

Time stepping methods for transient analysis of magnetodynamic problems

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This text presents research results of the Belgian programme on Interuniversity Poles of attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming. The scientific responsibility is assumed by its authors.

Abstract

The most widespread method used to solve transient magnetodynamic problems is the Crank-Nicolson (CN) method. For linear differential equations, the CN method is unconditionally stable. The behaviour of pure magnetodynamic systems is a simple damping even for non linear problems and the CN method is often stable and accurate enough. Nevertheless when the non-linear electromagnetic model is coupled to another model e.g. a mechanical model for moving parts or circuit equations for the feeding of conductors, it may exhibit a much more complicated behaviour. In this case, the CN method may become unstable. Another possibility is the more stable backward Euler method but it is less accurate and tends to overdamp waveforms. In this paper, alternative methods are presented.

An important point is that the finite element discretisation of magnetodynamic problems leads to systems of differential algebraic equations that can not be treated numerically like regular ordinary differential equations [1]. They are index 1 systems, i.e. with the coefficient matrix of the differential terms singular, and can only be treated by implicit methods. Candidate methods are the backward differentiation formulae, the implicit Runge-Kutta methods and the methods based on repeated Richardson extrapolation.

A numerical example of two-dimensional magnetodynamic problem coupled with electric circuits and hysteresis models is presented with an emphasis on energy conservation.

Introduction: magnetodynamics and differential/algebraic systems

The equation for the two-dimensional magnetostatics is [2]:

$$\text{div}(\nu \text{grad } A) = -J \quad (1)$$

where ν is the magnetic reluctivity and A and J are respectively the vector potential and the current density. These vectors have only the z component different from zero. Using Ohm's law, the case of eddy currents can be dealt with by introducing (2) as the expression for the current density in (1):

$$J = \sigma E = -\left(\frac{\partial A}{\partial t} + U\right) \quad (2)$$

The conductor is characterised by its electrical conductivity σ , and U can be interpreted as the terminal voltage of the conductor (per unit of length). The finite element formulation is based on the semi discrete Galerkin method. Weighted residuals (3) for the domain Ω of boundary Γ are constructed, where w is a weighting function (depending on space variables).

$$\iint_{\Omega} \left[\nu \text{grad } A \cdot \text{grad } w + \sigma \left(\frac{\partial A}{\partial t} + U \right) w \right] d\Omega - \oint_{\Gamma} w \nu \frac{\partial A}{\partial n} d\Gamma = 0 \quad (3)$$

The space discretisation of (3) using the functions w leads to a differential system of equations. For a classical choice of the weight functions, the unknowns of the problem are the nodal value of the vector potential. In a non conducting region, the conductivity is equal to zero and the concerned equations reduce to algebraic ones.

In the rest of the paper, the reference to electromagnetic quantities is given up and the general system of differential/algebraic equations (DAE) (4) is considered:

$$A \dot{y}(t) + B y(t) = b(t) \quad (4)$$

where A and B are coefficient matrices, b is the right hand member vector, y is the unknown vector, t is the time and the dot indicates the time derivative. Note that A and B may depend on y in the case of a non linear system e.g. B depends on y in the case of ferromagnetic saturation and one has B(y(t)).

The numerical solution of system (4) requires a time discretisation. A simple scheme is given by (5):

$$A \frac{y_{n+1} - y_n}{\Delta t} + B (\alpha y_{n+1} + (1-\alpha)y_n) = \alpha b_{n+1} + (1-\alpha)b_n \quad (5)$$

Indices n and n+1 refer to quantities at time t_n and $t_{n+1} = t_n + \Delta t$ respectively. Various choices of the parameter α lead to classical methods ($\alpha=1$. is implicit (backward) Euler, $\alpha=0$. is explicit (forward) Euler, $\alpha=1/2$ is Crank-Nicolson, $\alpha=2/3$ is Galerkin).

The matrix $A/\Delta t + \alpha B$ is involved in the solution of system (5). One obvious condition is that it must be invertible i.e. non singular. The study of DAE systems such as (5) involves the matrix pencil $A + \lambda B$ of matrices A and B where λ is an arbitrary parameter [1,3]. It must be regular i.e. $\det(A + \lambda B)$ must not vanish identically. If it is not the case, it is impossible to find a Δt and an α such that $A/\Delta t + \alpha B$ is not singular and the system is meaningless. On the other hand, if $A + \lambda B$ is a regular pencil, matrices E, F exist such that:

$$A^* = E A F = \text{diag}(I, J) \quad (6)$$

$$B^* = E B F = \text{diag}(W, I) \quad (7)$$

where diag denotes a square matrix constructed with the argument square matrices placed on its diagonal, I are unit matrices of the suitable dimension, W is a regular square matrix, and J is a nilpotent Jordan block matrix with blocks of the form:

$$\begin{bmatrix} 0 & & 0 \\ 1 & 0 & \\ & \dots & \dots \\ 0 & & 1 & 0 \end{bmatrix} \quad (8)$$

The size of the largest of these blocks is m, the nilpotency or index of the system (The nilpotency of a matrix J may alternatively be defined as the integer m such that $J^m = 0$ and $J^{m-1} \neq 0$). With $y^* = F^{-1}y$ and $b^* = E b$, the DAE system (4) is in his Kronecker normal form (KNF) [1]:

$$\begin{cases} \dot{u} + W u = q \\ J \dot{v} + v = r \end{cases} \quad \text{with} \quad y^* = \begin{pmatrix} u \\ v \end{pmatrix} \quad b^* = \begin{pmatrix} q \\ r \end{pmatrix} \quad (9)$$

The numerical behaviour of a DAE system depends strongly on its nilpotency. On the one hand, if the nilpotency is greater than one, the system is very difficult to solve and special techniques must be used. On the other hand, if the nilpotency is equal to one (J is identically equal to zero), the system is much easier to solve and some classical methods for ODE may work.

In the case of the magnetodynamic system (4), two kinds of equations must be considered. The first kind are the equations corresponding to nodes belonging to at least one conducting region. The time derivative of the corresponding nodal value of the vector potential gives a non zero term on the diagonal of A. Such equations do not influence the nilpotency of the system. The second kind are the equations corresponding to nodes belonging only to non conducting regions. Those equations do not have any differential term, they are purely algebraic and their contributions to the matrix A are null lines. They obviously lead to a system of nilpotency 1. Therefore, classical methods such as the scheme (5) may be used. Nevertheless, if the explicit Euler method is tried, the singular matrix $A/\Delta t$ is involved and the solution is impossible. In general, purely explicit methods may not be applied to solve (4) because they involve singular matrices and implicit methods must be used. Some of these methods are presented in the following sections.

Multistep methods: backward difference formulae

The general differential system (10) is considered:

$$\dot{y} = f(y, t) \quad (10)$$

To solve it by a backward difference formulae (BDF) method [4], the following replacements are made:

$$\dot{y} \rightarrow \frac{1}{\Delta t} \sum_{j=0}^p \alpha_j y_{n+1-j} \quad (11)$$

$$f \rightarrow \sum_{j=0}^q \beta_j f_{n+1-j} \quad (12)$$

Indices i refer to quantities at times t_i , e.g. $y_i = y(t_i)$ and $f_i = f(y_i, t_i)$, all the times t_i being spaced by a constant time step Δt . The solution being known up to time t_n , the solution at time t_{n+1} is computed by solving for y_{n+1} the system obtained by doing the replacements (11) and (12) in (10). A particular method is defined by giving $p+1$ coefficients α_i and $q+1$ coefficients β_i . Table 1 shows some classical methods. Adams Moulton and Gear methods appear as higher order generalisation of the scheme (5).

method	p	α_0	α_1	α_2	α_3	q	β_0	β_1	β_2
Implicit Euler	1	1	-1			0	1		
Crank Nicolson	1	1	-1			1	1/2	1/2	
Adams Moulton	1	1	-1			2	5/12	8/12	-1/12
two step Gear	2	3/2	-4/2	1/2		0	1		
three step Gear	3	11/6	-18/6	9/6	-2/6	0	1		

Table 1: Backward difference methods

In the case of the DAE system (4), with the following notations:

$$y_\alpha = \sum_{j=0}^p \alpha_j y_{n+1-j} \quad (13)$$

$$B(y)y|_\beta = \sum_{j=0}^q \beta_j B(y_{n+1-j})y_{n+1-j} \quad (14)$$

$$b_\beta = \sum_{j=0}^q \beta_j b_{n+1-j} \quad (15)$$

the computation of the next time step is the solution of the system (16):

$$A \frac{y_\alpha}{\Delta t} + B(y)y|_\beta = b_\beta \quad (16)$$

for y_{n+1} , where the dependence of B on y has been made explicit.

One step methods: Runge Kutta methods

The general differential system (10) is considered again. An approximation of the solution from the values y_n of the solution at the previous time step may be obtained by solving the following relations [5]:

form 1

$$\begin{cases} Y_i = y_n + \Delta t \sum_{j=1}^s \beta_{ij} f(Y_j, t_n + \alpha_i \Delta t) & i = 1 \dots s \\ y_{n+1} = y_n + \Delta t \sum_{i=1}^s \mu_i f(Y_i, t_n + \alpha_i \Delta t) \end{cases} \quad (17)$$

or form 2

$$\begin{cases} k_i = \Delta t \ f(y_n + \sum_{j=1}^s \beta_{ij} k_j, t_n + \alpha_i \Delta t) & i = 1 \dots s \\ y_{n+1} = y_n + \sum_{i=1}^s \mu_i k_i \end{cases} \quad (18)$$

The equivalence of both forms may be shown by remarking that $k_i = \Delta t \ f(Y_i, t_n + \alpha_i \Delta t)$. A s-step method is characterised by the coefficients α_i , β_{ij} and μ_i given in its Runge-Kutta tableau or Butcher diagram:

α_1	β_{11}	β_{12}	...	β_{1s}
α_2	β_{21}	β_{22}	...	β_{2s}
...
α_s	β_{s1}	β_{s2}	...	β_{ss}
	μ_1	μ_2	...	μ_s

with $\alpha_i = \sum_{j=1}^s \beta_{ij}$.

The Runge Kutta methods may be sorted according to the structure of their diagram. The classical explicit methods are the methods for which $\beta_{ij} = 0$ for $j \geq i$. If this condition is not fulfilled (if $\beta_{ij} \neq 0$ for $j \geq i$), the methods are called the implicit Runge Kutta methods (IRK). Special cases of IRK are the diagonally implicit Runge Kutta (DIRK) methods (if $\beta_{ij} = 0$ for $j > i$), a sub category of which are the singly diagonally implicit Runge Kutta (SDIRK) methods (if $\beta_{ij} = 0$ for $j > i$ and if $\beta_{ii} = \beta$ for all i).

The IRK methods may be applied to the DAE system (4). Nevertheless, only form 2 may be used because form 1 would involve the inverse of A in its last step. The adaptation of form 2 to the DAE systems gives the following relations [6]:

$$\begin{cases} Ak_i = \Delta t \left[b(t_n + \alpha_i \Delta t) - B(y_n + \sum_{j=1}^s \beta_{ij} k_j) (y_n + \sum_{j=1}^s \beta_{ij} k_j) \right] & \text{for } i = 1 \dots s \\ y_{n+1} = y_n + \sum_{i=1}^s \mu_i k_i \end{cases} \quad (19)$$

In this case, a $(s.n) \times (s.n)$ system has to be solved to find the k_i . Nevertheless, in the case of DIRK, k_1 depends only on y_n , k_2 only on y_n and k_1 , etc... and the relations (19) reduce to a set of s successive systems of size $n \times n$.

Here are some examples of DIRK:

• one step method [7]

λ	λ
	1

In the case of linear problems, this method reduces to scheme (5) excepted for the independent term b which is discretised as $b(t_n + \lambda \Delta t)$ instead of $(1-\lambda)b(t_n) + \lambda b(t_{n+1})$.

• two step method [7]

λ	λ	0
$1-\lambda$	$1-2\lambda$	λ
	1/2	1/2

• three step method [8]

$\alpha/2$	$\alpha/2$	0	0
1/2	α	$1/2-\alpha$	0
$1-\alpha/2$	α	$1-2\alpha$	$\alpha/2$
	α	$1-2\alpha$	α

with $\omega = \sqrt[3]{2}$ and $\alpha = \frac{1 + \omega + \omega^{-1}}{3}$.

Note that this method is symplectic i.e. for Hamiltonian systems, it conserves the symplectic two form. The weaker properties of volume conservation in phase space (Liouville's theorem) and of conservation of energy are a consequence of this symplecticness. Unfortunately the problems considered in this paper are not Hamiltonian and this scheme does not conserve energy.

The following algorithm may be used to implement the DIRK method to solve (4) and to keep the form of the non linear systems to be solved as close as possible to the ones involved in the other methods:

For $i = 1$ to $s-1$

- To find K_i , solve the system:

$$\left(\frac{A}{\Delta t \beta_{ii}} + B(K_i) \right) K_i = b(t_n + \alpha_i \Delta t) + \frac{A}{h \beta_{ii}} \left(y_n + \sum_{j=1}^{i-1} \beta_{ij} k_j \right)$$

- Compute $k_i = \frac{K_i - y_n - \sum_{j=1}^{i-1} \beta_{ij} k_j}{\beta_{ii}}$

Compute
$$y_{n+1} = y_n + \sum_{i=1}^s \mu_i k_i$$

One step methods: extrapolation methods

The principle of extrapolation methods is to perform computations with a basic scheme for a sequence of decreasing time steps and to extrapolate to zero time step (Richardson's deferred approach to the limit) [9]. This is made by considering the asymptotic development of the error of the basic scheme in powers of the time step and by annulling the successive terms. The following algorithm has been used for the computations performed in this paper: The chosen basic scheme is the implicit Euler method. Although the Crank Nicolson method is more accurate, small oscillations due to its weaker stability jeopardise the convergence of the extrapolation process. The implicit Euler method is very stable and its error development includes all the powers of the time step from the first one. The error development of CN includes the same terms excepted the first order one. The following sequence of time steps is considered [9]:

$$\{\Delta t, \Delta t/2, \Delta t/4, \Delta t/6, \Delta t/8, \Delta t/12, \dots, \Delta t/n_i \dots\}, \quad n_i = 2n_{i-2} \text{ for } i > 4 \quad (20)$$

Starting at time t_n with solution y_n , a solution T_{i1} at time $t_n + \Delta t$ is obtained by performing n_i steps of implicit Euler method with the time step $\Delta t/n_i$. The following sequence of approximations is obtained:

$$\{T_{11}, T_{21}, T_{31}, T_{41}, T_{51}, T_{61}, T_{71}, \dots, T_{i1} \dots\} \quad (21)$$

The higher order approximations are obtained by constructing the following diagram:

$$\begin{array}{ccccccc} T_{11} & & & & & & \\ T_{21} & T_{22} & & & & & \\ T_{31} & T_{32} & T_{33} & & & & \\ \vdots & & & \ddots & & & \\ T_{n1} & \dots & & & T_{nn} & & \end{array} \quad (22)$$

The first column is given by (21) and the other ones are computed using the following recurrence formula (polynomial extrapolation):

$$T_{i,k} = T_{i,k-1} + \frac{(T_{i,k-1} - T_{i-1,k-1})}{\left(\frac{n_i}{n_{i-k+1}}\right) - 1} \quad (23)$$

In practice, the diagram is constructed line by line by computing a new T_{i1} and performing the extrapolations. The process is stopped when T_{ii} and $T_{i,i-1}$ agree up to the required accuracy. The rational extrapolation may also give good results and is advised by some authors [9]. It requires only the use of (24) instead of (23):

$$T_{i,k} = T_{i,k-1} + \frac{(T_{i,k-1} - T_{i-1,k-1})}{\frac{n_i}{n_{i-k+1}} \left[1 - \frac{T_{i,k-1} - T_{i-1,k-1}}{T_{i,k-1} - T_{i-1,k-2}} \right] - 1} \quad (24)$$

In practice, both extrapolations are performed and the best is chosen. In numerical tests, the polynomial extrapolation seems to better converge.

Numerical examples

As an application, the discharge of a capacitor in a series RLC circuit is considered, where the inductive effects are computed by the finite element method [10]. This is a free oscillation problem, which is more sensitive to time integration method than forced oscillation ones. First, the linear case ($\mu_r = 1000$) is considered in order to study the behaviour of the various schemes. The reference solution is computed with the Crank-Nicolson method and with a very small stepsize $h = 4.10^{-4}$ sec. For comparing the various methods, a constant stepsize of $h = 4.10^{-3}$ sec is considered, which corresponds approximately to 20 stepsizes in one period. In figure 1, the multistep methods are compared. It appears that the implicit Euler method has a too important damping effect. The Crank-Nicolson scheme is very accurate (the energy of the system is approximately conserved), but a small frequency error is introduced. The more precise scheme is the Adams Moulton one, but it becomes unstable after a few periods. Finally, two step Gear is not of great interest because frequency error as well as numerical damping effect are present. It must be noticed that all these methods are similar in computation time. Higher order methods (such as two step Gear and Adams Moulton) only require the storage of intermediate vectors.

The one step methods are compared in figure 2. It appears that the symplectic DIRK method as well as the extrapolation method give excellent results. Convergence of the extrapolation process is generally reached after 3 subdivisions of the stepsize. This means that this method is more expensive than Crank-Nicolson scheme. Larger stepsizes have been tried. For the largest stepsize ($4h$), the extrapolation process converges after 4 subdivisions. The solution is still excellent, but seems discontinuous. This can explain why the Crank-Nicolson method is preferred for linear problems.

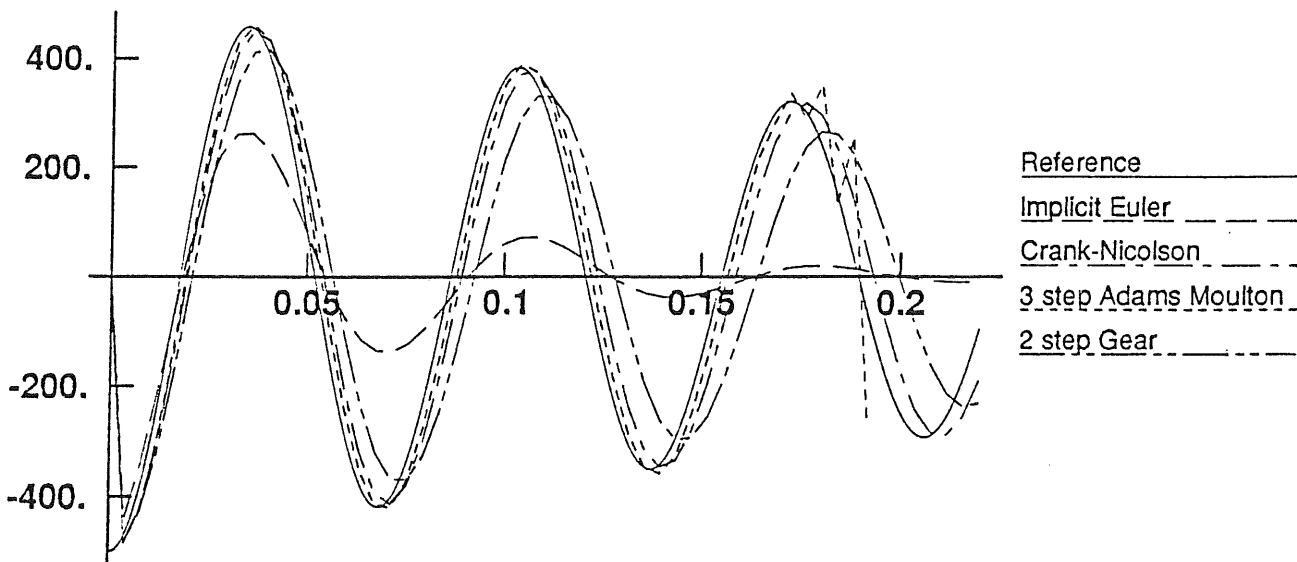


Figure 1 : Time evolution (s) of capacitor voltage (V) - Multistep methods.

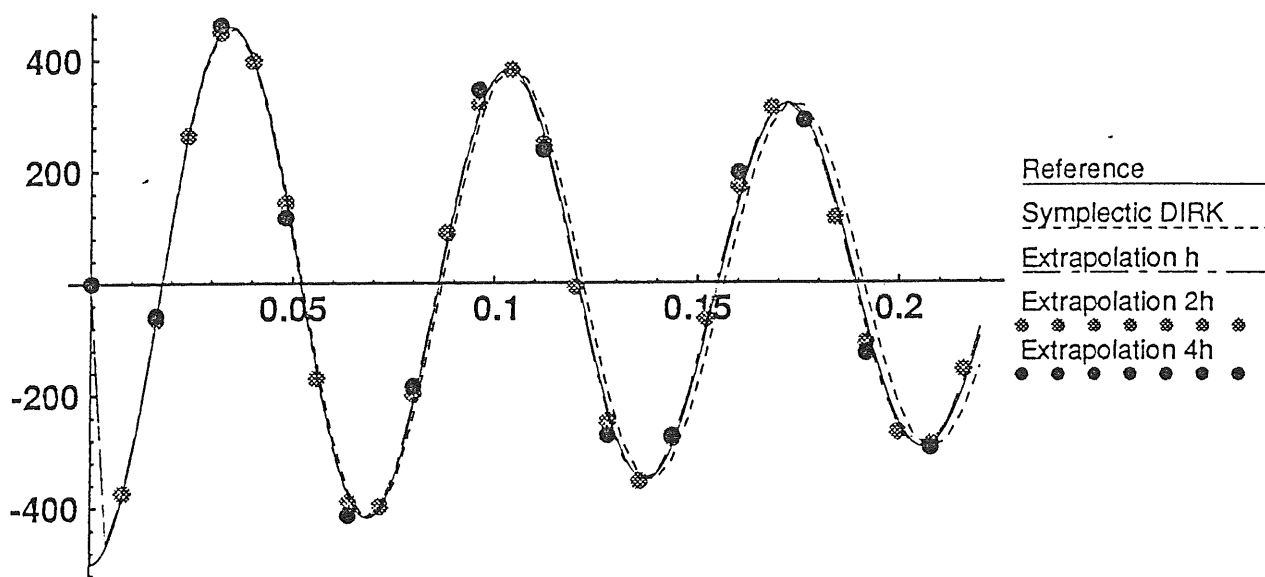


Figure 2 : Time evolution (s) of capacitor voltage (V) - One step methods.

As a non linear test, the same problem with a hysteretic core is considered. As a criterion of quality of the solution, the energy balance is checked. In the beginning, only electrostatic energy is present. This energy is converted to magnetostatic energy and dissipated into Joule and hysteretic losses. The summation of these four types of energy must give the total energy of the system and must stay constant. The implementation of the Preisach hysteresis model in the finite element equations with the computation of hysteretic losses and of magnetostatic energy is given in reference [10].

In this case, it is shown that Crank-Nicolson method does not conserve the total energy of the system (figure 3), the opposite of the extrapolated solution (figure 4). Further developments will concern the application of IRK methods to non linear problems.

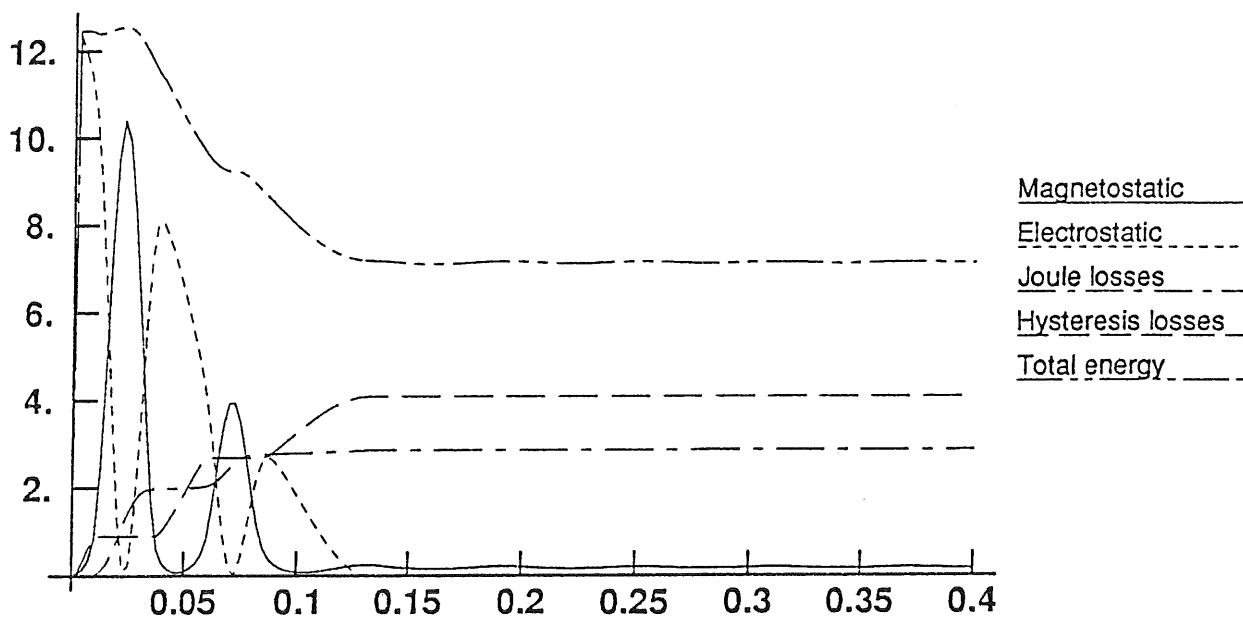


Figure 3 : Time evolution (s) of the energies (J) computed with Crank-Nicolson scheme.

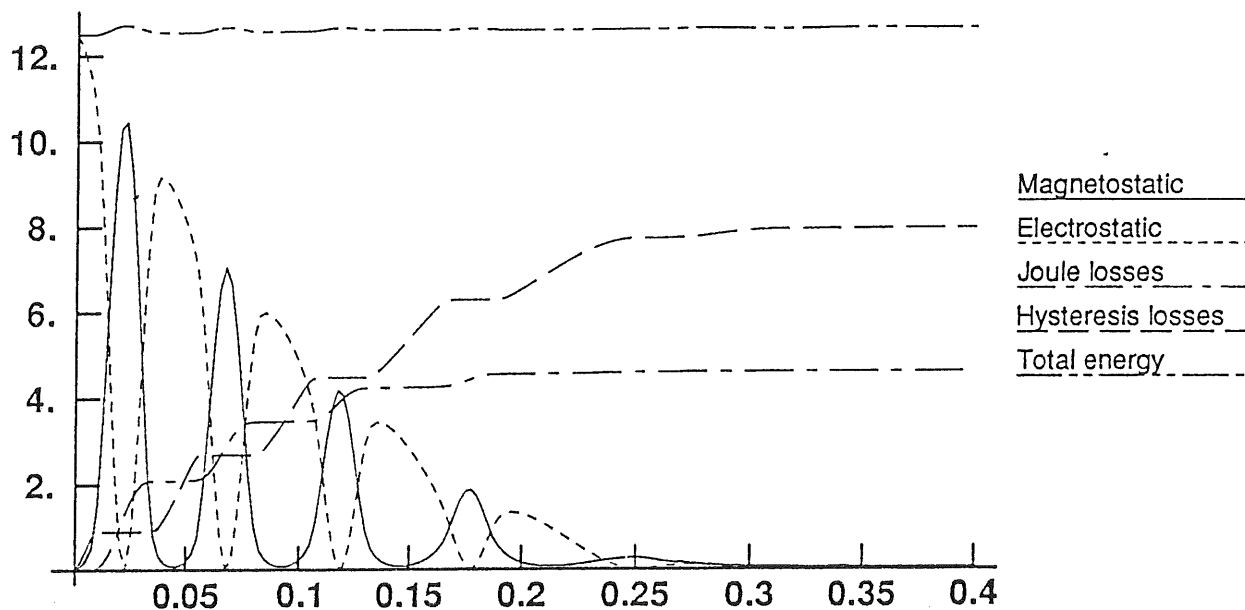


Figure 4 : Time evolution (s) of the energies (J) computed with extrapolation method.

Conclusion

The Crank-Nicolson method appears to be a simple and accurate method. Nevertheless it may fail in some problems. An interesting alternative are the extrapolation methods. Although they are time consuming, extrapolation methods are very robust and provide a natural mechanism of step control. Energy conservation appears to be a good test of the validity of the method for very complex problems.

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