Implicit Runge-Kutta methods in magnetodynamic problems

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1. Introduction

The most widespread method used to solve transient magnetodynamic problems is the Crank-Nicolson (CN) method. For linear differential equations, the CN method is unconditionally stable. The behaviour of pure magnetodynamic systems is a simple damping even for nonlinear problems and the CN method is often stable and accurate enough. Nevertheless, when the nonlinear electromagnetic model is coupled to another model, e.g. a mechanical model for moving parts or circuit equations for the feeding of conductors, it may exhibit a much more complicated behaviour. In this case, the CN method may become unstable. Another possibility is the more stable backward Euler method but it is less accurate and tends to overdamp waveforms. An important point is that the finite element discretization of magnetodynamic problems leads to systems of differential algebraic equations (DAE) that cannot be treated numerically like regular ordinary differential equations[1]. The coefficient matrix of the differential terms is singular, and the systems can only be treated by implicit methods.

The equation for the two-dimensional magnetostatics is:

$$\mathrm{div}(\nu \,\mathrm{grad}\, A) = -J \tag{1}$$

where μm is the magnetic reluctivity, and A and J are respectively the vector potential and the current density. These vectors have only the z-component different from zero.

Using Ohm's law, the case of eddy currents can be dealt with by introducing (2) as the expression for the current density in (1):

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$$J = \sigma E = -\sigma\left(\frac{\partial A}{\partial t} + U\right) \tag{2}$$

The conductor is characterized by its electrical conductivity σ, and U can be interpreted as the terminal voltage of the conductor (per unit of length). The finite element formulation [2] is based on the semi discrete Galerkin method. Weighted residuals (3) for the domain Ω of boundary Γ are constructed, where w is a weighting function (depending on space variables).

$$\iint_{\Omega}\left[\nu \mathrm{grad}A.\mathrm{grad}w + \sigma\left(\frac{\partial A}{\partial t} + U\right)w\right]d\Omega \; - \oint_{\Gamma} w\nu\frac{\partial A}{\partial n}d\Gamma = 0 \tag{3}$$

The space discretization of (3) using the functions w leads to a differential system of equations. For a classical choice of the weight functions, the unknowns of the problem are the nodal value of the vector potential. In a nonconducting region, the conductivity is equal to zero and the concerned equations reduce to algebraic ones.

In the rest of the paper, the reference to electromagnetic quantities is given up and the general system of differential/algebraic equations (DAE) (4) is considered:

$$\mathbf{A}\ \dot{y}(t) + \mathbf{B}\ y(t) = b(t) \tag{4}$$

(where **A** and **B** are coefficient matrices, b is the right hand member vector, y is the unknown vector, t is the time and the dot indicates the time derivative. Note that **A** and **B** may depend on y in the case of a nonlinear system, e.g. **B** depends on y in the case of ferromagnetic saturation and one has **B**(y(t)). The numerical solution of system (4) requires a time discretization. A simple scheme is given by (5):

$$\mathbf{A}\ \frac{y_{n+1} - y_n}{\Delta t} + \mathbf{B}\ (\alpha y_{n+1} + (1-\alpha)y_n) = \alpha b_{n+1} + (1-\alpha)b_n \tag{5}$$

Indices n and n+1 refer to quantities at time *tn* and $t_{n+1} = t_n + \Delta t$ respectively. Various choices of the parameter α lead to classical methods (α=1 is implicit (backward) Euler, α=0 is explicit (forward) Euler, α=1/2 is Crank-Nicolson, α=2/3 is Galerkin).

The matrix $A/\Delta t + \alpha B$ is involved in the solution of system (5). One obvious condition is that it must be invertible, i.e. nonsingular. In the case of the magnetodynamic system (3), two kinds of equations must be considered. The first kind are the equations corresponding to nodes belonging to at least one conducting region. The time derivative of the corresponding nodal value of the vector potential gives a nonzero term on the diagonal of **A**. The second kind are the equations corresponding to nodes belonging only to nonconducting regions.

Those equations do not have any differential term, they are purely algebraic and their coefficients in the matrix **A** are zero. Classical methods such as scheme (5) may be used. Nevertheless, if the explicit Euler method is tried, the singular matrix $\mathbf{A}/\Delta t$ is involved and the solution is impossible. In general, purely explicit methods may not be applied to solve (3) because they involve singular matrices and implicit methods must be used. We have tested several implicit schemes[3]. Backward difference methods such as Adams Moulton or Gear methods did not give satisfactory results because none of them was both stable and accurate enough. Extrapolation methods are very robust but are also extremely time consuming. Another possibility is to use the implicit Runge-Kutta methods which are presented in the next section.

2. Runge-Kutta methods

The general differential system $\dot{y} = f(y, t)$ is considered. An approximation of the solution y_{n+1} at time t_{n+1} from the values y_n of the solution at the previous time step may be obtained by solving the following relations[4]:

$$\begin{cases} k_i = \Delta t \ \ f(y_n + \sum_{j=1}^{s} \beta_{ij} \ \ k_j, \ t_n + \alpha_i \Delta t) \qquad i = 1 \ldots s \\ y_{n+1} = y_n + \sum_{i=1}^{s} \mu_i \ \ k_i \end{cases} \tag{6}$$

A s-step method is characterized by the coefficients α_i, β_{ij} and μ_i with $\alpha_i = \sum_{j=1}^{s} \beta_{ij}$.

The Runge-Kutta methods may be sorted according to their coefficient structure. The classical explicit methods are the methods for which $\beta_{ij} = 0$ for $j \geq i$. If this condition is not fulfilled (if $\beta_{ij} \neq 0$ for $j \geq i$), the methods are called the implicit Runge-Kutta methods (IRK). Special cases of IRK are the diagonally implicit Runge-Kutta (DIRK) methods (if $\beta_{ij} = 0$ for $j > i$), a subcategory of which are the singly diagonally implicit Runge-Kutta (SDIRK) methods (if $\beta_{ij} = 0$ for $j > i$ and if $\beta_{ii} = \beta$ for all i).

The IRK methods may be applied to a DAE system. If the following system is considered: $\mathbf{A}\,\dot{y} + \mathbf{B}(y)y = b(t)$ where **A** may be a singular matrix and which is nonlinear because **B** depends on y, the following scheme may be applied:

$$\begin{cases} \mathbf{A}k_i = \Delta t \left[b(t_n + \alpha_i \Delta t) - \mathbf{B}(y_n + \sum_{j=1}^{s} \beta_{ij} \ \ k_j) \ (y_n + \sum_{j=1}^{s} \beta_{ij} \ \ k_j) \right] \qquad \text{for} \ \ i = 1 \ldots s \\ y_{n+1} = y_n + \sum_{i=1}^{s} \mu_i \ \ k_i \end{cases} \tag{7}$$

The algorithm of Figure 1 may be used to implement the DIRK method to solve the previous DAE system and to keep the form of the nonlinear systems to be numerically solved as close as possible to the ones involved in the other

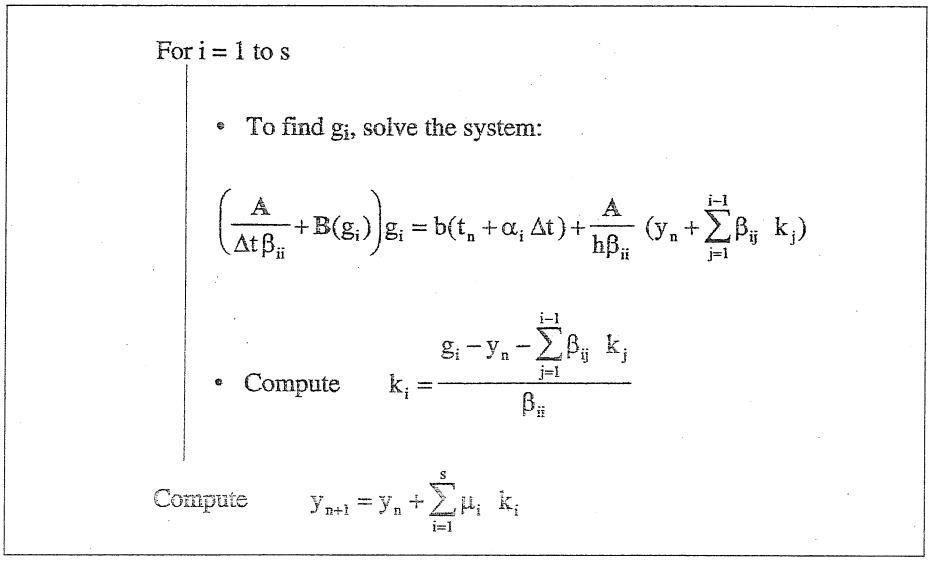

Figure 1.
Algorithm for DIRK method

methods[5]. It is very important to note that the algebraic systems to be solved are very similar to the classical ones: they only differ by the right-hand member and because the time step involved in the left-hand member is multiplied by β_{ii}. There is no reason to lose, for instance, the symmetry or the sparsity obtained with other schemes. The use of this algorithm makes the introduction of DIRK in existing computer code straightforward. The advantage of DIRK is that the global system (7) for IRK reduces to a sequence of s systems as indicated in the algorithm.

Here are some examples of DIRK:

- *one-step method*[6]

  $\alpha_1 = \beta_{11} = \alpha, \mu_1 = 1$

  In the case of linear problems, this method reduces to scheme (5) except for the independent term b(t) which is discretized as $b(t_n+a\Delta t)$ instead of $(1-\alpha)b(t_n)+\alpha\, b(t_{n+1})$. Therefore in the particular cases $\alpha = 0$ and $\alpha = 1$, it reduces to forward and backward Euler respectively. In the case $a = 0.5$, it is called the implicit midpoint method[7] which is only distinct from the CN method by the discretization of the independent term $b(t)$ and which have similar stability and order of accuracy properties.

- *two-step method*[6]

  $\alpha_1 = \beta_{11} = \alpha, \mu_1 = 1/2$

  $\alpha_2 = 1-\alpha, \beta_{21} = 1-2\alpha, \beta_{22} = \alpha, \mu_2 = 1/2$

  This 2-step scheme leads to more interesting methods than the previous 1-step one. This scheme is A-stable for $\alpha \geq 1/4$[8] and it is L-stable and of the order 2 for $\alpha = (2-\sqrt{2}) / 2$[6]. It is of the order 3 for $\alpha = (3 + \sqrt{3})/6$. L-

stability includes the A-stability and moreover guarantees that phenomena with small time constants will be rapidly damped, even when large time steps are used. Thanks to this property, this method is well suited for stiff problems. Another interesting feature of this method is that its first step corresponds to that of the 1-step scheme. It is therefore possible to combine both methods in an adaptive scheme. Once the intermediate quantities k_1 and k_2 are computed, two approximations $\bar{y}^{(1)}$ and $\bar{y}^{(2)}$ are produced and an estimation of the error $\bar{e}$ is computed:

$$\begin{cases} \bar{y}^{(1)}_{n+1} = \bar{y}^{(2)}_{n} + k_1 \\ \bar{y}^{(2)}_{n+1} = \bar{y}^{(2)}_{n} + (k_1 + k_2)/2 \\ \bar{e}_{n+1} = \bar{y}^{(2)}_{n+1} - \bar{y}^{(1)}_{n+1} = (k_2 - k_1)/2 \end{cases} \tag{8}$$

The error estimator can therefore be used to control the time step.

3. Example

As an example, the case of a nonlinear inductance is considered. It is connected to a capacitor to form a RLC circuit (Figure 2). The free damped oscillations of the discharge of the capacitor are studied[5,9]. The space discretization is made by linear triangular finite elements and the time integration is performed with the second order 2-step DIRK method ($\alpha = (2-\sqrt{2})/2$) with the adaptive feature. The non linear character allows a better testing of adaptivity as the time constant of the system evolves with the kernel saturation. The DIRK methods have proved to be the only effective ones when hysteresis is considered[3].

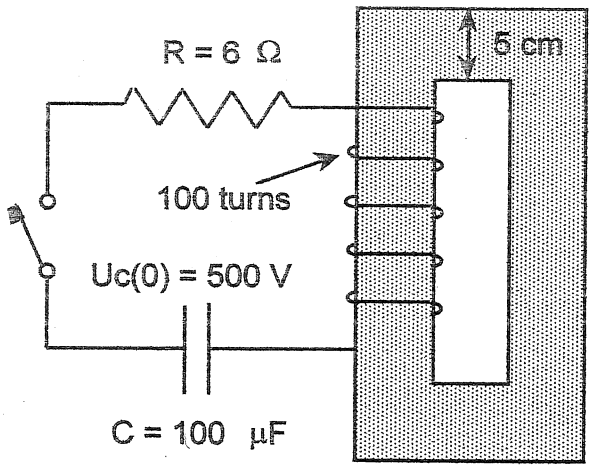

Figure 2.
Test problem (discharge of a capacitance in an inductance)

Figure 3 shows the numerical results: the time evolutions of various energies (i.e. magnetostatic and electrostatic energies) and cumulated Joule losses are given. Their sum, the total energy, must remain constant, as is the case, up to small numerical variations (less than 2 per cent). The fluctuations seem to be related to the saturation of the kernel but the reason is not obvious. The time steps are marked on the total energy curve to show the adaptivity of the scheme. Small time steps are used when the magnetostatic energy is high (i.e. when the magnetic core is saturated) while large time steps are used at lower magnetostatic energy levels (i.e. when the material is linear). The adaption uses a norm of the error vector and the sensitivity of the method to local error depends on the choice of this norm.

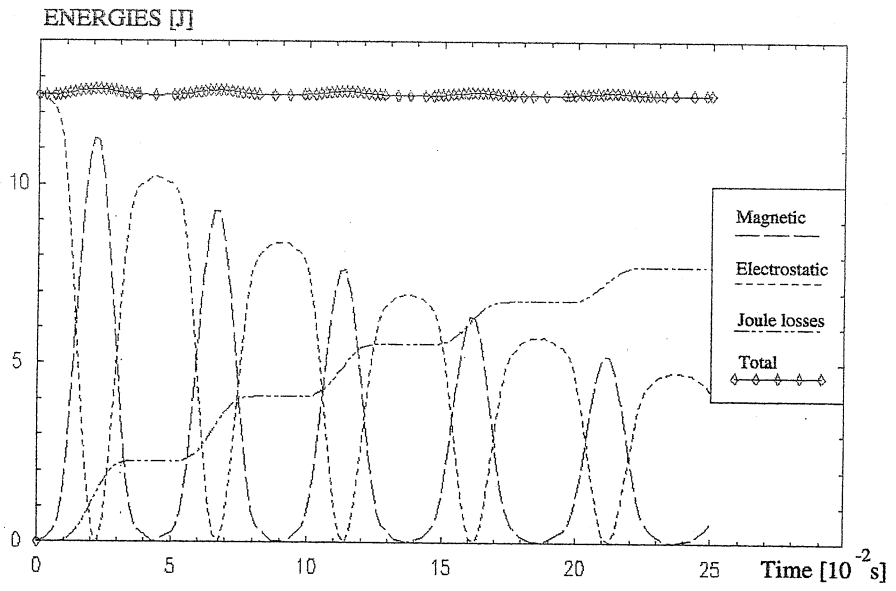

Figure 3.
Time evolution of energies and losses during free damped oscillations of a RLC non linear circuit using a DIRK method

4. Conclusion

Contrary to generally accepted ideas, implicit Runge-Kutta methods do not necessitate an important modification of computation codes. In particular, it is possible, using the proposed algorithm, to preserve the symmetry of the algebraic systems arising in the solution. They are not too expensive from a CPU time point of view and therefore they provide a set of methods with interesting properties such as high accuracy, stability, and error control. It can be claimed that adaptive DIRK methods achieve a good performance-cost ratio.

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