ANALOGY BETWEEN THE FINITE ELEMENT METHOD AND THE CIRCUIT EQUATIONS

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***Abstract*—Classical electromagnetism, which is a continuous media theory, leads to two kinds of discrete models: on the one hand circuit theory with lumped elements and on the other hand discrete numerical schemes such as the finite element method. The purpose of this paper is to emphasize the common structure of the three models.**

I. Introduction

The finite element method is nowadays a classical method to discretize the partial derivative equations of electromagnetism. Nevertheless, if this approach became common with the use of computers for large numerical computations, another discretization of electromagnetic systems was well known even before the advent of Maxwell's theory: electrical circuits with lumped elements. Though from apparently unrelated origin, both techniques appear amazingly close when presented from the right point of view. In this paper we draw a parallel between: Classical electrodynamic model for continuous media based on differential forms, general matrix (e.g. node or mesh) formulations for circuit equations, and Whitney finite elements, the most appropriate context to present edge element techniques. We will therefore consider the following paradigmatic examples: For continuous electromagnetism, the *full Maxwell system* and a *simple electrokinetic model*, for electrical circuits, a *resistance network*, and for the numerical discrete case, *finite element method* applied to the electromagnetic model presented above using the Whitney forms [1].

II. Topology

The basic geometric framework of each model is a topological structure i.e. a set of points with no reference to either distance or angle.

- The topological structure required for electromagnetism is the classical $\mathbb{R}^3$ *topology* given by open sets. In fact, a little bit more is necessary and the differentiable structure of $\mathbb{R}^3$ with the suitable regularity is usually considered (differentiable manifold structure). Special subsets are useful such as surfaces (2-dimensional submanifolds), curves (1-dimensional submanifolds) and isolated points (0-dimensional submanifolds). Roughly speaking, a p-dimensional submanifold is a map from a domain Ω of $\mathbb{R}^p$ (parameter space) to $\mathbb{R}^3$.
- A circuit model is basically a *graph* [2]. A graph G is a set N of elements called vertices or *nodes* together with a set B of unordered pairs of nodes called edges or *branches.* The following concept is also useful: a *mesh* is a set of branches forming a loop. View nodes, branches and meshes respectively as 0-, 1- and 2-dimensional geometrical objects of the model.
- Finite element models require a *meshing* of the geometrical domain. Simplicial meshes are considered here. The domain is assumed to be tiled by a set T of 3-simplices (tetrahedra) called the *elements.* F is the set of 2-simplices (triangles) that are the *faces* of the tetrahedra in T, E is the set of 1-simplices (segments) that are the *edges* of the triangles in F and N is the set of 0-simplices (points) called the *nodes* or vertices that are the extremities of the edges in E.

It may be useful to consider *oriented* elements in order to determine the sign of physical quantities associated to geometrical elements. For instance, graphs must therefore be directed what is done by ordering pairs of nodes associated to the edges. It is also convenient to number the elements in discrete sets with successive integers starting at 0 or 1.

III. Physical quantities

At this stage, the physical quantities may already be defined by duality with the topological elements. A *physical quantity* is a map from a given class of (oriented) geometrical objects to (real or complex) numbers.

- In the continuous theory, the language of *differential forms* is adopted here. 0-forms such as the electric scalar potential V are maps from points to numbers i.e. the usual definition of a function on $\mathbb{R}^3$. 1-forms such as the electric field E, the magnetic field H, the vector potential A are maps from curves to numbers. 2-forms such as the electric displacement D, the magnetic flux density B, the current density J are maps from surfaces to numbers. 3-forms such as the charge density ρ are maps from volumes (three-dimensional subsets) to numbers. To emphasize the duality, we write (M, α) the number associated to the p-submanifold M (p-dimensional geometric object) and the p-form α. For instance, (S, B) is an abstract notation for the surface integral of flux density B across the surface S. This taxonomy is obviously more accurate than the one of vector analysis that merges 0-forms and 3-forms in scalar fields and 1-forms and 2-forms in vector fields.

Forms appear here as set functions and are in a way connected to the concept of Lebesgue measure. A p-form may indeed be considered as the integrand on a p-dimensional submanifold.If such definitions may seem abstract at first sight, they are in fact deeply related to real life experimental processes [3]. A magnetic flux density is never directly accessible but only known through measurements on magnetic fluxes across finite size loops in various positions.

- On graphs, one has nodal quantities such as node potential V mapping nodes to numbers and branch quantities such as branch currents I and voltages U mapping branches to numbers. It will also be useful to define mesh currents I_m that map meshes to numbers. Note that branch quantities are of course related to the continuous physical quantities. Physically a branch is made of a device such as a piece of conducting material. The current is the flux integral of the current density on a cross section of the device and the voltage is the line integral along the mean fiber. The hypothesis that such branch quantities are sufficient to represent accurately enough the physical behaviour of the device on the branch is the basis of the *lumped element model.*
- For the numerical discrete model, *discrete p-field* correspond to maps from p-simplices to numbers. For instance the electric field is represented by a set of value associated to edges. Of course, a practical mesh is placed in continuous physical space such that the tetrahedra are embeddings of a reference tetrahedron. One therefore may define a *projection* from a continuous p-form to a discrete p-field by taking the integral on the corresponding p-simplices. For instance, the continuous electric field is projected on the mesh by taking its line integrals along all the edges.

IV. Topological operators

To manipulates quantities one has to define operators that map then to each others. A first class of operators rely only on the topological structure.

- In the continuous model, one defines the *exterior derivative* d of a p-form. Consider the *boundary* of a p-dimensional submanifold. The boundary $\partial\Omega$ of a set Ω of the parameter space $\mathbb{R}^p$ is the set of points x such that any open set containing the point x contains both points that are in Ω and points that are not in Ω. If a p-submanifold M is the image of Ω in $\mathbb{R}^n$, its boundary ∂M is the image of $\partial\Omega$ in $\mathbb{R}^n$. The boundary operator ∂ is a map from p-dimensional sets of points to (p-1)-dimensional sets of points. If ∂M is regular enough, it is a map from (p-1)-forms α to numbers $(\partial M, \alpha)$. We now take the general Stokes theorem [4] as a definition of the exterior derivative i.e. d is a linear map from p-forms α to (p+1)-forms $d\alpha$ such that for any p-dimensional submanifold M one has $(M, d\alpha) = (\partial M, \alpha)$. It may be shown that this operator corresponds to *grad*, *curl* and *div* of vector analysis when acting on 0-, 1- and 2-forms respectively. The fact that the boundary of a boundary is an empty set ($\partial\partial M = \emptyset$ for any M) leads immediately to the fact that $dd\alpha = 0$ for any p-form α.
- The corresponding operators on graphs are *incidence matrices*. Consider V the column array (of size $\#V$) of nodal value of the potential and U the column array (of size $\#U$) of voltages associated to branches. Build now the following $\#U \times \#V$ rectangular array such that every line corresponds to a branch and every column to a node. It is now possible to encode the connectivity of the oriented graph (i.e. what nodes are boundaries of what branches) in this array the following way: consider an element of the array corresponding to a given node and a given branch, if the node is the end point of the branch set the element value to +1 or if it is the starting point set the value to -1 else if the branch is not connected to the node set the value to 0. Considered as a matrix A, this array is a linear operator from nodal quantities to branch quantities e.g. the fact that voltages across a branch are differences of the potentials on the nodes at the extremities of the branches is expressed by the matrix product $U = AV$. The matrix C is associated to the meshes: lines correspond to meshes and columns to branches, elements of the matrix are different from zero if the branch is on the mesh, +1 if the orientations of the branch and the mesh match and -1 if they are opposite. The fundamental property of the incidence matrices is that the matrix product $CA = 0$.
- The *discrete operators* on the meshing are also defined as incidence matrices that are representations of the boundary operator. Here the boundary operator associates to a p-simplex the (p-1)-simplices that constitute its boundary e.g. it associates to a tetrahedron the four triangles that are its faces. This can be easily encoded in incidence matrices. G is the node-edge incidence matrix, the lines correspond to edges and the column to nodes, whose elements are +1, -1 or 0 in a way exactly similar to the graph case. R is the edge-face incidence matrix, the lines correspond to faces and the column to edges, whose elements are +1, -1 or 0 indicating that the given column corresponds to an edge that is on the boundary of the face corresponding to the given line with a similar or opposite orientation. D is the face-tetrahedron incidence matrix, the lines correspond to tetrahedra and the column to faces, whose elements are +1, -1 or 0 indicating that the given column corresponds to a face that is on the boundary of the tetrahedron corresponding to the given line with a similar or opposite orientation. The fact that the boundary of a boundary is an empty set is now encoded in the matrix products $RG = 0$ and $DR = 0$.

The topological operators now available already allow a set up of fundamental equations.

- Using exterior derivative, *Maxwell equations* are: $dH = J + \partial_t D$, $dE = -\partial_t B$, $dD = \rho$, and $dB = 0$ where ∂_t is the partial derivative with respect to time. The electrokinetic model is obtained from the full system by assuming some simplifying hypotheses: time variations are neglected and one focuses on E and J what gives $dJ = 0$, $dE = 0$.
- In circuits equations, *Kirchhoff laws* are a direct consequence of the Maxwell equations assuming simplifying hypotheses. Mainly, dynamical terms are neglected outside lumped elements (roughly Faraday induction law is confined inside inductances and displacement currents are only present inside capacitors) and the current density is supposed perfectly confined inside lumped elements and ideal connections (i.e. without voltage drop). Conser-

vation of current $dJ = 0$ implies directly the Kirchhoff current law (KCL) for nodes given in matrix form by $A^T I = 0$. and $dE = 0$ implies directly the Kirchhoff voltage law (KVL) for meshes given in matrix form by $C^T U = 0$.

- One of the prominent feature of the Whitney element approach is that exact formulation of discrete Maxwell equations for discrete fields (denoted here by the boldface lowercase corresponding to the continuous field) is obtained thanks to incidence matrices, namely: $R\mathbf{h} = \mathbf{j} + \partial_t \mathbf{d}$, $R\mathbf{e} = -\partial_t \mathbf{b}$, $D\mathbf{d} = \rho$, and $D\mathbf{b} = 0$.

V. Metric

In the previous sections, only topological properties have been involved. This section is now devoted to the introduction of metric concepts.

- The *Hodge star operator* $*$ is a linear map from p-forms to (3-p)-forms. It is necessary to express the usual material *constitutive laws* that relate 1-forms such as electric and magnetic field to 2-forms such as current density, electric displacement and magnetic flux density. For instance $D = \varepsilon * E$, $B = \mu * H$, and $J = \sigma * E$ where ε is the dielectric permittivity, μ the magnetic permeability , and σ the electric conductivity. In vector analysis, metric and topological aspects are interlaced and usually completely hidden in the use of Cartesian coordinates.
- In circuit theory, metric properties are encoded in the *impedance matrix* Z, a square matrix that relates branch voltages to branch currents. Consider for instance a resistance network. Z is therefore a diagonal matrix containing the resistance of each branch. On the one hand, if the shape of a particular resistive element is modified, its section or length, the resistance value is modified indicating its metric nature. On the other hand, changing the shape of an ideal connection without changing the topology of the circuit does not modify the circuit equations.
- The introduction of *Whitney elements* [1] allows the interpolation of discrete p-fields to continuous p-forms and is used in setting up the *finite element method.* To a p-simplex s of the mesh defined by $p+1$ vertex points of indices $i_0, i_1, \ldots, i_p$, is associated a *shape p-form*: $\mathbf{w}^s = \sum_\sigma (-1)^\sigma \lambda_{\sigma(i_0)} (d\lambda_{\sigma(i_1)} \wedge \ldots \wedge d\lambda_{\sigma(i_p)})$ where σ is a permutation of the indices, $(-1)^\sigma$ its sign, $\wedge$ the exterior product [4] and λ_i the barycentric coordinate associated to node i.

Given a discrete p-field, the interpolated discrete p-form or Whitney form is the linear combination of shape p-forms with the number associated to the corresponding p-simplex as coefficient. Beyond the trivial case of 0-forms that corresponds to classical nodal elements, 1-forms are represented with the help of the so-called *edge elements.* For instance $E = \sum_{edges\ e_{ij}} e^{e_{ij}} (\lambda_i grad\lambda_j - \lambda_j grad\lambda_i)$ where $e^{e_{ij}}$ is the line integral of E along the edge e_{ij} from node i to node j. The interesting properties of such interpolation for finite element modelling are nowadays well known [1]. Another example are face elements where to each triangle f_{ijk} is associated to the vector shape function $\mathbf{w}^{f_{ijk}} = 2(\lambda_i grad\lambda_j \times grad\lambda_k + \lambda_j grad\lambda_k \times grad\lambda_i + \lambda_k grad\lambda_i \times grad\lambda_j)$. The main role of Whitney interpolation shape functions is the construction of a *discrete Hodge operator.* This involves the Euclidean scalar (i.e. dot) product of $\mathbb{R}^3$ used in the construction of a matrix that corresponds to the impedance matrix of circuit models (but is usually only sparse and not diagonal). Let α denote a scalar field representing a material property such as ε, μ, or $\sigma \ldots$ For two p-simplices s and s', the matrix $M_p(\alpha)$ is defined such that its elements are $m_p(\alpha)_{s,s'} = (\Omega, \alpha \mathbf{w}^s \wedge * \mathbf{w}^{s'})$ where Ω is the support of the meshing in $\mathbb{R}^3$. For instance, using vector analysis notations, coefficients of $M_2(\sigma^{-1})$ are given by $m_2(\sigma^{-1})_{s,s'} = \int_\Omega \sigma^{-1} \mathbf{w}^s . \mathbf{w}^{s'} d\Omega$ [5].

VI. Potentials and gauge

Introducing potentials allows to verify automatically homogeneous equations.

- Homogeneous Maxwell equations $dE + \partial_t B = 0$ and $dB = 0$ are verified if one introduces *potentials* as auxiliary quantities, namely a 'magnetic' vector potential A (1-form) and an 'electric' scalar potential V (0-form) such that $B = dA$ and $E = -\partial_t A - dV$. In the case of the electrokinetic model, the conservation of current $dJ = 0$ may be insured by the introduction of an 'electric' vector potential T such that $J = dT$.
- The equivalent feature in circuit model is the introduction of *mesh currents* I_m. To a mesh is associated a current circulating with the same intensity in all the branches of the loop. Such a current certainly respects the KCL as it enters and leaves a node on the mesh exactly once. The total current in a branch is the sum of all the currents of the meshes in which the branch is involved. This can be written in matrix form as $I = CI_m$. Similarly, given nodal potential V, the set of branch voltages $U = AV$ obtained by the differences of potential respects the KVL.
- In the finite element model, *discrete potentials* mimic exactly continuous potentials excepted that discrete operators are used. For instance, a conservative discrete current density $\mathbf{j}$ (associated to the faces) is obtained if a discrete 1-form $\mathbf{t}$ (associated to the edges) such that $\mathbf{j} = R\mathbf{t}$ is introduced.

The central problem of using potentials is that they are not unique and explicit computations very often require a gauge fixing condition.

- Rather than the classical Coulomb and Lorenz gauge of classical electrodynamics textbooks, the *axial gauge* will be considered here. A common setting of this gauge is to impose the third component of the vector potential (T or A) to be equal to zero i.e. $T_z = 0$. A more general setting is to introduce an arbitrary (contravariant) *vector field* $\mathbf{v}$ such that its integral curves (i.e. the curves such that the tangent vector to the curve at a point is precisely the value of the vector field at this point) have no loop. In this case, $\mathbf{v}.T = 0$ may be taken as a gauge (note that here the dot product does not involve the metric and represents in fact a duality product). We force as much as possible the line integrals of A (they are imposed to be zero along any part of an integral curve of $\mathbf{v}$) keeping the line integrals along any closed loop (i.e. the current through the loop) free.
- A similar phenomena occurs in circuit theory. If all the possible meshes are considered in a circuit, the vector I_m

is much too large to represent uniquely the branch currents satisfying the KCL. The classical method to define a suitable set of meshes is to use a *spanning tree* i.e. a connected subgraph such that it spans all the nodes of the original graph and has no loop (note the similarity with vector $\mathbf{v}$ used to define the axial gauge). The branches of the graph not included in the tree form the co-tree. The following set of suitable meshes is selected: to each branch of the co-tree is associated a mesh that include this very branch together with branches exclusively taken in the tree. From now, I_m will refer only to the currents associated to this set of meshes.

- In the Whitney finite element practice, the most natural gauge is the *spanning tree gauge.* A spanning tree is build on the set of edges of the meshing. The values of the discrete 1-form potential (e.g. $\mathbf{a}$ or $\mathbf{t}$) associated to the tree branches are set to zero while the ones associated to the co-tree are kept as unknowns. This introduces a maximum set of constrains that keep free the flux of the curl of the potential through the faces.

VII. Models

All the tools are now available to define the models of interest.

- The classical general model is of course Maxwell full system with more or less sophisticated constitutive laws. We focus here on the more specific electrokinetic model. To define a model, a geometrical domain have to be defined together with boundary conditions e.g. defining the trace (normal component from vector analysis point of view) of J on the boundary what corresponds to fixing the amount of current injected in the system. Putting the various equations altogether, one has to solve $d\sigma^{-1} * dT = 0$ ($curl\ \sigma^{-1} curl\ T = 0$ in vector analysis formalism) for the potential T submitted to the boundary conditions and constrained by the chosen gauge condition.
- In circuits equations, the *method of meshes* is considered here. The KCL $A^T I = 0$ is satisfied if $I = CI_m$. Beside the impedance, active voltage sources E are considered in the branches so that voltages in the branches are $U = ZI - E$ (the sign is determined by the power convention) submitted to the KVL $C^T U = 0$. All together this gives the linear system: $C^T ZCI_m = CE$.
- The matrix system for the finite element model of the electrokinetic model is : $R^T M_2(\sigma^{-1})R\mathbf{t} = \alpha$ where $\mathbf{t}$ is the unknown vector associated to the edges such that the current through the faces are $\mathbf{j} = R\mathbf{t}$ and α comes from the boundary conditions or any other prescribed sources.

It is useful here to consider *dual* structures. In the case of a planar circuit (i.e. that can be drawn on a plane without crossing branches), its dual is defined the following way: an obvious method to choose the suitable mesh set is to take the connected surfaces of the plane determined by branch loops. The dual graph is obtained by exchanging the role of nodes and meshes so that to each branch of the dual corresponds a branch of the primal graph. Incidence matrices A and C (and therefore KCL and KVL) exchange their role and simple rules (e.g. an impedance corresponds to the inverse impedance on the dual branch) make that the branch current in the primal circuit is numerically equal to the branch voltage in the dual one and vice versa. In the case of the finite element meshing, the dual is obtained by switching nodes with tetrahedra and edges with faces. The incidence matrices G^*, R^*, D^* of the dual meshing correspond to transposed of the primal ones $G^* = D^T, R^* = R^T, D^* = G^T$. The key to understand the discrete setting of Maxwell system is to associate discrete 'magnetic' quantities $\mathbf{h}, \mathbf{j}, \mathbf{d}, \rho, \mathbf{t}$ with a meshing and 'electric' quantities $\mathbf{e}^*, \mathbf{a}^*, \mathbf{b}^*, \mathbf{v}^*$ with its dual meshing. Therefore one has $D\mathbf{j} = 0$ i.e. $\mathbf{j} = R\mathbf{t}$ on the primal meshing and $R^T\mathbf{e}^* = 0$ on the dual meshing. The discrete Hodge operator is the square matrix that relate the current through the faces of the primal meshing to the electromotive force along the corresponding branches of the dual meshing: $M_2(\sigma^{-1})\mathbf{j} = \mathbf{e}^*$. This set of discrete equations provides directly the system of equations usually found thanks to the Galerkin method.

The general theory of electrical circuits may be directly transposed to *magnetic circuits* similarly represented by graphs. To the branches are associated the magnetic fluxes Φ and the magnetomotive forces H (circulation of the magnetic field along a branch) that are related by the *reluctance matrix* $\mathcal{R}$ such that $H = \mathcal{R}\Phi$. The conservation of the magnetic flux is expressed as $\mathcal{A}^T\Phi = 0$ where $\mathcal{A}$ is the branch-node incidence matrix of the magnetic circuit. Again this is satisfied by the introduction of mesh magnetic fluxes Φ_m and a mesh-branch incidence matrix $\mathcal{C}$ such that $\Phi = \mathcal{C}\Phi_m$. A magnetomotive force $\mathcal{M}$ can be associated to the meshes so that $\mathcal{C}^T H = \mathcal{M}$. The matrix equation for the magnetic circuit is $\mathcal{C}^T\mathcal{R}\mathcal{C}\Phi_m = \mathcal{M}$. We introduce now the electric-magnetic circuit coupling by remarking that the magnetomotive force is produced by currents circulating in electric circuits and by transforming the KVL, bringing back the Faraday induction law $C^T U = e$ where e is the f.e.m. induced on the meshes. Consider a magnetic circuit interlaced with an electric circuit. The coupling of the circuits is a purely topological property described by two matrices (They are not unique but are taken among the possible equivalent ones by choosing cutting surfaces for meshes): the magnetic mesh-electric branch incidence matrix $\mathcal{F}_m$ and the electric mesh-magnetic branch incidence matrix $\mathcal{F}_e$. The Faraday induction law is now $C^T U = -\partial_t \mathcal{F}_e \Phi$ and the Ampere law writes $\mathcal{C}^T H = \mathcal{F}_m I$. Finally, the coupling equations for the electric and magnetic circuits are: $C^T ZCI_m = -\partial_t \mathcal{F}_e \mathcal{C}\Phi_m$ and $\mathcal{C}^T\mathcal{R}\mathcal{C}\Phi_m = \mathcal{F}_m CI_m$.

References

[1] A. Bossavit, "Computational Electromagnetism, Variational formulations, Edge elements, Complementarity", Academic Press, Boston, 1998.

[2] A. Calvaer, *Electicité Théorique, régimes et circuits électriques, fascicule 2*, Derouaux, Liège, 1978.

[3] E. Tonti, On the Geometrical Structure of Electromagnetism, in *Gravitation, Electromagnetism and Geometrical Structures, for the 80th birthday of A. Lichnerowicz*, G. Ferrarese ed., Pitagora Editrice Bologna, pp. 281-308. (1995)

[4] K. Jänich, Vector Analysis, Springer Verlag, New York, 2001.

[5] A. Bossavit,L. Kettunen, Yee-like schemes on a tetrahedral mesh with diagonal lumping, *Int. J. Numer. Modell.*, 12:129-142, 1999.