} \mathbf{r} \times (\mathbf{F}_n + \mathbf{F}_t) \, \mathrm{d}\Gamma \tag{12}$$

where $\mathbf{r}$ is the position vector relative to the considered centre of rotation considered.

One of the interesting properties of the coupling method is that it yields the values of the vector potential and of the tangential field on the boundary of the objects. It permits an accurate use of formulae (11) and (12).

On the one hand, in the classical finite element method, only the nodal values of the vector potential are involved and $H_t$ is discontinuous on the boundaries between the elements. In this case, the numerical application of formula (11) requires to choose an integration line $\Gamma'$ passing through the elements (Fig. 1). The choice of $\Gamma'$ is sometimes difficult when various objects are in contact.

On the other hand, the coupling method yields directly the value of $H_t$ on the boundaries of the objects (from this point of view this method may be qualified of « mixed »). $H_t$ is continuous along the boundary and, in the case of first order elements, is piecewise linear. As $B_n$ is well defined on the boundaries using (10), the very boundary $\Gamma$ (Fig. 1) may be used to compute formula (11). This choice is easy, automatic, and is always possible even in the case of various objects in contact.

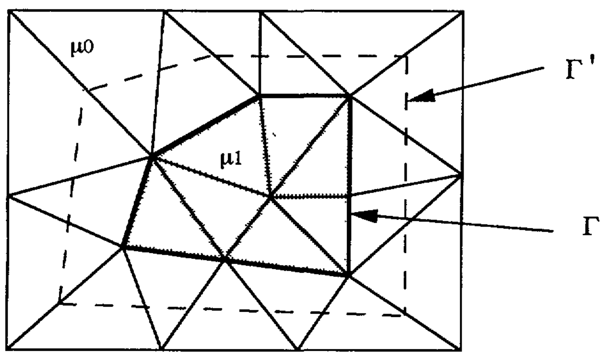


Fig. 1. — Integration path for force calculation.

**Movement equation.**

In the case of a natural movement, the displacement of an object is computed by solving the equation of the movement involving the magnetic forces.

Consider for instance a translation along the axis $\mathbf{e}_x$.

The following quantities are defined :

- position : $x$ ;
- speed : $v = \mathrm{d}x/\mathrm{d}t$ ;
- mass : $M$ ;
- elastic constant of a spring : $k$ ;
- friction coefficient : $f$ ;
- other forces including the magnetic forces : $F(t)$.

The equation of the movement is :

$$M \frac{\mathrm{d}v}{\mathrm{d}t} = F(t) - fv - kx . \tag{13}$$

The time discretization of this equation by the backward Euler method gives (Stoer [9]) :

$$v_{t+\Delta t} = \frac{Mv_t + F(t)\,\Delta t - kx_t\,\Delta t}{k\,\Delta t^2 + f\,\Delta t + M} \tag{14}$$

$$x_{t+\Delta t} = \frac{Mx_t + Mv_t\,\Delta t + F(t)\,\Delta t^2 + fx_t\,\Delta t}{k\,\Delta t^2 + f\,\Delta t + M} \tag{15}$$

Equations (14) and (15) relate the position $x_{t+\Delta t}$ and the speed $v_{t+\Delta t}$ at time $t + \Delta t$ to the position $x_t$ and to the speed $v_t$ at time $t$.

The movement is determined if an initial position $x_0$ and an initial speed $v_0$ are specified. Limits of movement can also be specified to avoid non physical interpenetration of moving parts. The other degrees of freedom are dealt with in a similar way (mutatis mutandis).

**Introduction of the movement in the software.**

The basis of the modelization of the movement in electromagnetic devices is an existing 2D magnetic field computation software called LUCIE. The central feature of the software is the use of the coupled FEM-BEM. The modular structure of the software is designed to make further developments and modifications as easy as possible.

The algorithm used to model the movement is the following :

1) Compute the electromagnetic state of the system.
2) Compute the magnetic forces and torques for this state.
3) Compute the displacements of the moving bodies with equations (14) and (15).
4) Modify the geometry in the database of the software using the computed displacements.
5) Start again at step 1.

This procedure involves some implicit hypotheses. The time varying electromagnetic force is considered as constant on a time step in equations (14) and (15), the displacements (i.e. the geometry variations) must not be too large on a time step.

Practical considerations show that an adaptative time step is necessary in order to keep the computational time within reasonable limits while remaining as close as possible to the hypotheses.

**Result : modelization of a relay.**

The modelization of an electromechanical relay requires a transient modelization because there are no periodic phenomena like in a rotating machine (Juffer [10]). Moreover, the movement plays a fundamental role in the electromagnetic modelization of the closure of the relay.

Figure 2 shows the geometry of the 2D model of a relay.

The magnetic core is composed of an upper moving part and of a lower static part. The coil has 1 430 turns and its resistance is 18 Ω. The mass of the moving part is 112 g and the elastic constant of the spring supporting the moving part is $k = 137$ N/m. Friction is neglected ($f = 0$). The depth of the structure is 1.6 cm and, for the modelization, all the characteristics of the relay are translated into equivalent characteristics for a standard depth of 1 m.

A special treatment to force a uniform current density is applied to the coil to take into account the small section of the wire that prevents eddy currents. Instead of equation (5), the following relation is used :

$$NI = -\sigma \int_{\text{coil}} \frac{\partial A}{\partial t}\, \mathrm{d}S - \sigma S U \tag{16}$$

where $I$ is the current in the coil, $N$ is the number of turns, $U$ the terminal voltage of the coil, $\sigma$ the equivalent conductivity and $S$ the section of one turn. The current density is now given by $J = I/S$ and there is no skin effect, only a global inductive effect is allowed (Delincé [11]). Therefore $\sigma$ may be chosen in order to have the correct resistance for the coil.

The relay is fed by a step of voltage of 12 V at the terminals of the coil.

The coil and the saturable magnetic parts are modelized with finite elements. The surrounding air is modelized with boundary elements (Fig. 2).

The computation time is about 12 CPU hours on a DecStation 3100 and this high value is due to the fact that for each time step a complete non linear problem must be solved. The average computation time is about 4 min for a time step and 50 s for a Newton-Raphson iteration. The non-linearity prevents simple tabulation of force and inductance with respect to position that can be used for a simpler modelling (Koltermann [12]).

Figure 3 shows the field lines in the open relay.

Figure 4 shows the time variation of the position of the moving part.

Figure 5 compares the computed (continuous line) and the measured (dashed line) time variation of the current in the coil.

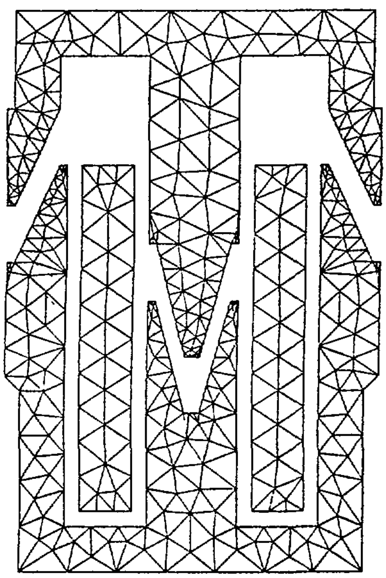

Fig. 2.

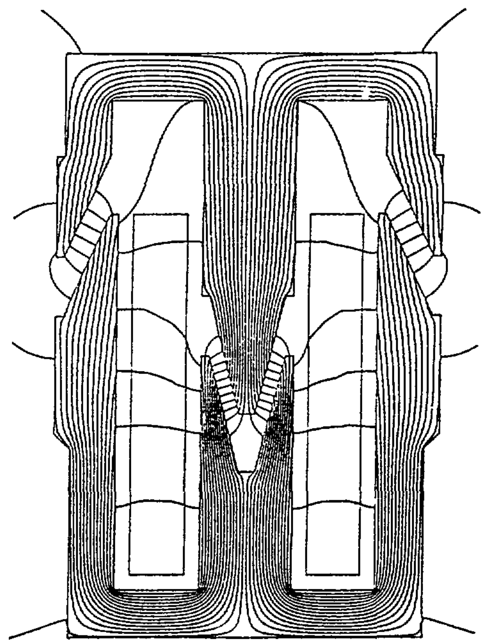

Fig. 3.

Fig. 2. — Meshing of an electromechanical relay.

Fig. 3. — Field lines in the relay.

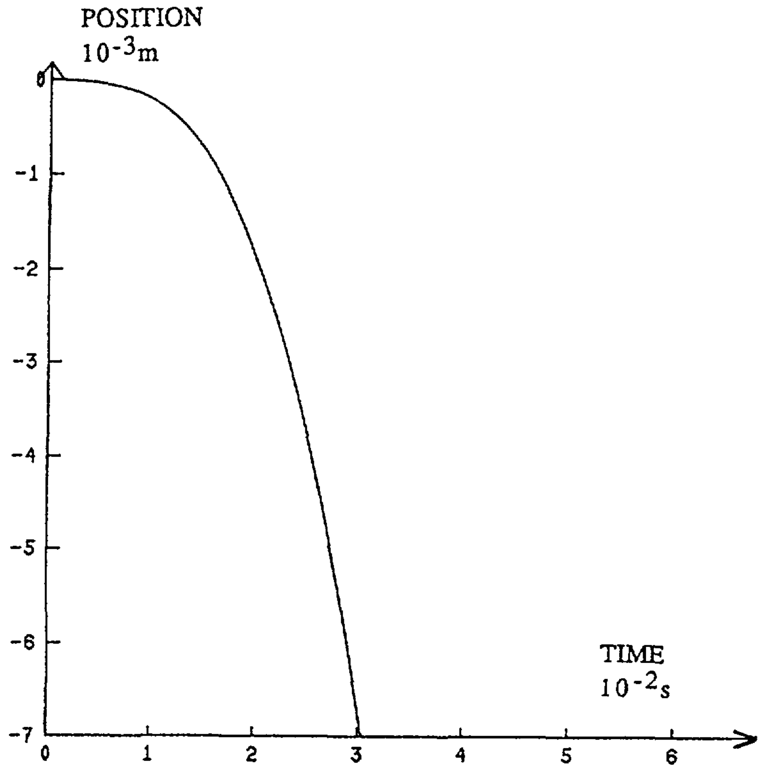


Fig. 4. — Displacement of the moving part.

The motional induced electromotive force is responsible for the decrease of the current just before the closure is completed.

**Conclusions.**

The coupling between the finite elements and the boundary elements appears to be a natural and workable method to model the movement in electromagnetic devices.

The rigid moving parts involving materials with complicated behaviour are easily modelled with the finite elements. Most of the time, the media wherein rigid parts move is linear magnetic and non-conducting (air) and very often open. Its treatment with boundary elements is therefore natural and avoids all the problems of mesh deformations. The paper shows that the motional induced electromotive force is easily taken into account in the eddy current formulation. The basis of the modelization is the 2D magnetic field computation software called LUCIE using the coupling between the boundary element and the finite element methods. Its modular structure has allowed an efficient and safe modification of the software.

The algorithm of the movement modelization shows that the modifications inside the program have been kept to a minimum. Most of the computations proper to the movement are gathered in an independent module.

The numerical computations show the accuracy of the method. Further developments will be the extension to three-dimensional computations.

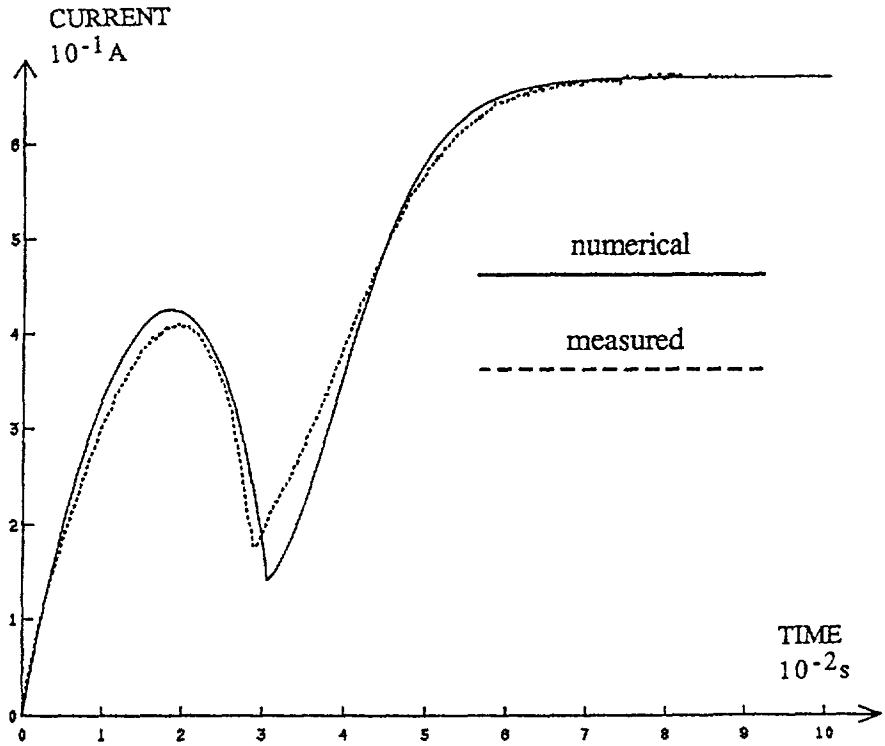


Fig. 5. — Current in the coil of the relay.

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