Implicit Runge-Kutta Methods for Transient Magnetic Field Computation

A. Nicolet
Ecole Supérieure d'Ingénieurs de Marseille - Department of Electrical Engineering
IMT-Technopôle de Château-Gombert- F-13451 Marseille Cedex 20 (France)

F. Delincé
University of Liège - Department of Electrical Engineering
Institut Montéfiore - Sart Tilman Bât. B28 - B-4000 Liège (Belgium)

Abstract — **This paper presents the implicit Runge-Kutta methods as an interesting alternative to Crank-Nicolson and backward Euler methods to solve differential algebraic systems arising in nonlinear transient magnetodynamic problems.**

I. Introduction

The most widespread method used to solve transient magnetodynamic problems is the Crank-Nicolson (CN) method. For linear differential equations, the CN method is unconditionally stable. The behaviour of pure magnetodynamic systems is a simple damping even for nonlinear problems and the CN method is often stable and accurate enough. Nevertheless when the nonlinear electromagnetic model is coupled to another model e.g. a mechanical model for moving parts or circuit equations for the feeding of conductors, it may exhibit a much more complicated behaviour. In this case, the CN method may become unstable. Another possibility is the more stable backward Euler method but it is less accurate and tends to overdamp waveforms. An important point is that the finite element discretisation of magnetodynamic problems leads to systems of differential algebraic equations (DAE) that can not be treated numerically like regular ordinary differential equations [1]. They are index 1 systems, i.e. with the coefficient matrix of the differential terms singular, and can only be treated by implicit methods. Candidate methods are the implicit Runge-Kutta methods.

II. Differential Algebraic Equations

The equation for the two-dimensional magnetostatics is [2]:

$$\mathrm{div}(\nu \,\mathrm{grad}\, A) = -J \tag{1}$$

where ν is the magnetic reluctivity, and A and J are respectively the vector potential and the current density. These vectors have only the z-component different from zero. Using Ohm's law, the case of eddy currents can be dealt with by introducing (2) as the expression for the current density in (1):

$$J = \sigma E = -\left(\frac{\partial A}{\partial t} + U\right) \tag{2}$$

The conductor is characterised by its electrical conductivity σ, and U can be interpreted as the terminal voltage of the conductor (per unit of length). The finite element formulation is based on the semi discrete Galerkin method. Weighted residuals (3) for the domain Ω of boundary Γ are constructed, where w is a weighting function (depending on space variables).

$$\iint_{\Omega}\left[\nu \,\mathrm{grad}A.\,\mathrm{grad}w + \sigma\left(\frac{\partial A}{\partial t} + U\right)w\right]d\Omega - \oint_{\Gamma} w\nu\frac{\partial A}{\partial n}d\Gamma = 0 \tag{3}$$

The space discretisation of (3) using the functions w leads to a differential system of equations. For a classical choice of the weight functions, the unknowns of the problem are the nodal value of the vector potential. In a nonconducting region, the conductivity is equal to zero and the concerned equations reduce to algebraic ones.

In the rest of the paper, the reference to electromagnetic quantities is given up and the general system of differential/algebraic equations (DAE) (4) is considered:

$$\mathbf{A}\ \dot{y}(t) + \mathbf{B}\ y(t) = b(t) \tag{4}$$

where **A** and **B** are coefficient matrices, b is the right hand member vector, y is the unknown vector, t is the time and the dot indicates the time derivative. Note that **A** and **B** may depend on y in the case of a nonlinear system e.g. **B** depends on y in the case of ferromagnetic saturation and one has **B**(y(t)). The numerical solution of system (4) requires a time discretisation. A simple scheme is given by (5):

$$\mathbf{A}\ \frac{y_{n+1} - y_n}{\Delta t} + \mathbf{B}\ (\alpha\, y_{n+1} + (1-\alpha)y_n) = \alpha\, b_{n+1} + (1-\alpha)b_n \tag{5}$$

Indices n and n+1 refer to quantities at time t_n and $t_{n+1} = t_n + \Delta t$ respectively. Various choices of the parameter α lead to classical methods (α=1. is implicit (backward) Euler, α=0. is explicit (forward) Euler, α=1/2 is Crank-Nicolson, α=2/3 is Galerkin).

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A. Nicolet E-mail: nicolet@esim.imt-mrs.fr, Fax +33-91054343, Phone +33-91054456
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The matrix $\mathbf{A}/\Delta t + \alpha\mathbf{B}$ is involved in the solution of system (5). One obvious condition is that it must be invertible i.e. nonsingular. The study of DAE systems such as (5) involves the matrix pencil $\mathbf{A} + \lambda\,\mathbf{B}$ of matrices **A** and **B** where λ is an arbitrary parameter [1],[3]. It must be regular i.e. $\det(\mathbf{A} + \lambda\,\mathbf{B})$ must not vanish identically. If it is not the case, it is impossible to find a Δt and an α so that $\mathbf{A}/\Delta t + \alpha\mathbf{B}$ is not singular and the system is meaningless. On the other hand, if $\mathbf{A} + \lambda\,\mathbf{B}$ is a regular pencil, matrices **E**, **F** exist so that:

$$\mathbf{A}^* = \mathbf{E}\,\mathbf{A}\,\mathbf{F} = \mathrm{diag}(\mathbf{I},\mathbf{J}) \tag{6}$$

$$\mathbf{B}^* = \mathbf{E}\,\mathbf{B}\,\mathbf{F} = \mathrm{diag}(\mathbf{W},\mathbf{I}) \tag{7}$$

where diag denotes a square matrix constructed with the argument square matrices placed on its diagonal, **I** are unit matrices of the suitable dimension, **W** is a regular square matrix, and **J** is a nilpotent Jordan block matrix with blocks of the form:

$$\begin{bmatrix} 0 & & & 0 \\ 1 & 0 & & \\ & \cdots & \cdots & \\ 0 & & 1 & 0 \end{bmatrix} \tag{8}$$

The size of the largest of these blocks is m, the nilpotency or index of the system (The nilpotency of a matrix **J** may alternatively be defined as the integer m so that $\mathbf{J}^m = 0$ and $\mathbf{J}^{m-1} \neq 0$). With $y^* = \mathbf{F}^{-1}y$ and $b^* = \mathbf{E}\,b$, the DAE system (4) is in its Kronecker normal form (KNF)[1]:

$$\begin{cases} \dot{u} + \mathbf{W}\,u = q \\ \mathbf{J}\,\dot{v} + v = r \end{cases} \quad \text{with} \quad y^* = \begin{pmatrix} u \\ v \end{pmatrix} \quad b^* = \begin{pmatrix} q \\ r \end{pmatrix} \tag{9}$$

The numerical behaviour of a DAE system depends on its nilpotency. On the one hand, if the nilpotency is greater than one, the system is very difficult to solve and special techniques must be used. On the other hand, if the nilpotency is equal to one (**J** is identically equal to zero), the system is much easier to solve and some classical methods for ordinary differential equations may work.

In the case of the magnetodynamic system (4), two kinds of equations must be considered. The first kind are the equations corresponding to nodes belonging to at least one conducting region. The time derivative of the corresponding nodal value of the vector potential gives a nonzero term on the diagonal of **A**. Such equations do not influence the nilpotency of the system. The second kind are the equations corresponding to nodes belonging only to nonconducting regions. Those equations do not have any differential term, they are purely algebraic and their contributions to the matrix **A** are null lines. They obviously lead to a system of nilpotency 1. Therefore, classical methods such as scheme (5) may be used. Nevertheless, if the explicit Euler method is tried, the singular matrix $\mathbf{A}/\Delta t$ is involved and the solution is impossible. In general, purely explicit methods may not be applied to solve (4) because they involve singular matrices and implicit methods must be used. One possibility is to use the implicit Runge-Kutta methods which are presented in the next section.

III. Runge-Kutta Methods

The general differential system $\dot{y} = f(y,t)$ is considered. An approximation of the solution y_{n+1} at time t_{n+1} from the values y_n of the solution at the previous time step may be obtained by solving the following relations [4]:

$$\begin{cases} k_i = \Delta t\ f\left(y_n + \sum_{j=1}^{s} \beta_{ij}\ k_j,\ t_n + \alpha_i \Delta t\right) \qquad i = 1 \ldots s \\ y_{n+1} = y_n + \sum_{i=1}^{s} \mu_i\ k_i \end{cases} \tag{10}$$

A s-step method is characterised by the coefficients α_i, β_{ij} and μ_i given in its Runge-Kutta tableau or Butcher diagram:

α_1	β_{11}	β_{12}	...	β_{1s}
α_2	β_{21}	β_{22}	...	β_{2s}
...	...	...		...
α_s	β_{s1}	β_{s2}	...	β_{ss}
	μ_1	μ_2	...	μ_s

with $\alpha_i = \sum_{j=1}^{s} \beta_{ij}$.

The Runge-Kutta methods may be sorted according to the structure of their coefficient diagram. The classical explicit methods are the methods for which $\beta_{ij} = 0$ for $j \geq i$. If this condition is not fulfilled (if $\beta_{ij} \neq 0$ for $j \geq i$), the methods are called the implicit Runge-Kutta methods (IRK). Special cases of IRK are the diagonally implicit Runge-Kutta (DIRK) methods (if $\beta_{ij} = 0$ for $j > i$), a subcategory of which are the singly diagonally implicit Runge-Kutta (SDIRK) methods (if $\beta_{ij} = 0$ for $j > i$ and if $\beta_{ii} = \beta$ for all i).

The IRK methods may be applied to a DAE system . If the following system is considered: $\mathbf{A}\,\dot{y} + \mathbf{B}(y)\,y = b(t)$ where **A** may be a singular matrix and which is nonlinear because **B** depends on y, the following scheme may be applied:

$$\begin{cases} \mathbf{A}k_i = \\ \Delta t \left[b(t_n + \alpha_i \Delta t) - \mathbf{B}\left(y_n + \sum_{j=1}^{s} \beta_{ij}\ k_j\right)\left(y_n + \sum_{j=1}^{s} \beta_{ij}\ k_j\right) \right] \\ \qquad\qquad \text{for } i = 1 \ldots s \\ y_{n+1} = y_n + \sum_{i=1}^{s} \mu_i\ k_i \end{cases} \tag{11}$$

The algorithm of table I may be used to implement the DIRK method to solve the previous DAE system and to keep the form of the nonlinear systems to be numerically solved as close as possible to the ones involved in the other methods [9]. It is very important to note that the algebraic systems to be solved are very similar to the classical ones: they only differ by the right-hand member and because the time step involved in the left-hand member is multiplied by β_{ii}. There is no reason to lose, for instance, the symmetry or the sparsity obtained with other schemes. The use of this algorithm makes the introduction of DIRK in existing computer code straightforward.

TABLE I
ALGORITHM FOR DIRK METHODS

For i = 1 to s

- To find K_i, solve the system:

$$\left(\frac{\mathbf{A}}{\Delta t \beta_{ii}} + \mathbf{B}(K_i)\right) K_i = b(t_n + \alpha_i \Delta t) + \frac{\mathbf{A}}{h\beta_{ii}} \left(y_n + \sum_{j=1}^{i-1} \beta_{ij}\ k_j\right)$$

- Compute $$k_i = \frac{K_i - y_n - \sum_{j=1}^{i-1} \beta_{ij}\ k_j}{\beta_{ii}}$$

Compute $$y_{n+1} = y_n + \sum_{i=1}^{s} \mu_i\ k_i$$

Here are some examples of DIRK:

• one step method [7]

α	α
	1

In the case of linear problems, this method reduces to scheme (5) except for the independent term b(t) which is discretised as $b(t_n+\alpha\Delta t)$ instead of $(1-\alpha)b(t_n)+\alpha\, b(t_{n+1})$. Therefore in the particular cases $\alpha=0$ and $\alpha=1$, it reduces to forward and backward Euler respectively. In the case $\alpha=0.5$, it is called the implicit midpoint method [5] which is only distinct from the CN method by the discretisation of the independent term b(t) and which have similar stability and order of accuracy properties.

• two step method [7]

α	α	0
$1-\alpha$	$1-2\alpha$	α
	1/2	1/2

This 2-step scheme leads to more interesting methods than the previous 1-step one. This scheme is A-stable for $\alpha \geq 1/4$ [6] and it is L-stable and of the order 2 for $\alpha = (2-\sqrt{2})/2$ [7]. It is of the order 3 for $\alpha = (3+\sqrt{3})/6$. L-stability includes the A-stability and moreover guarantees that phenomena with small time constants will be rapidly damped, even when large time steps are used. Thanks to this property, this method is well suited for stiff problems. Another interesting feature of this method is that the first line of its tableau corresponds to that of the 1-step scheme. It is therefore possible to combine both methods in an adaptive scheme. Once the intermediate quantities k1 and k2 are computed, two approximations $\bar{y}^{(1)}$ and $\bar{y}^{(2)}$ are produced and an estimation of the error $\bar{e}$ is computed:

$$\begin{cases} \bar{y}^{(1)}_{n+1} = \bar{y}^{(2)}_n + k_1 \\ \bar{y}^{(2)}_{n+1} = \bar{y}^{(2)}_n + (k_1 + k_2)/2 \\ \bar{e}_{n+1} = \bar{y}^{(2)}_{n+1} - \bar{y}^{(1)}_{n+1} = (k_2 - k_1)/2 \end{cases} \tag{12}$$

The error estimator can therefore be used to control the time step.

IV. EXAMPLE

As an example, the case of a nonlinear inductance is considered. It is connected to a capacitor to form a RLC circuit (Fig. 1). The free damped oscillations of the discharge of the capacitor are studied [8],[9]. The space discretisation is made by linear triangular finite elements and the time integration is performed with the second order 2-step DIRK method ($\alpha = (2-\sqrt{2})/2$) with the adaptive feature.

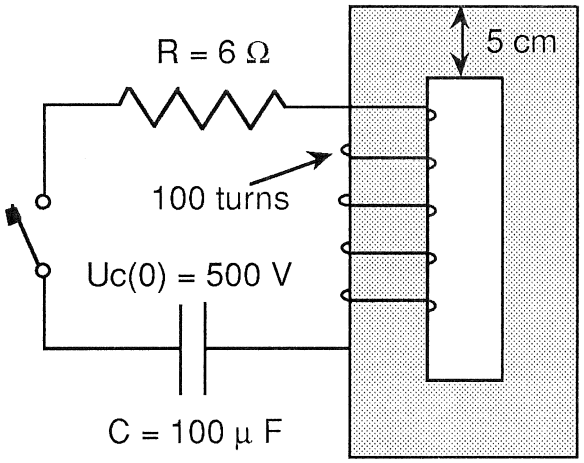

Fig. 1. Test problem (discharge of a capacitance in an inductance).

Fig. 2 shows the numerical results: The time evolutions of various energies (i.e. magnetostatic and electrostatic energies) and cumulated Joule losses are given. Their sum, the total energy, must remain constant, as is the case, up to small numerical variations. The time steps are marked on the total energy curve to show the adaptivity of the scheme.

Small time steps are used when the magnetostatic energy is high (i.e. when the magnetic core is saturated) while large time steps are used at lower magnetostatic energy levels (i.e. when the material is linear).

It is difficult to give an absolute comparison of respective performances of the CN method and of DIRK methods. In the case of the 1-step method with α=0.5 and a fixed time step, the computation work, the stability and the accuracy are almost the same as the CN method. In the case of the second order 2-step DIRK method, with a fixed time step, this method is of the same order but stabler (L-stable) than the CN method for twice as much work. Therefore the comparison of both methods depends greatly on the problem and on the chosen time step. On the other hand, an adaptive scheme brings some new qualitative advantages because it is almost independent of the initial choice of the time step and guarantees a required accuracy or at least detects numerical problems. It is difficult to define the computation cost of such methods. For simple problems, the time step will be increased and the method will be quite economical but anyway less than a simpler method with a smart choice of the time step. For more difficult problems such as stiff problems, the time step will be reduced and the computation cost will be high. In real problems, the adaptivity leads to a constant variation in the time step. The computation cost associated with such methods is directly related to their robustness and reliability. Therefore, adaptive methods are incommensurable with constant time step methods. It can be claimed that adaptive DIRK methods achieve a good performance-cost ratio.

V. Conclusion

Contrary to generally accepted ideas, implicit Runge-Kutta methods do not necessitate an important modification of computation codes and, specially, it is possible, using the proposed algorithm, to preserve the symmetry of the algebraic systems arising in the solution. They are not too expensive from a CPU time point of view and therefore they provide a set of methods with interesting properties such as high accuracy, stability, and error control.

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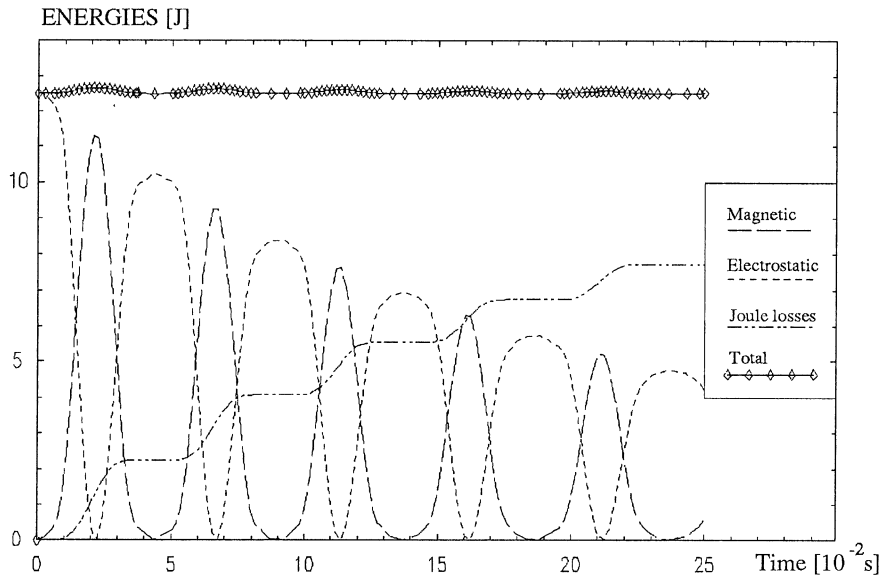

Fig. 2. Time evolution of energies and losses during free damped ocillations of a RLC noninear circuit using a IRK method.