Transformation methods in computational electromagnetism

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The representation of electromagnetic quantities by differential forms allows the use of nonorthogonal coordinate systems. A judicious choice of coordinate system facilitates the finite element modeling of infinite or very thin domains.

DIFFERENTIAL GEOMETRY

Because of the propagation of electromagnetic fields in free space (or in the air), electromagnetic problems are often open, i.e., characterized by the decay of the fields at infinity, the opposite to closed problems with boundary conditions at a finite distance. As the finite element computations can only involve a finite number of degrees of freedom, various methods have been proposed to overcome this difficulty. One of the most interesting ones is the transformation method: the infinite domain is transformed into a finite one used for the discretization and the resolution.[1] The problem has then to be formulated in this domain, where the coordinate system is not orthogonal. The differential geometry allows the use of any coordinate system and is the right tool to formulate the method clearly in a general way.

From a differential geometric point of view,[2–5] the vectors are the first-order linear differential operators on functions. They have a vector space structure, one basis of which is the set $\{\partial/\partial x^i\}$ of partial derivatives with respect to coordinates. The action of a vector v on a function f is noted $v(f)$. A 1-form α is a linear map from vectors v to real numbers $\alpha(v)$ (also noted $\langle\alpha,v\rangle$ to emphasize duality). A special 1-form associated to a function f is its differential df, defined such that $df(v)=v(f)$. One basis for the vector space of 1-forms is the set $\{dx^i\}$ of the differentials of the coordinates. A p-form ω is a multilinear totally skew symmetric map from p vectors $v_1,\ldots,v_p$ to real numbers $\omega(v_1,\ldots,v_p)$. Functions are identified with 0-forms. In three-space only 0-, 1-, 2-, and 3-forms are not identically equal to zero (because of skew symmetry). The 0- and 3-form spaces are one-dimensional vector spaces, while 1- and 2-forms are three-dimensional vector spaces. From this point of view, scalar from vector analysis are 0- or 3-forms, depending on their physical meaning: 0-forms are pointwise relevant functions, while 3-forms are densities to be integrated on volumes. The "vectors" from the vector analysis are 1-forms and 2-forms: 1-forms are integrands of line integrals, while 2-forms are flux densities. Operations on forms include the following.

(i) The exterior or wedge product $\wedge$ that maps pairs of a p-form ω_1 and a q-form ω_2 on the $(p+q)$-form $\omega_1\wedge\omega_2$, defined by

$$(\omega_1\wedge\omega_2)(v_1,\ldots,v_{p+q})$$
$$=\frac{1}{p!q!}\sum_{\pi\in S_{p+q}}[\mathrm{Sgn}(\pi)\,\omega_1(v_{\pi(1)},\ldots,v_{\pi(p)})$$
$$\times\,\omega_2(v_{\pi(p+1)},\ldots,v_{\pi(p+q)})], \tag{1}$$

where π runs over the set of permutations of $p+q$ indices. The set $\{dx^{i_1}\wedge\cdots\wedge dx^{i_p}\}$ of the linearly independent exterior products of p differentials of the coordinates is a basis for the $n!/[p!(n-p)!]$-dimensional vector space of p-forms. Any p-form can be expressed as a linear combination of such p-monomials.

(ii) The exterior derivative d that maps p-forms,

$$\omega=\frac{1}{p!}\sum_{i_1\cdots i_p=1}^{n}\omega_{i_1\cdots i_p}\,dx^{i_1}\wedge\cdots\wedge dx^{i_p}$$

on $(p+1)$-forms,

$$d\omega=\frac{1}{p!}\sum_{i_1\cdots i_p=1}^{n}(d\omega_{i_1\cdots i_p})\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_p}.$$

From this definition it is obvious that the exterior derivative of a function is its differential df.

(iii) the integration of an n-form $\omega=f(x^1\cdots x^n)dx^1\wedge\cdots\wedge dx^n$ on a n-dimensional domain M is defined by

$$\int_M\omega=\int_{R^n}f(x^1\cdots x^n)dx^1\cdots dx^n, \tag{2}$$

where f is supposed to be zero outside M.

These objects and operations only involve the topology and the differential structure of the ambient space, i.e., they are independent of any notion of angle and/or distance. Those notions are introduced by giving a metric g, i.e., a symmetric bilinear map from two vectors v,w to real numbers $g(v,w)$. The metric allows the definition of the Hodge star operator $\star$, which maps p-forms on $(n-p)$-forms, where n is the dimension of the ambient space. In local coordinates, the star operator is defined for an exterior p-monomial by (using the Einstein summation convention on repeated indices)

$$\star dx^{i_1}\wedge\cdots\wedge dx^{i_p}=g^{i_1j_1}\cdots g^{i_pj_p}\,dx^{j_{p+1}}\wedge\cdots\wedge dx^{j_n}\,\epsilon_{j_1\cdots j_n}\frac{\sqrt{|g|}}{(n-p)!}, \tag{3}$$

where $\epsilon_{j_1\cdots j_n}$ is the Levi–Civita symbol. If the matrix, the elements of which are $g_{ij}=g(\partial/\partial x^i,\partial/\partial x^j)$ is considered, the g^{ij} are the components of its inverse and $g=\det(g_{ij})$ is its determinant. By linearity, the definition of the star operator may be extended to any form. In three-space, the Hodge star operator maps 0-forms on 3-forms, 1-forms on 2-forms, and conversely. This is why only functions and "vectors" are used in the vector analysis of the three-space with the Euclidean metric.

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The interesting property of differential forms is their behavior under the mapping of domains. If ϕ is a differential map from a domain M to a domain N, a function f on N is mapped (pulled back) on a function $\phi^\star(f)=f\circ\phi$ on M by composition with ϕ. Vectors on M are mapped (pushed forward) on vectors on N by considering their action on pulled-back functions. Since a vector v on M maps the pulled-back functions $\phi^\star(f)$ to numbers $v[\phi^\star(f)]$, it defines a linear differential operator for the functions f on N, i.e., a vector $\phi_\star(v)$ on N, whose action on a function f on N is given by $\phi_\star(v)(f)=v[\phi^\star(f)]$. A 1-form on N is mapped (pulled back) on M by considering its action on pushed-forward vectors from M to N. The pull-back $\phi^\star(\alpha)$ on M of a 1-form α on N is defined by $\langle\phi^\star(\alpha),v\rangle=\langle\alpha,\phi_\star(v)\rangle$. Any purely covariant object such as a p-form or the metric can be pulled back from M to N by the following definitions, for a p-form $\phi^\star(\omega)(v_1,\ldots,v_p)=\omega[\phi_\star(v_1),\ldots,\phi_\star(v_p)]$, and for the metric $\phi^\star(g)(v_1,v_2)=g[\phi_\star(v_1),\phi_\star(v_2)]$. As g allows the definition of the star operator $\star_g$ on N, $\phi^\star(g)$ allows the definition of the star operator $\star_{\phi^\star(g)}$ on M. In the preceding definitions, it is important to remark how the duality reverses maps leading to an alternation of push-forward and pull-back maps for the domains, the functions, the vectors, and the forms. The fundamental point for the setting of transformation methods is that the operations on forms previously defined commute with the pull-back. For forms α,β, one has

$$\phi^\star(\alpha\wedge\beta)=\phi^\star(\alpha)\wedge\phi^\star(\beta), \tag{4}$$

$$\phi^\star(d\alpha)=d\phi^\star(\alpha), \tag{5}$$

$$\int_M\phi^\star(\alpha)=\int_{N=\phi(M)}\alpha, \tag{6}$$

$$\star_{\phi^\star(g)}\phi^\star(\alpha)=\phi^\star(\star_g\alpha). \tag{7}$$

The electromagnetic fields and sources may be represented by differential forms: the magnetic field H and the electric field E are 1-forms; the magnetic flux density B, the electric flux density D, and the current density J are 2-forms; and the charge density ρ is a 3-form. In this representation, Maxwell's equations are $dH=J+\partial_tD$, $dE=-\partial_tB$, $dB=0$, and $dD=\rho$, where ∂_t is for partial derivation with respect to time. The 1-form A and the 0-form V may be introduced as potentials, such that $B=dA$ and $E=-\partial_tA-dV$. All those equations are obviously independent of the metric. Nevertheless, this one is involved in the definition of the constitutive relations: the free space electromagnetic characteristics are, with $c=1$, $\mu_0=1$, and $\epsilon_0=1$, given by $D=\star E$ and $B=\star H$.

FINITE ELEMENTS AND TRANSFORMATION METHOD

In order to set up the finite element method, a variational form is introduced. For example, the magnetostatic Lagrangian is given by the integration of the 3-form L, the magnetostatic Lagrangian density, on the domain M^4 (the coefficient ν is the magnetic reluctivity):

$$\int_M L(A)=\int_M\left(-\frac{1}{2}\ dA\wedge\nu\star dA-A\wedge J\right). \tag{8}$$

The finite element method consists of approximating A by $\underline{A}=\Sigma A_i\omega_i$, where A_i are parameters and ω_i are 1-forms corresponding to "shape functions" obtained by assuming a simple behavior on elements from a meshing of M. The current density J is approximated by $\underline{J}=\Sigma J_i\star\omega_i$.

The equations for the parameters are found by expressing the extremum conditions for the discretized Lagrangian:

$$\frac{\partial}{\partial A_i}\int_M L(\underline{A})=0. \tag{9}$$

The first term of $L(\underline{A})$ leads to terms in the finite element equations, such that the unknowns are the A_j and the coefficients a_{ij} are the integrals of the discrete 3-forms $(-\nu/2)\ d\omega_i\wedge\star d\omega_j$ on the elements. If a domain $M^\star$ is mapped on the domain M, the forms and the metric on M may be pulled-back on $M^\star$, and the formulation of the problem on $M^\star$ is immediate. The transformation method is thus the following one: map the "transformed domain" $M^\star$ on the original domain M (and not the opposite!), pull-back the Lagrangian and apply the finite element method with the elements obtained by meshing $M^\star$. It does not matter if the transformed domain no longer fits the geometry because it has only to be connected topologically with the rest of the problem.

In the case of a two-dimensional magnetostatic problem, invariant by translation along the z axis, the vector potential is the 1-form $A(x,y)dz$, and the geometry is described by its trace on the x-y plane. The shape 1-forms are $\omega_i=\alpha_i(x,y)dz$, where $\alpha_i(x,y)$ are the classical shape functions and $\star\omega_i=\alpha_i(x,y)\star dz=\alpha_i(x,y)dx\wedge dy$. With these Cartesian coordinates, it gives exactly the traditional method. The mapping of a domain $M^\star$ with coordinates $\{X,Y\}$ on the original domain M with coordinates $\{x,y\}$ is given by two functions, such that $\{X,Y\}\rightarrow\{x,y\}=\{f_1(X,Y),f_2(X,Y)\}$. The contribution of an element to the coefficient a_{ij} of A_j in the ith equation is the integral on the element of

$$-\nu\ \ (\partial_X\alpha_i\ \partial_Y\alpha_i)$$

$$\times\begin{bmatrix}\dfrac{\partial_Yf_1{}^2+\partial_Yf_2{}^2}{\partial_Xf_1\ \partial_Yf_2-\partial_Xf_2\ \partial_Yf_1} & -\dfrac{\partial_Xf_1\ \partial_Yf_1+\partial_Xf_2\ \partial_Yf_2}{\partial_Xf_1\ \partial_Yf_2-\partial_Xf_2\ \partial_Yf_1}\\ -\dfrac{\partial_Xf_1\ \partial_Yf_1+\partial_Xf_2\ \partial_Yf_2}{\partial_Xf_1\ \partial_Yf_2-\partial_Xf_2\ \partial_Yf_1} & \dfrac{\partial_Xf_1{}^2+\partial_Xf_2{}^2}{\partial_Xf_1\ \partial_Yf_2-\partial_Xf_2\ \partial_Yf_1}\end{bmatrix}$$

$$\times\begin{pmatrix}\partial_X\alpha_j\\ \partial_Y\alpha_j\end{pmatrix}, \tag{10}$$

where ∂_X and ∂_Y indicate the partial derivative with respect to X and Y, respectively, and $\alpha_i(X,Y)$, $\alpha_j(X,Y)$ are the shape functions on the transformed element. This contribution of the transformed element is equal to the nontransformed one up to the central matrix. If the transformation is trivial $[f_1(X,Y)=X,f_2(X,Y)=Y]$ or corresponds to a conformal transformation $[f_1(X,Y)+if_2(X,Y)$ is analytic, i.e., $\partial_xf_1=\partial_yf_2$ and $\partial_xf_2=-\partial_yf_1]$, this matrix reduces to the unit matrix. As for the term involving the current density, the contribution to the ith equation of an element is the product of J_j by the integral of $\omega_i\wedge\star\omega_j=\alpha_i\alpha_j\theta$ on the element. θ is

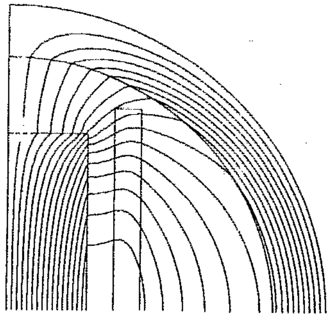

FIG. 1. Modeling of an infinite domain mapped on a corona.

the volume form associated to the metric, i.e., $\star 1=\theta=(\partial_X f_1\ \partial_Y f_2-\partial_X f_2\ \partial_Y f_1)dX\wedge dY$ and the integral transforms simply according to the formula for the change of coordinates in multiple integrals.

In the three-dimensional case, a family of discrete forms are the Whitney forms on tetrahedra.[6] They correspond to nodal elements for 0-forms (e.g., scalar magnetic potential ϕ) and to edge elements for 1-forms (e.g., A or H). Using such discretizations entirely based on differential forms allows a straightforward formulation of transformation methods by pull-back of the corresponding discrete Lagrangian density or weighted residuals.

NUMERICAL EXAMPLES

Infinite domain: The most obvious application of the transformation method is the case of open boundary problems. In the present example, the magnetic field around a coil with a ferromagnetic kernel is computed. A fictitious circular boundary of radius A is defined around the problem and the outside of this circle constitutes an infinite domain. The method consists of mapping a corona $M^\star$ on M.[1] The corona, a finite domain, has an inner radius A and an outer radius B, all the circles considered here having the same center. The Cartesian coordinates on M are x and y, and the coordinates on $M^\star$ are X and Y. The transformation is given by the two functions (with $R = \sqrt{X^2+Y^2}$):

$$x=f_1(X,Y)=X[A(B-A)]/[R(B-R)], \tag{11}$$

$$y=f_2(X,Y)=Y[A(B-A)]/[R(B-R)]. \tag{12}$$

Figure 1 shows the field lines computed by this method. Thanks to the symmetry of the problem, only one quarter has been computed.

Thin plate: Very thin objects and very small skin depths are other examples of problems difficult to model with finite elements. In this case it is interesting to stretch the geometry along the small dimension. The example considered here is a steel band heated by induction. The steel band is 1.2 mm thick and is heated by a one-turn rectangular coil (240 mm long and 3 mm thick), placed 28 mm above the band and fed by a 3490 A current. The thickness of the band is small with respect to the characteristic length of the problem, but the skin depth is even smaller: the frequency of the current is 275 kHz, the conductivity of the plate is 10^7 S/m, and the relative permeability is 405. In those conditions, the skin depth is equal to about 15 μm. To model the plate, surface layers of 45 μm are considered. Those layers are stretched by a factor 15 000 and then become layers of 675 mm in the transformed problem. Thus, the transformation used is simply $\{x=X,\ y=Y/15\ 000\}$. Although almost trivial, this transformation makes the problem more tractable. Moreover, the use of a general formalism allows the extension of this technique to curved plates or to nonlinear transformations, e.g., taking into account the exponential decay of the current density and field inside the plate. Figure 2 shows the field lines in the lower stretched layer at the phase-0 degree of the excitation current.

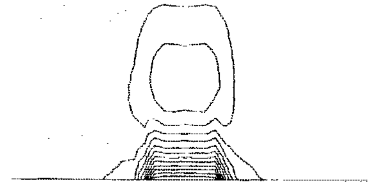

FIG. 2. Field lines in a stretched layer phase: 0 degree.

CONCLUSION

The use of the differential geometry in computational electromagnetics has been advocated by various authors.[5,7] It is the natural framework to set up the transformation method that becomes almost trivial in this context. Although the first examples of application of this method have been introduced with the help of the vector calculus more familiar to engineers, its formulation with the help of differential geometry allows a systematic generalization, and leads to a quasiautomatic implementation method, where all the steps are clearly defined.

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