EPJ manuscript No.
(will be inserted by the editor)

Modelling of twisted optical waveguides with edge elements.

A. Nicolet[1], F. Zolla[2], and S. Guenneau[3]

[1] Institut Fresnel, UMR 6133, Université Aix-Marseille III, case 162, F13397 Marseille Cedex 20, France
[2] Institut Fresnel, UMR 6133, Université Aix-Marseille I, case 162, F13397 Marseille Cedex 20, France
[3] Blackett Laboratory, Imperial College London, London SW7 2EZ, UK

Received: 26/11/2003/ Revised version: 26/06/2004

Abstract. We present the modelling of twisted electromagnetic waveguides using helicoid co-ordinates. This amounts to introducing equivalent inhomogeneous anisotropic materials which are however taken into account easily by the finite element method. An interesting property of such helicoid co-ordinates is to preserve the intrinsically two-dimensional nature of the problem.

PACS. 03.40.-Kf Waves and wave propagation: general mathematical aspects – 02.70.-Dh Finite-element and Galerkin methods

1 Introduction

1.1 Industrial motivations

The object of our study came out from scientific collaboration we maintained with the opto-electronic group at Bath University [6] (led by Prof. Russell), over the past three years. There is indeed a keen interest nowadays in Photonic Crystal Fibres [5,7], but it seems that nobody addressed as yet the effect of a twist on the dispersive properties of an electromagnetic signal localised in some central defect in a PCF. Still, this issue is of importance, since Prof. Russell and his co-workers observed experimentally this phenomenon and expressed their concern on its practical consequences: It is rather obvious that one should make clear whether or not this effect becomes preponderant on the long run. Here, of course, we pursue a more humble goal: We only intend to show a simple way to cope with such a spectral problem; But our study will yet pave the way for further analysis in this exciting direction.

Two companies, one of them being based in UK at Bath University ('http://www.blazephotonics.com'), have already distributed sample quantities and will soon begin volume production of PCF leading to a new generation of overseas optical telecommunications.

1.2 The spirit of our modelling

In this study, we depart from fairly well covered material on structures which present some translational invariance (on both geometrical and material aspects) along one axis, say z. Therefore, care has to be taken on the definition of a mode (e.g. waveguides exhibiting some curvature or with variation in their cross-section).

Let us consider a twisted closed waveguide of cross-section Ω (in the micro-wave regime for instance, the model of the metallic waveguide with infinite conducting walls is fairly accurate). The analysis of modes propagating in such waveguides amounts now to looking at non-trivial square integrable solutions of the Maxwell system in the following form

$$\mathcal{E}(x,y,z,t) = \Re e\{\mathbf{E}(x,y)e^{-i(\omega t-\beta z)}\} \quad (1)$$

with ω the pulsation, β the propagation constant and $\mathbf{E}(x,y)$ the variation of the electric field in the $(x-y)$ plane in $z=0$ at time $t=0$[7].

Here, we choose the electric field as the unknown, since its tangential trace is null on the outer boundary of the waveguide, unlike its magnetic counterpart. The differential operator associated to this spectral problem has a compact resolvent and thus we only need look at a countable set of eigenvalues and associated eigenvectors to perform the analysis (they generate so-called dispersion curves) [3,7].

The key point in our study is that the differential operators consist of products by functions independent of z so that we can perform some Fourier transform in the z direction. Hence, we just need worry about derivatives with respect to x and y (differentiating in z amounts simply to multiplying by $i\beta$). Therefore, if this situation arises in other co-ordinate systems, we can still speak of modes.

This is obviously the case for cylindrical co-ordinates and axi-symmetric structures: one can definitely find some modes propagating along the azimutal co-ordinate θ.

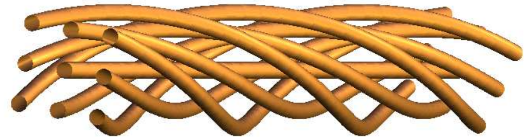

Fig. 1. The helicoidal structure under consideration.

2 Covariant approach of the problem

2.1 Helicoidal co-ordinates

Let us now play with a co-ordinate system u, v, w deduced from Cartesian co-ordinates (x, y, z) in the following way

$$\begin{cases} x = u\cos(\alpha w) + v\sin(\alpha w) \\ y = -u\sin(\alpha w) + v\cos(\alpha w) \\ z = w \end{cases} \quad (2)$$

where α is a parameter which characterises the torsion of the waveguide. The idea is to define the electromagnetic problem of twisted geometry as a problem whose geometry (i.e. the limit conditions together with the material properties ε, μ) only depends on co-ordinates u and v.

This change of co-ordinates amounts to replacing the different materials (often homogeneous and isotropic, which corresponds to the case of scalar piecewise constant permittivities and permeablilities) by equivalent inhomogeneous anisotropic materials. This general co-ordinate system is characterised by the Jacobian of the tranformation (2):

$$\mathbf{J} = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{pmatrix} \cos(\alpha w) & \sin(\alpha w) & \alpha v\cos(\alpha w) - \alpha u\sin(\alpha w) \\ \sin(\alpha w) & \cos(\alpha w) & -\alpha u\cos(\alpha w) - \alpha v\sin(\alpha w) \\ 0 & 0 & 1 \end{pmatrix} . \quad (3)$$

Even though this Jacobian matrix depends explicitely on the third co-ordinate w, it is noticeable that the system of helicoidal co-ordinates leads to a transformation matrix $\mathbf{T}$ (describing the change in the material properties) which does no longer depend on the third co-ordinate w, hence the possibility to define properly the notion of propagating modes in helicoidal structures.

2.2 Geometric Transformations

A neat way to derive $\mathbf{T}$ is to use differential geometry. Let $\mathbf{E}$ be a 1-form and $\mathbf{D}$ a 2-form. They are expressed in the (non-orthogonal) helicoidal co-ordinate system by:

$$\begin{aligned} \mathbf{E} &= E_u du + E_v dv + E_w dw \\ \mathbf{D} &= D_u dv \wedge dw + D_v dw \wedge du + D_w du \wedge dv \ , \end{aligned} \quad (4)$$

where $\wedge$ denotes the exterior product. Also, the scalar product between p-forms is expressed with $\int \alpha \wedge \star\gamma$ where $\star$ is the Hodge operator. A simple way to evaluate this product is to project our forms back onto familiar Cartesian co-ordinates:

$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \mathbf{J}^{-1} \begin{pmatrix} E_u \\ E_v \\ E_w \end{pmatrix} \text{ and } \begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \frac{\mathbf{J}}{det(\mathbf{J})} \begin{pmatrix} D_u \\ D_v \\ D_w \end{pmatrix} , \quad (5)$$

and to use the good old scalar product in Cartesian co-ordinates (keeping in mind the factor $det(\mathbf{J})^{-1}$ for the volume form). We obtain the following expression for the scalar products:

$$\int \mathbf{E} \wedge \star\mathbf{E}' = \int (\mathbf{TE}).\mathbf{E}' d\Omega \ , \quad (6)$$

and

$$\int \mathbf{D} \wedge \star\mathbf{D}' = \int (\mathbf{T}^{-1}\mathbf{D}).\mathbf{D}' d\Omega \ , \quad (7)$$

where the right members involve the column vectors of Cartesian co-ordinates and the matrix:

$$\mathbf{T} = \frac{\mathbf{J}^T\mathbf{J}}{det(\mathbf{J})} = \begin{pmatrix} 1+\alpha v^2 & -\alpha^2 uv & -\alpha v \\ -\alpha^2 uv & 1+\alpha u^2 & \alpha u \\ -\alpha v & \alpha u & 1 \end{pmatrix} . \quad (8)$$

In the particular case of an helicoidal transformation $det(\mathbf{J}) = 1$, which means that there is no change in volume. On a geometric point of view, the matrix $\mathbf{T}$ plays the role of the metric tensor. The only thing to do for a finite element formulation of the helicoidal problem is to replace the materials (often homogeneous and isotropic) by inhomogeneous ones (their characteristics are no longer piecewise constant but merely depend on u, v co-ordinates) and anisotropic ones (tensorial nature) whose properties are given by $\underline{\underline{\varepsilon'}} = \varepsilon\mathbf{T}$ and $\underline{\underline{\mu'}} = \mu\mathbf{T}$. We note that there is no change in the impedance of the media, since the permittivity and permeability suffer the same transformation.

In what follows, we will adopt a variational point of view for our mathematical modelling. The salient consequence of (6) and (7) is that we can work in Cartesian coordinates, the Hodge operator $\star$ being contained within the scalar product.

3 Mathematical set up of the eigenvalue problem

3.1 Governing equations in covariant co-ordinates

We consider a metallic waveguide with heterogeneous permittivity and permeablity, of constant section $\Omega \subset \mathbb{R}^2$ invariant along the z axis. We are looking for electromagnetic fields $(\mathcal{E}, \mathcal{H})$ solutions of the following Maxwell equations

$$\begin{cases} \operatorname{rot} \mathcal{H} = \varepsilon \dfrac{\partial \mathcal{E}}{\partial t} \\ \operatorname{rot} \mathcal{E} = -\mu \dfrac{\partial \mathcal{H}}{\partial t} \end{cases} \quad (9)$$

Here, ε and μ are two second order real symmetric tensor fields, defined in Ω, with bounded coefficients satisfying the ellipticity conditions for every vector $\xi \in \mathbb{R}^3$

$$\sum_{i,j=1,3} \varepsilon_{ij}\xi_i\xi_j \geq \alpha|\ \xi\ |^2 \quad \text{and} \quad \sum_{i,j=1,3} \mu_{ij}\xi_i\xi_j \geq \gamma|\ \xi\ |^2 \ ,$$

with $\alpha, \gamma > 0$. Furthermore, choosing a time dependance in $e^{-i\omega t}$, and taking into account the invariance of the guide along its z axis, we define time-harmonic two dimensional electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$ by:

$$\begin{cases} \mathcal{E}(x,y,z,t) &= \Re e(\mathbf{E}(x,y)e^{-i(\omega t-\beta z)}) \\ \mathcal{H}(x,y,z,t) &= \Re e(\mathbf{H}(x,y)e^{-i(\omega t-\beta z)}) \end{cases} \tag{10}$$

where ω is the angular frequency in the vacuum and β denotes the propagating constant of the guided mode.

For $(\mathbf{E},\mathbf{H})$ satisfying (10), (9) can be written as:

$$\begin{cases} \operatorname{rot}_\beta \mathbf{H} = -i\omega\varepsilon(x,y)\mathbf{E} \\ \operatorname{rot}_\beta \mathbf{E} \ = i\omega\mu(x,y)\mathbf{H} \end{cases} \tag{11}$$

where we define the curl of a vector field $\mathbf{U}$ as

$$\operatorname{rot}_\beta \mathbf{U} = \left(\frac{\partial U_z}{\partial y} - i\beta U_y\right)\mathbf{e}_x - \left(\frac{\partial U_z}{\partial x} - i\beta U_x\right)\mathbf{e}_y + \left(\frac{\partial U_y}{\partial x} - \frac{\partial U_x}{\partial y}\right)\mathbf{e}_z \ . \tag{12}$$

3.2 The spectral problem

We say that $(\mathbf{E},\mathbf{H})$ is a guided mode if:

$$\begin{cases} (\beta,\omega) \in \mathbb{R}^2 \\ (\mathbf{E},\mathbf{H}) \neq (\mathbf{0},\mathbf{0}) \\ \mathbf{E},\mathbf{H} \in [L^2(\Omega)]^3 \end{cases} \tag{13}$$

where the tangential trace $n \times \mathbf{E}$ of $\mathbf{E}$ vanishes on the boundary $\partial\Omega$ of Ω whereas the tangential trace of $\mathbf{H}$ gives rise to an (unknown) surface current $\mathbf{J}_s = n \times \mathbf{H}$.

We choose an electric field formulation [3] because the tangential trace of $\mathbf{E}$ is null, contrary to the trace of $\mathbf{H}$. Thus, div_β being an operator defined in a similar way to rot_β in (12) as

$$\operatorname{div}_\beta \mathbf{U} = \frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + i\beta U_z \ , \tag{14}$$

it is clear that $\operatorname{div}_\beta \operatorname{rot}_\beta \mathbf{E} = 0$, for all $\mathbf{E}$ in Ω such that $n \times \mathbf{E} = 0$. Hence, we are led to the following system of Maxwell's type:

$$\begin{cases} \operatorname{rot}_\beta(\mu^{-1}\operatorname{rot}_\beta \mathbf{E}) = \omega^2\varepsilon\mathbf{E} \\ \operatorname{div}_\beta(\varepsilon\mathbf{E}) \qquad\quad = 0 \ . \end{cases} \tag{15}$$

Let (β,ω) be a solution of the spectral problem (15) and $\mathbf{E}$ its associated eigenvector. Then $(\beta,-\omega)$ is a solution of (15) with the same eigenvector $\mathbf{E}$. Physically speaking, this is induced by the time-invariance of the wave equation when dealing with non dissipative media (ω is not complex). Furthermore, $(-\beta,\omega)$ and $(-\beta,-\omega)$ are also solutions of (15) with the eigenvector $(E_1, E_2, -E_3)$: this is a space-invariance induced by the symmetry of the guide along the z-axis (even in an helicoidal structure). Roughly speaking, the physical nature of the problem remains unchanged if a wave propagates along the z-positive or negative, provided that the cross section of the guide is constant. We therefore look solely for $(\beta,\omega) \in \mathbb{R}^+ \times \mathbb{R}^+$.

3.3 Weak form of the problem

The variational formulation associated to (15) is:

$$\int_\Omega \operatorname{rot}_\beta(\mu^{-1}\operatorname{rot}_\beta \mathbf{E}) \cdot \overline{\mathbf{E}'} d\Omega = \int_\Omega \omega^2\varepsilon\mathbf{E} \cdot \overline{\mathbf{E}'} d\Omega \ . \tag{16}$$

Thanks to the Stokes theorem, we see that the left member of (16) defines a bilinear form

$$a(\beta;\mathbf{E},\mathbf{E}') = \int_\Omega \mu^{-1}\operatorname{rot}_\beta \mathbf{E} \cdot \overline{\operatorname{rot}_\beta \mathbf{E}'} d\Omega \ . \tag{17}$$

3.4 Characterisation of the spectrum

We want to make it clear that this form is bilinear, symmetric, continuous and cœrcive on the Sobolev space

$$V(\beta) = \{\mathbf{F} \in [L^2(\Omega)]^3 \ ; \ \operatorname{rot}_\beta \mathbf{F} \in [L^2(\Omega)]^3 \\ \operatorname{div}_\beta \mathbf{F} = 0 \ , \ n \times \mathbf{F} = 0 \ \text{ on } \partial\Omega\} \ .$$

We shall use the two following results:

Lemma 1:

Let $\mathbf{F}$ be a vector field in $[H^1(\Omega)]^3$ such that $n \times \mathbf{F} = 0$ on the boundary $\partial\Omega$ of Ω. We then have the Green formula:

$$\int_\Omega (|\operatorname{rot}_\beta \mathbf{F}\ |^2 + |\operatorname{div}_\beta \mathbf{F}\ |^2) d\Omega = \int_\Omega (|\operatorname{grad} \mathbf{F}\ |^2 + \beta^2|\ \mathbf{F}\ |^2) d\Omega$$

The functional space $V(\beta)$ is then isomorphic to the Hilbert space $[H_0^1(\Omega)]^3$.

Since $n \times \mathbf{F}$ vanishes on $\partial\Omega$, the result follows from an integration by parts [3, 7].

Lemma 2:

Let s be a positive real. Then, the two following systems are equivalent in $V(\beta)$:

$$\begin{cases} \operatorname{rot}_\beta(\mu^{-1}\operatorname{rot}_\beta \mathbf{E}) = \omega^2\varepsilon\mathbf{E} \\ \operatorname{div}_\beta(\varepsilon\mathbf{E}) \qquad\quad = 0 \end{cases} \tag{18}$$

$$\operatorname{rot}_\beta(\mu^{-1}\operatorname{rot}_\beta \mathbf{E}) - s\operatorname{grad}_\beta(\operatorname{div}_\beta \varepsilon\mathbf{E}) = \omega^2\varepsilon\mathbf{E} \ .$$

A proof for this can be found in [3, 7].

Our bilinear form $a(\beta,.,.)$ in (17) is therefore bounded and coercive on $V(\beta) \times V(\beta)$ thanks to the added term

$$\int_\Omega \operatorname{div}_\beta(\varepsilon\mathbf{E})\ \overline{\operatorname{div}_\beta(\mathbf{E}')} d\Omega \ , \tag{19}$$

which acts in fact as a constraint which forces the nullity of $\mathrm{div}_\beta(\varepsilon\mathbf{E})$. It is worth noting that the operator associated to the variational problem (16) has a compact resolvent (compact embedding of $H^1(\Omega)$ in $L^2(\Omega)$ by Rellich lemma). It can be shown that the spectrum consists of a discrete set of eigenvalues belonging to $[\frac{\beta^2}{\varepsilon\mu};+\infty[$. This provides us with a numerical criterion to eliminate non physical modes [3, 7].

4 Discretisation with finite element method

From (18), we clearly see that the finite element formulation amounts to minimizing the following residue (Galerkin method):

$$\int_\Omega \Big((\underline{\underline{\mu'}})^{-1}\,\mathrm{rot}_\beta\,\mathbf{E}\cdot\overline{\mathrm{rot}_\beta\,\mathbf{E'}} - \omega^2 \underline{\underline{\varepsilon'}}\mathbf{E}\cdot\overline{\mathbf{E'}}\Big) d\Omega = 0 \;, \quad (20)$$

where $\mathbf{E'}$ are the (vector) weight functions, rot_β is defined by (12), and $\underline{\underline{\mu'}} = \mu\mathbf{T}$ and $\underline{\underline{\varepsilon'}} = \varepsilon\mathbf{T}$.

Although we are no longer in a Cartesian co-ordinate system, the scalar product stressed by a dot is indeed what we would write with the classical recipe which amounts to adding the product of corresponding components of two fields. This, because the matrix $\mathbf{T}$ contains already within it the good entries, thanks to the modification of one of the factors.

The weight vector field $\mathbf{E'}$ is chosen in the same discrete Hilbert space as the unknown field $\mathbf{E}$ i.e. a space with finite dimension equal to the number of numerical parameters to be determined. This formulation involves both a transverse field $\mathbf{E}_t$ in the section of the guide and a longitudinal field E_l along its axis such that:

$$\mathbf{E} = \mathbf{E}_t(u,v) + E_l(u,v)\mathbf{e}_z \;. \quad (21)$$

In the isotropic case, the terms in (20) involving the scalar product of a longitudinal component by a transversal component cancel out. In the twisted case, all media become anisotropic in nature and the matrix $\mathbf{T}$ change the orientation of the components. We thus have to keep all the terms in the scalar product in (20). This leads to slightly more complicated expressions in the finite element package. It implies an increased number of non zero entries in the matrices. Of course, the number of unknowns of the numerical problem remains unchanged.

The section of the guide is meshed with triangles and Whitney finite elements [1] are used i.e. edge elements for the transverse field and node elements for the longitudinal field:

$$\mathbf{E} = \begin{cases} \mathbf{E}_t = \sum\limits_{edges\ i} \chi_i \mathbf{w}_i^{\mathbf{e}}(u,v) \\ E_l = \sum\limits_{nodes\ j} \gamma_j w_j^n(u,v) \end{cases} , \quad (22)$$

where χ_i denotes the line integral of the transverse component $\mathbf{E}_t$ on the edges, and γ_j denotes the line integral of the longitudinal component E_l along one unit of length of the axis of the guide (what is equivalent to a nodal value). Besides, $w_j^n(u,v) = \lambda_j(u,v)$ and

$$\mathbf{w}_i^{\mathbf{e}}(u,v) = \lambda_k(u,v)\,\mathrm{grad}\,\lambda_l(u,v) - \lambda_l(u,v)\,\mathrm{grad}\,\lambda_k(u,v) \;,$$

(where λ_j is the barycentric coordinate of node j and the edge i has nodes k and l as extremities) are respectively the basis functions of Whitney 1-forms (edge element discrete space W^1) and Whitney 0-forms (nodal element discrete space W^0) [1].

Moreover, the use of the Whitney elements solves the spurious mode problem in a way similar to the one of the cavities [1]. To see that, it has to be noticed that the penalty term (19) involving the divergence is not introduced in the discrete formulation (20) because the use of Whitney elements guarantees the nullity of the divergence in a weak sense.

As the eigenvalue problem involves, on the one side, ω^2 only and, on the other side, both β and β^2, a more classical (though generalized) eigenvalue problem is obtained by fixing $\beta \in \mathbb{R}_+$ (rather than ω^2) and looking for $(\omega^2, \mathbf{E})$ satisfying the discrete spectral problem $\mathbf{AE} = \omega^2\mathbf{BE}$

Such problems involving large sparse Hermitian matrices can be solved using Lanczos algorithm that gives the largest eigenvalues. Physically we are in fact interested in the smallest eigenvalues and therefore $\mathbf{A}^{-1}$, the inverse of $\mathbf{A}$, instead of $\mathbf{A}$ itself must be used in the iterations. Of course, the inverse is never computed explicitly but the matrix-vector products are replaced by system solutions thanks to a GMRES method. It is therefore obvious that the numerical efficiency of the process relies strongly on Krylov subspace techniques and the Arnoldi iteration algorithm. The practical implementation of the model has been performed thanks to the *GetDP* software [2].

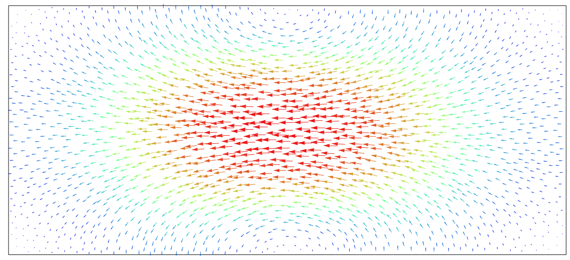

Fig. 2. Transverse component of the electric field for a resonance mode of angular frequency $\omega c = 3.78m$ ($\beta = 0$) in an helicoidal metallic stucture of rectangular cross-section (thickness = $1.02m$, length = $2.29m$, torsional parameter $\alpha = 1m^{-1}$). The picture suggests that the twist confines the resonant mode within the disc inscribed into the rectangular cross-section.

5 Perspectives

In this paper, we have presented a simple algorithm to analyse twisted heterogeneous waveguides of constant cross-section. We have illustrated our procedure with a simple case of a homogeneous metallic rectangular waveguide, but it goes without saying that our algorithm can tackle more complex geometries such as micro-structured fibres. On the long run, we wish also to relax somehow the assumptions on the matrices of permittivity and permeability to be able to explore the properties of waveguides with negative refractive index such as so-called thin-loop wires. This could be done by considering some small imaginary part for ε and μ (hence complex frequencies).

References

1. A. Bossavit, IEEE Transactions on Magnetics **26**, No. 2 (1990) 702-705.
2. P. Dular, C. Geuzaine, F. Henrotte, W. Legros, IEEE Transactions on Magnetics **34**, No. 5 (1998). See also the Internet address + http://www.geuz.org/getdp/.
3. S. Guenneau, A. Nicolet, F. Zolla, S. Lasquellec IEEE Transactions on Magnetics **38**, No. 2 (2002) 1261-1264.
4. F. Henrotte, B. Meys, H. Hedia, P. Dular, W. Legros, IEEE Transactions on Magnetics, **35**, No. 3 1434-1437 (1999) 1434-1437.
5. J. C. Knight, J. Broeng, T. A. Birks, P. St. J. Russell, SCIENCE **282** (1999) 1476-1478.
6. P. St. J. Russell, E. Marin, A. Diez, S. Guenneau, A. B. Movchan, Optics Express **11**, No. 20 (2003) 2555-2560.
7. F. Zolla, G. Renversez, A. Nicolet, B. Kuhlmey, S. Guenneau, D. Felbacq, *Foundations of Photonic Crystal Fibres.* (World Scientific/ICP Press, London 2004).