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Boundary elements and singular integrals in 3D magnetostatics*

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Vector and scalar potential formulations of magnetostatics are compared and the meaning of vector singular kernels is explained. The numerical computation of the integrals involving those kernels is discussed and a method to avoid computation of nearly singular kernels arising in the computation of thin structures is presented.

Key words: magnetostatics, vector potential, singular kernels, thin plates.

INTRODUCTION

The boundary element method[1] is now a well established method in computational electromagnetics[2] and particularly in magnetostatics. There are several formulations of this problem depending on the choice of the unknowns: fields, potentials, equivalent charges or dipoles. Due to the structure of electromagnetism, for each of these choices there are two dual possibilities. For instance, potential formulations may be based on a magnetic vector potential $\mathbf{A}$ such that $\mathbf{B} = \operatorname{curl}\mathbf{A}$, or on a magnetic scalar potential ϕ such that $\mathbf{H} = -\operatorname{grad}\phi$.

Note that this latter formulation is only possible in a current-free region. Moreover, topological constraints must be imposed on this region in order to have a single valued potential. The corresponding indirect methods are based on a single layer of electric current for the vector potential formulation and on a single layer of magnetic monopoles for the scalar potential formulation. The main difference between the two formulations is that one is a vector formulation and the other is a scalar formulation. While the scalar case may be considered as a paradigmatic problem of boundary element analysis, the vector case is less studied. Nevertheless, the meaning and the numerical evaluation of the singular kernels require attention.

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SCALAR AND VECTOR FORMULATION OF 3D LINEAR MAGNETOSTATICS

The problem of linear magnetostatics in a current-free region is considered here. Topological problems are required to be solved in order to have a single valued scalar potential. In such a region, the equations to be solved are:

$$\operatorname{curl}\mathbf{H} = 0 \tag{1a}$$

$$\operatorname{div}\mathbf{B} = 0 \tag{1b}$$

The scalar potential ϕ is defined by:

$$\mathbf{H} = -\operatorname{grad}\phi \tag{2}$$

in order to satisfy eqn (1a) and the equation for ϕ is found by introducing eqn (2) in eqn (1b):

$$-\operatorname{div}\operatorname{grad}\phi = -\Delta\phi = 0 \tag{3}$$

The vector potential $\mathbf{A}$ is defined by:

$$\mathbf{B} = \operatorname{curl}\mathbf{A} \tag{4}$$

in order to satisfy eqn (1b) and the equation for $\mathbf{A}$ is found by introducing eqn (4) in eqn (1a):

$$\operatorname{curl}\operatorname{curl}\mathbf{A} = -\Delta\mathbf{A} + \operatorname{grad}\operatorname{div}\mathbf{A} = 0 \tag{5a}$$

In magnetostatics, the Coulomb gauge $\operatorname{div}\mathbf{A} = 0$ is often chosen in order to have unicity of $\mathbf{A}$. In this case, the equation for $\mathbf{A}$ is:

$$-\Delta\mathbf{A} = 0 \tag{5b}$$

The corresponding boundary integral formulations for eqns (3) and (5b) are, for a domain D of boundary

∂D, respectively:

$$h(p)\phi(p) = \oint_{\partial D} [G(p,q)(\text{grad}\, \phi(q) \cdot \mathbf{n}(q)) - (\text{grad}_q\, G(p,q) \cdot \mathbf{n}(q))\phi(q)]\, \text{d}\partial D \quad (6)$$

$$h(p)\mathbf{A}(p) = \oint_{\partial D} \begin{bmatrix} -G(p,q)(\text{curl}\, \mathbf{A}(q) \times \mathbf{n}(q)) \\ +(\mathbf{A}(q) \cdot \mathbf{n}(q))\, \text{grad}_q\, G(p,q) \\ +(\mathbf{A}(q) \times \mathbf{n}(q)) \times \text{grad}_q\, G(p,q) \end{bmatrix} \text{d}\partial D \quad (7)$$

where p is the field point; q the source point; and $G(p,q)$ is the free space Green function of the 3D (scalar) Laplace operator (Fig. 1):

$$G(p,q) = G(q,p) = \frac{1}{4\pi r}$$

$$\text{grad}_q\, G(p,q) = -\text{grad}_p\, G(q,p) = \frac{\mathbf{r}}{4\pi r^3}$$

with $\mathbf{r} = p - q$ and $r = |p - q|$.

The various terms in eqn (7) may be rearranged to obtain the following form:[3]

$$h(p)\mathbf{A}(p) = \oint_{\partial D} \begin{bmatrix} -G(p,q)(\text{curl}\, \mathbf{A}(q) \times \mathbf{n}(q)) \\ -\mathbf{A}(q)(\mathbf{n}(q) \cdot \text{grad}_q\, G(p,q)) \\ +\mathbf{A}(q) \times (\mathbf{n}(q) \times \text{grad}_q\, G(p,q)) \end{bmatrix} \text{d}\partial D \quad (8)$$

$h(p)$ is a coefficient depending on the position of point p with respect to the domain D; $h(p)$ is equal to the solid angle under which the domain D is seen from point p, expressed in steradians, and divided by 4π. If the point is inside the domain $h(p) = 1$, if the point is outside $h(p) = 0$ and if the point is on a smooth part of the boundary $h(p) = 1/2$.

The two terms in the second member of eqn (6) may be interpreted respectively as the contributions of a single and a double layer of magnetic monopoles. The three terms in the second member of eqn (7) may be interpreted respectively as the contribution of an equivalent single layer of current, a term producing no magnetic field (its curl is equal to zero and it is related to the gauge choice) and the contribution of an equivalent double layer of current.

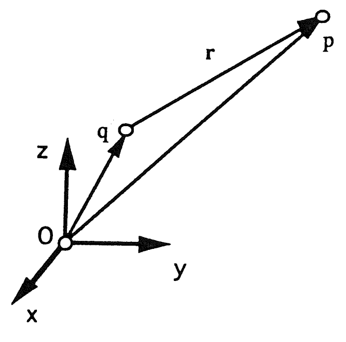

Fig. 1. 3D Green function.

SINGULAR KERNELS

Integrals involving singular kernels must be carefully defined. It is well known that the function $1/x^\theta$ is integrable on $]0, a]$ if $\theta < 1$.

Nevertheless, an integral as:

$$\int_a^b \frac{1}{x} \text{d}x \qquad a < 0, b > 0 \quad (9)$$

may be given a definite meaning.

The Cauchy principal value is defined as:[4]

$$vp \int_a^b \frac{1}{x} \text{d}x = \lim_{\epsilon \to 0} \left(\int_a^{-\epsilon} \frac{1}{x} \text{d}x + \int_\epsilon^b \frac{1}{x} \text{d}x \right) \quad (10)$$

Although the individual terms of the right-hand member are meaningless, the skewsymmetry of the integrand with respect to the singularity leads to a cancellation of the divergent parts and allows the convergence of the whole expression.

Note that if the integrand is integrable in the classical sense, the Cauchy principal value corresponds to the value of the classical integral.

The problem is to generalise the concept of Cauchy principal value to the multiple integrals of the vector integral formula (7).

The Cauchy principal value is defined as:[5]

$$vp \int_{\partial D} K(p,q) f(q)\, \text{d}\partial D = \lim_{\epsilon \to 0} \int_{\partial D - \{r \le \epsilon\}} K(p,q) f(q)\, \text{d}\partial D \quad (11)$$

where:

p is a point in space, q is a point of ∂D,

∂D is the boundary of a domain D,

$K(p,q)$ is a (possibly vector) kernel singular for $p = q$,

$f(q)$ is a smooth and bounded function on ∂D.

This definition of the principal value is similar to the one-dimensional one in the sense that an infinitesimal symmetrical neighbourhood of the singularity is removed where the divergent contributions are supposed to cancel each other. Here, this neighbourhood is the set of points of ∂D whose distance from p is less than ϵ. The principal value is the limit for ϵ tending to zero. Again, this definition corresponds to the classical integral when this one exists.

Definition (11) may be straightforwardly generalised to the case of vector integrands by considering each component separately.[3] In the case of a singular kernel it is interesting to study the contribution of the ϵ-neighbourhood in order to obtain information on the nature of the integral.

The boundary ∂D is smooth and ϵ is supposed to be small enough in order to consider the neighbourhood of the singularity as a disc D_ϵ of radius ϵ (Fig. 2).

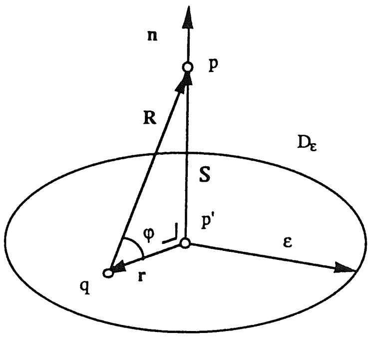

Fig. 2. Neighbourhood of the singularity.

As the function $f(q)$ is smooth and bounded it may be developed in series of the radial coordinate r:

$$f(q) = f(p) + rg(\theta) + O(r^2) \tag{12}$$

The interesting term is the constant $f(p)$ because higher-order terms in r weaken the singularity of the integrand. In this case, the constant factor $f(p)$ may be taken out of the integral. As the neighbourhood of the singularity is a flat disc, $\mathbf{n}(q)$ is a constant vector and it is only necessary to discuss the behaviour of the kernels for $\mathbf{n}(q) = \mathbf{n}(p) = \mathbf{n}$.

In the case of expression (8), it is only necessary to consider the integral of the kernels $G(p,q)$, $\mathbf{n}\cdot\mathrm{grad}_q\, G(p,q)$ and $\mathbf{n}\times\mathrm{grad}_q\, G(p,q)$. The kernel $\mathrm{grad}_q\, G(p,q)$ which will be encountered in the indirect formulation will also be discussed here.

The point p is at a distance S from the boundary and p' is its orthogonal projection on the boundary; $\mathbf{n}$ is the exterior normal vector at point p'; D_ϵ is the disc of radius ϵ and centre p'; and q is a point of the disc whose distances respectively with p and p' are R and r (Fig. 2). The contribution of D_ϵ to the integrals of eqn (8) will be evaluated and the singular case will be considered by taking the limit for p tending to p', i.e. S tending to zero. Then the limit for ϵ tending to zero will be taken. Note that the order of evaluation of those limits is important.[3]

Kernel G(p, q)

The contribution of D_ϵ for this kernel is:

$$C_\epsilon = \int_{D_\epsilon} G(p,q)\,\mathrm{d}S = \int_0^{2\pi}\int_0^{\epsilon} \frac{1}{4\pi R} r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \frac{1}{2}\int_0^{\epsilon} \frac{r}{\sqrt{r^2+S^2}}\,\mathrm{d}r = \left[\frac{1}{2}\sqrt{r^2+S^2}\right]_0^{\epsilon}$$

$$= \tfrac{1}{2}\left(\sqrt{\epsilon^2+S^2} - |S|\right) \tag{13}$$

$$\lim_{\epsilon\to 0}\lim_{S\to 0} C_\epsilon = \lim_{\epsilon\to 0}\frac{|\epsilon|}{2} = 0 \tag{14}$$

The singular kernel $G(p,q)$ is integrable and does not lead to any discontinuity across the boundary.

Kernel n · grad$_q$ G(p, q)

The contribution of D_ϵ for this kernel is:

$$C_\epsilon = \int_{D_\epsilon} \mathbf{n}\cdot\mathrm{grad}_q\, G(p,q)\,\mathrm{d}S = \int_0^{2\pi}\int_0^{\epsilon} \frac{\mathbf{n}\cdot\mathbf{R}}{4\pi R^3} r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \frac{1}{2}\int_0^{\epsilon} \frac{Sr}{(\sqrt{r^2+S^2})^3}\,\mathrm{d}r = \left[-\frac{1}{2}\frac{S}{\sqrt{r^2+S^2}}\right]_0^{\epsilon}$$

$$= \frac{1}{2}\left(\mathrm{sgn}(S) - \frac{S}{\sqrt{\epsilon^2+S^2}}\right) \tag{15}$$

$$\lim_{\epsilon\to 0}\lim_{S\to 0} C_\epsilon = \lim_{\epsilon\to 0}\tfrac{1}{2}\mathrm{sgn}(S) = \tfrac{1}{2}\mathrm{sgn}(S) \tag{16}$$

The singular kernel $\mathbf{n}\cdot\mathrm{grad}_q\, G(p,q)$ is integrable but it introduces a discontinuity across the boundary. Indeed, for an interior point:

$$\lim_{S\to 0^-}\tfrac{1}{2}\mathrm{sgn}(S) = -\tfrac{1}{2} \tag{17}$$

and for an exterior point:

$$\lim_{S\to 0^+}\tfrac{1}{2}\mathrm{sgn}(S) = +\tfrac{1}{2} \tag{18}$$

The integral $\oint_{\partial D} \mathbf{A}(q)(\mathbf{n}(q)\cdot\mathrm{grad}_q\, G(p,q))\,\mathrm{d}\partial D$ has a discontinuity equal to the one of $h(p)\mathbf{A}(p)$, the left-hand member of eqn (7). Physically, this may be interpreted as a double layer of current that introduces a discontinuity of the potential.

When the point p is on the boundary, $\mathrm{sgn}(S) = 0$.

The kernel $\mathbf{n}(q)\cdot\mathrm{grad}_q\, G(p,q)$ will be denoted $\partial G/\partial\mathbf{n}$ and the following notation is introduced for the integrals:

$$\int^*_{\partial D} \mathbf{A}(q)\frac{\partial G}{\partial \mathbf{n}}\mathrm{d}\partial D = \frac{1}{2}\left[\lim_{S\to 0^+}\int_{\partial D}\mathbf{A}(q)\frac{\partial G}{\partial \mathbf{n}}\mathrm{d}\partial D + \lim_{S\to 0^-}\int_{\partial D}\mathbf{A}(q)\frac{\partial G}{\partial \mathbf{n}}\mathrm{d}\partial D\right] \tag{19}$$

Then:

$$\lim_{S\to 0^-}\int_{\partial D}\mathbf{A}(q)\frac{\partial G}{\partial \mathbf{n}}\mathrm{d}\partial D = \int^*_{\partial D}\mathbf{A}(q)\frac{\partial G}{\partial \mathbf{n}}\mathrm{d}\partial D - \frac{1}{2}\mathbf{A}(p) \quad \text{(interior limit)} \tag{20}$$

$$\lim_{S\to 0^+}\int_{\partial D}\mathbf{A}(q)\frac{\partial G}{\partial \mathbf{n}}\mathrm{d}\partial D = \int^*_{\partial D}\mathbf{A}(q)\frac{\partial G}{\partial \mathbf{n}}\mathrm{d}\partial D + \frac{1}{2}\mathbf{A}(p) \quad \text{(exterior limit)} \tag{21}$$

Kernels n × grad$_q$ G(p, q) and grad$_q$ G(p, q)

The kernels $\mathbf{n}\times\mathrm{grad}_q\, G(p,q)$ and $\mathrm{grad}_q\, G(p,q)$ are not integrable. Nevertheless for every point q of D_ϵ with a position $\mathbf{r}$ corresponds a point q' of D_ϵ with a position $-\mathbf{r}$ and the kernels $\mathbf{n}\times\mathrm{grad}_q\, G(p,q)$ and $\mathrm{grad}_q\, G(p,q)$ are skewsymmetrical because $\mathrm{grad}_q\, G(p,q) = -\mathrm{grad}_{q'}\, \mathrm{G}(\mathrm{p},\mathrm{q}')$.

Thanks to this skewsymmetry, the global contribution

C_ϵ of D_ϵ is equal to zero by cancellation of the divergent contributions. This is a particular case of an important class of improper integrals with a skewsymmetrical kernel for which a principal value may be defined.

Contributions involving such kernels must be taken in the Cauchy principal value sense. Note that this kind of term arises only in the vector case.

In the case of a non-smooth boundary, the Cauchy principal value may still be defined for a point on an edge or on a corner.[6]

INDIRECT BOUNDARY ELEMENT METHOD

The principle of the indirect boundary element method is to consider not the vector potential and its normal derivative as unknowns, but rather the equivalent surface currents as unknowns. It is therefore necessary to find an integral equation for these currents.

A boundary ∂D between a domain D_1 and a domain D_2 is considered. The quantities related to these domains are noted with the indices 1 and 2 of the corresponding domain. The outer normal of the domain 1 is chosen as the common normal ($\mathbf{n} = \mathbf{n}_1 = -\mathbf{n}_2$) and, for a smooth part of the boundary, equation (7) can be written:

$$\begin{aligned}\tfrac{1}{2}\mathbf{A}_1(p) = &\int_{\partial D} [-G(p,q)(\operatorname{curl}\mathbf{A}_1(q) \times \mathbf{n}(q))]\,\mathrm{d}\partial D \\ &+ \int^*_{\partial D} [(\mathbf{A}_1(q)\cdot\mathbf{n}(q))\,\operatorname{grad}_q G(p,q) \\ &+ (\mathbf{A}_1(q) \times \mathbf{n}(q)) \times \operatorname{grad}_q G(p,q)]\,\mathrm{d}\partial D \quad (22)\end{aligned}$$

$$\begin{aligned}\tfrac{1}{2}\mathbf{A}_2(p) = &\int_{\partial D} [+G(p,q)(\operatorname{curl}\mathbf{A}_2(q) \times \mathbf{n}(q))]\,\mathrm{d}\partial D \\ &- \int^*_{\partial D} [(\mathbf{A}_2(q)\cdot\mathbf{n}(q))\,\operatorname{grad}_q G(p,q) \\ &+ (\mathbf{A}_2(q) \times \mathbf{n}(q)) \times \operatorname{grad}_q G(p,q)]\,\mathrm{d}\partial D \quad (23)\end{aligned}$$

The vector potential is continuous across the boundary: $\mathbf{A} = \mathbf{A}_1 = \mathbf{A}_2$, but its derivatives are not and the sum of eqns (22) and (23) is:

$$\begin{aligned}\mathbf{A}(p) = &\int_{\partial D} G(p,q)[\operatorname{curl}\mathbf{A}_2(q) \times \mathbf{n}(q) \\ &- \operatorname{curl}\mathbf{A}_1(q) \times \mathbf{n}(q)]\,\mathrm{d}\partial D \quad (24)\end{aligned}$$

The following notations are introduced:

$$\begin{aligned}&\operatorname{curl}\mathbf{A}_1 \times \mathbf{n} = \mathbf{B}_{t1} \\ &\operatorname{curl}\mathbf{A}_2 \times \mathbf{n} = \mathbf{B}_{t2} \\ &\mathbf{B}_{t1} - \mathbf{B}_{t2} = \Delta\mathbf{B}_t \quad (25)\end{aligned}$$

Note that the $\mathbf{B}_t$ are not the tangent components of flux densities, i.e. their projections on the tangent plane, but their projections rotated by 90° in this plane. Equation (24) can be written:

$$\mathbf{A}(p) = \int_{\partial D} G(p,q)\Delta\mathbf{B}_t(q)\,\mathrm{d}\partial D \quad (26)$$

In order to eliminate the vector potential from eqn (26), the vector product of the curl of expression (26) is taken with $\mathbf{n}(p)$:

$$\begin{aligned}&\operatorname{curl}_p \mathbf{A}(p) \times \mathbf{n}(p) \\ &= \int_{\partial D} \operatorname{curl}_p[G(p,q)\Delta\mathbf{B}_t(q)] \times \mathbf{n}(p)\,\mathrm{d}\partial D \\ &= \int_{\partial D} [\operatorname{grad}_p G(p,q) \times \Delta\mathbf{B}_t(q)] \times \mathbf{n}(p)\,\mathrm{d}\partial D \\ &= -\int_{\partial D} [\operatorname{grad}_q G(p,q) \times \Delta\mathbf{B}_t(q)] \times \mathbf{n}(p)\,\mathrm{d}\partial D \quad (27)\end{aligned}$$

The integrand can be written:

$$\begin{aligned}&(\operatorname{grad}_q G(p,q) \times \Delta\mathbf{B}_t(q)) \times \mathbf{n}(p) \\ &= (\mathbf{n}(p)\cdot\operatorname{grad}_q G(p,q))\Delta\mathbf{B}_t(q) \\ &\quad - (\mathbf{n}(p)\cdot\Delta\mathbf{B}_t(q))\operatorname{grad}_q G(p,q) \quad (28)\end{aligned}$$

According to the discussion of singular kernels, the first term of the right-hand member of eqn (28) leads to a discontinuity across the boundary while the second term must be understood in the Cauchy principal value sense. (Nevertheless, $\mathbf{n}(p)\cdot\Delta\mathbf{B}_t(q)$ vanishes when $p = q$ and this weakens the singularity.) Introducing the starred integrals, eqn (27) splits in:

$$\begin{aligned}\mathbf{B}_{t1}(p) = &-\int^*_{\partial D} [\operatorname{grad}_q G(p,q) \times \Delta\mathbf{B}_t(q)] \\ &\times \mathbf{n}(p)\,\mathrm{d}\partial D + \frac{\Delta\mathbf{B}_t(p)}{2} \quad (29a)\end{aligned}$$

$$\begin{aligned}\mathbf{B}_{t2}(p) = &-\int^*_{\partial D} [\operatorname{grad}_q G(p,q) \times \Delta\mathbf{B}_t(q)] \\ &\times \mathbf{n}(p)\,\mathrm{d}\partial D - \frac{\Delta\mathbf{B}_t(p)}{2} \quad (29b)\end{aligned}$$

The average value on the boundary of the tangential flux density is defined by:

$$\begin{aligned}\mathbf{B}_{t0}(p) = \frac{\mathbf{B}_{t1} + \mathbf{B}_{t2}}{2} = &-\int^*_{\partial D} [\operatorname{grad}_q G(p,q) \times \Delta\mathbf{B}_t(q)] \\ &\times \mathbf{n}(p)\,\mathrm{d}\partial D \quad (30)\end{aligned}$$

The tangential magnetic field is continuous and that can be written:

$$\frac{\mathbf{B}_{t1}}{\mu_1} = \frac{\mathbf{B}_{t0} + \frac{\Delta\mathbf{B}_t}{2}}{\mu_1} = \frac{\mathbf{B}_{t2}}{\mu_2} = \frac{\mathbf{B}_{t0} - \frac{\Delta\mathbf{B}_t}{2}}{\mu_2} \quad (31)$$

Equation (31) shows that the value of the discontinuity of the tangential flux density is proportional to the average value:

$$\Delta\mathbf{B}_t = -2\frac{\mu_2 - \mu_1}{\mu_2 + \mu_1}\mathbf{B}_{t0} = -2\Omega\mathbf{B}_{t0} \quad (32)$$

One poses $\Delta\mathbf{B}_t = \mu_0\mathbf{K}$, with μ_0 the free space magnetic permeability, in eqn (30) in order to bring about equivalent currents (the dimension of $\mathbf{K}$ is the one of a surface current density):

$$\frac{\mu_0\mathbf{K}(p)}{2} = \Omega\int_{\partial D}^{*}[\mathrm{grad}_q\, G(p,q)\times\mu_0\mathbf{K}(q)]\times\mathbf{n}(p)\,\mathrm{d}\partial D \qquad (33)$$

This is a Fredholm vector integral equation of the second kind for the equivalent currents. This result may be interpreted in terms of an equivalent system: if the medium D_1 of permeability μ_1 and the medium D_2 of permeability μ_2 are replaced by a uniform medium of permeability μ_0 (free space) and by a surface current density satisfying eqn (33), then the vector potential and the magnetic flux density will be the same in both cases (but not the magnetic field).

Note: $\mathbf{K} = 0$ is obviously a solution of eqn (33). This is because no excitation has been introduced in the problem. The influence of sources is taken into account by introducing their contribution B_{t0}^{s} to B_{t0}:

$$\frac{\mu_0\mathbf{K}(p)}{2} = \Omega\left[\int_{\partial D}^{*}[\mathrm{grad}_q\, G(p,q)\times\mu_0\mathbf{K}(q)]\times\mathbf{n}(p)\,\mathrm{d}\partial D - B_{t0}^{s}(p)\right] \qquad (34)$$

In the case of the scalar potential, the indirect formulation leads to a single layer of magnetic monopoles ρ_m. In this scalar case, the computations are strictly identical to the two-dimensional ones. They can be found for the two-dimensional vector potential in Ref. 7. The difference between the scalar potential and the vector potential is that the vector potential $\mathbf{A}$, the magnetic flux density $\mathbf{B}$, the tangential flux density $\mathbf{B}_t$, the magnetic permeability μ and the equivalent current density $\mathbf{K}$ exchange their roles with the scalar potential ϕ, the magnetic field $\mathbf{H}$, the normal magnetic field H_n, the magnetic reluctivity ν (the inverse of the magnetic permeability) and the equivalent magnetic monopole density ρ_m, respectively. Moreover:

$$\frac{\nu_2-\nu_1}{\nu_2+\nu_1} = \frac{\dfrac{1}{\mu_2}-\dfrac{1}{\mu_1}}{\dfrac{1}{\mu_2}+\dfrac{1}{\mu_1}} = \frac{\mu_1-\mu_2}{\mu_1+\mu_2} = -\Omega \qquad (35)$$

and the integral formula for ρ_m is:

$$\frac{\rho_m(p)}{2\mu_0} = -\Omega\left[\int_{\partial D}^{*}[\mathrm{grad}_q\, G(p,q)\cdot\mathbf{n}(p)]\times\frac{\rho_m(q)}{\mu_0}\mathrm{d}\partial D - H_{n0}^{s}(p)\right] \qquad (36)$$

NUMERICAL INTEGRATION

The practical use of formulae (6) and (7) in numerical computations involves the numerical integration in the Cauchy principal value sense.

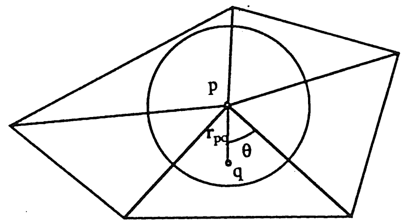

Fig. 3. Triangular meshing and neighbourhood of p.

As an example consider a surface mesh of triangular linear elements and a point p located on one of its nodes. The surface around p is then made with pieces of planes (Fig. 3). On each of them, the vector potential $\mathbf{A}$ varies linearly and may be expressed as:

$$\mathbf{A}(q) = \mathbf{A}(p) + r\mathbf{a}(\theta) \qquad (37)$$

where $\mathbf{A}(p)$ is a constant vector; $\mathbf{a}(\theta)$ is a vector field that depends on the nodal values of $\mathbf{A}$; and r is the distance between the points p and q.

The term of eqn (7) that requires a principal value evaluation may be expressed on an element Δ as:

$$\begin{aligned}
&\int_\Delta \mathbf{A}(q)\times\frac{\mathbf{n}(q)\times\mathbf{r}}{r^3}\mathrm{d}\Delta \\
&= \int_\Delta \mathbf{A}(p)\times\frac{\mathbf{n}(q)\times\mathbf{r}}{r^3}\mathrm{d}\Delta + \int_\Delta r\mathbf{a}(\theta)\times\frac{\mathbf{n}(q)\times\mathbf{r}}{r^3}\mathrm{d}\Delta \\
&= \mathbf{A}(p)\times\int_\Delta \frac{\mathbf{n}(q)\times\mathbf{r}}{r^3}\mathrm{d}\Delta + \int_{\theta_0}^{\theta_1}\int_0^{l(\theta)} r\mathbf{a}(\theta)\times\frac{\mathbf{n}(q)\times\mathbf{r}}{r^3}r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \mathbf{A}(p)\times\int_\Delta \frac{\mathbf{n}(q)\times\mathbf{r}}{r^3}\mathrm{d}\Delta + \int_{\theta_0}^{\theta_1}\int_0^{l(\theta)} \mathbf{a}(\theta)\times\frac{\mathbf{n}(q)\times\mathbf{r}}{r}\mathrm{d}r\,\mathrm{d}\theta
\end{aligned} \qquad (38)$$

Because of the cancellations by skewsymmetry, a section of a finite disc may be removed from the integration domain of the first term. This operation is only meaningful when all the elements are gathered together with a unique given radius for the sections removed. Considering all the elements, this corresponds to removing a finite part of the integration domain whose contribution is equal to zero. The remaining part of the integration domain does not contain any singularity and may be easily evaluated. The second term is an integral that may be taken in the classical sense. Another possibility to evaluate numerically such a principal value is to fold the integration domain such that the divergent terms cancel each other.[8]

For the other kinds of integrals that must not be considered in the Cauchy principal value sense, there exist several more or less classical methods based on changes of variable and adaptive quadrature rules.[9]

ELIMINATION OF NEARLY SINGULAR INTEGRALS IN THIN MAGNETIC STRUCTURES

Nearly singular integrals arise when analysing thin structures. Moreover, care is necessary in the numerical evaluation of these integrals because the equations for nodes close together are very similar and the algebraic system is ill-conditioned. The error on the numerical coefficients must be at least one order of magnitude inferior to their small difference. While there exist efficient methods for the evaluation of nearly singular integrals,[9] the most efficient and robust way to deal with thin structures is probably to eliminate those nearly singular integrals in the theoretical formulation of the method.[2,7,10] The purpose of this section is to design such a method in the case of vector and scalar three-dimensional magnetostatics.

The starting point is the application of the indirect boundary element method to a thin magnetic plate (Fig. 4). In this case, a thin magnetic plate of thickness e and of permeability μ_3, which separates two media of respective permeabilities μ_1 and μ_2, is replaced by a surface current density $\mathbf{K}_1$ on one side and by a surface current density $\mathbf{K}_2$ on the other side.

As the two layers of current are extremely close to each other, the approximation is to consider those two layers $\mathbf{K}_1$ and $\mathbf{K}_2$ as the superposition of a single layer of current $\mathbf{K}_S$ and a double layer of current $\mathbf{K}_D$ at the middle of the plate and defined by (see the signs of $\mathbf{K}_1$ and $\mathbf{K}_2$ on Fig. 4):

$$\mathbf{K}_S = \mathbf{K}_1 - \mathbf{K}_2 \tag{39}$$

$$\mathbf{K}_D = (\mathbf{K}_1 + \mathbf{K}_2)/2 \tag{40}$$

The equations for $\mathbf{K}_S$ and $\mathbf{K}_D$ must of course take into account the mutual influence of $\mathbf{K}_1$ and $\mathbf{K}_2$.

Using starred integrals with singular kernels, integral equations for $\mathbf{K}_1$ and $\mathbf{K}_2$ can be written (see eqns (33) and (34)):

$$\frac{\mu_0\mathbf{K}_1(p_1)}{2} = \Omega_{13}\left[\begin{array}{l}\int^*_{\Gamma_1}[\mathrm{grad}_q\, G(p_1,q)\times\mu_0\mathbf{K}_1(q)]\\ \quad\times\,\mathbf{n}_1(p_1)\,\mathrm{d}\Gamma\\ -\int_{\Gamma_2}[\mathrm{grad}_q\, G(p_1,q)\times\mu_0\mathbf{K}_2(q)]\\ \quad\times\,\mathbf{n}_1(p_1)\,\mathrm{d}\Gamma - \mathbf{B}^s_{t1}(p_1)\end{array}\right] \tag{41}$$

$$\frac{\mu_0\mathbf{K}_2(p_2)}{2} = \Omega_{23}\left[\begin{array}{l}\int^*_{\Gamma_1}[\mathrm{grad}_q\, G(p_2,q)\times\mu_0\mathbf{K}_2(q)]\\ \quad\times\,\mathbf{n}_2(p_2)\,\mathrm{d}\Gamma\\ -\int_{\Gamma_2}[\mathrm{grad}_q\, G(p_2,q)\times\mu_0\mathbf{K}_1(q)]\\ \quad\times\mathbf{n}_2(p_2)\,\mathrm{d}\Gamma - \mathbf{B}^s_{t2}(p_2)\end{array}\right] \tag{42}$$

with

$$\Omega_{13} = \frac{\mu_1-\mu_3}{\mu_1+\mu_3} \tag{43}$$

$$\Omega_{23} = \frac{\mu_2-\mu_3}{\mu_2+\mu_3} \tag{44}$$

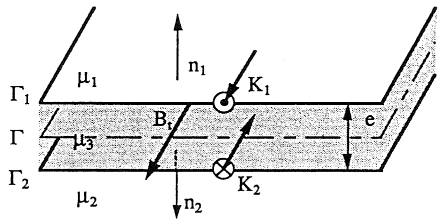

Fig. 4. Equivalent currents for a thin magnetic plate.

In eqns (41) and (42), the first term of the right-hand member represents the self influence of the current layer of the corresponding side, while the second term is the influence of the other layer on the opposite side. The third term is the influence of all the other sources.

In order to obtain a dipole approximation, the limit is taken of the thickness e of the plate tending to zero. In this case, the two sides Γ_1 and Γ_2 tend towards the middle surface Γ of the plate and points p_1 and p_2 tend towards a common point p. As the two sides merge, the integrals of the second term of the right-hand member of eqns (41) and (42) become singular. This is the point of the method: the nearly singular integrals are replaced by singular integrals but which are already involved in the boundary element formulation. The normal $\mathbf{n}_1 = -\mathbf{n}_2$ to face 1 is chosen as the common normal $\mathbf{n}$ for both equations. As the points p_1 and p_2 merge, the exterior tangential flux density due to external sources is the same on both sides:

$$\lim_{p_1\to p}\mathbf{B}^s_{t1}(p_1) = \lim_{p_2\to p}\mathbf{B}^s_{t2}(p_2) = \mathbf{B}^s_t(p) \tag{45}$$

Equations (41) and (42) can be written with starred integrals:

$$\frac{\mu_0\mathbf{K}_1(p)}{2} = \Omega_{13}\left[\begin{array}{l}+\int^*_{\Gamma}[\mathrm{grad}_q\, G(p,q)\times\mu_0\mathbf{K}_1(q)]\\ \quad\times\,\mathbf{n}(p)\,\mathrm{d}\Gamma - \dfrac{\mu_0\mathbf{K}_2(p)}{2}\\ -\int^*_{\Gamma}[\mathrm{grad}_q\, G(p,q)\times\mu_0\mathbf{K}_2(q)]\\ \quad\times\,\mathbf{n}(p)\,\mathrm{d}\Gamma - \mathbf{B}^s_t(p)\end{array}\right] \tag{46}$$

$$\frac{\mu_0\mathbf{K}_2(p)}{2} = \Omega_{23}\left[\begin{array}{l}-\int^*_{\Gamma}[\mathrm{grad}_q\, G(p,q)\times\mu_0\mathbf{K}_2(q)]\\ \quad\times\,\mathbf{n}(p)\,\mathrm{d}\Gamma - \dfrac{\mu_0\mathbf{K}_1(p)}{2}\\ +\int^*_{\Gamma}[\mathrm{grad}_q\, G(p,q)\times\mu_0\mathbf{K}_1(q)]\\ \quad\times\,\mathbf{n}(p)\,\mathrm{d}\Gamma - \mathbf{B}^s_t(p)\end{array}\right] \tag{47}$$

Common terms in eqns (46) and (47) are put together and definition (48) is used:

$$\mathbf{B}_{\mathrm{t\,out}}(p) = \left[\begin{array}{l}-\int^*_{\Gamma}[\mathrm{grad}_q\, G(p,q)\times\mu_0\mathbf{K}_1(q)]\\ \quad\times\,\mathbf{n}(p)\,\mathrm{d}\Gamma\\ +\int^*_{\Gamma}[\mathrm{grad}_q\, G(p,q)\times\mu_0\mathbf{K}_2(q)]\\ \quad\times\,\mathbf{n}(p)\,\mathrm{d}\Gamma + \mathbf{B}^s_t(p)\end{array}\right] \tag{48}$$

The quantity $\mathbf{B}_{t\,out}$ may be physically interpreted as the external tangential flux density on Γ, i.e. due to all the sources except the local influence of $\mathbf{K}_1$ and $\mathbf{K}_2$. As the discontinuity of the tangential flux density is due to this local influence, $\mathbf{B}_{t\,out}$ is continuous and well defined on Γ. Expressions (46) and (47) can be simply written:

$$\mathbf{K}_1/2 = -\Omega_{13}(\mathbf{B}_{t\,out}/\mu_0 + \mathbf{K}_2/2) \tag{49a}$$

$$\mathbf{K}_2/2 = -\Omega_{23}(\mathbf{B}_{t\,out}/\mu_0 + \mathbf{K}_1/2) \tag{49b}$$

After performing some algebra,[2,7] the definitions (43), (44), (39) and (40) of Ω_{13}, Ω_{23}, $\mathbf{K}_S$ and $\mathbf{K}_D$ and formulae (49a) and (49b) for $\mathbf{K}_1$ and $\mathbf{K}_2$ lead to the following final expressions for the equivalent single and double layers $\mathbf{K}_S$ and $\mathbf{K}_D$:

$$\mathbf{K}_S = -2\frac{\mu_1 - \mu_2}{\mu_1 + \mu_2}\frac{\mathbf{B}_{t\,out}}{\mu_0} \tag{50}$$

$$\mathbf{K}_D = -\frac{\frac{\mu_1 + \mu_2}{2} - \mu_3}{\frac{\mu_1 + \mu_2}{2}}\frac{\mathbf{B}_{t\,out}}{\mu_0} \tag{51}$$

A model for thin magnetic plates has been obtained in terms of equivalent currents. It is now possible to extend this model to the direct boundary method by finding equivalent transmission conditions for the vector potential and tangential flux density. Those conditions may be directly deduced from the current densities. The single layer density $\mathbf{K}_S$ in the equivalent situation corresponds to a discontinuity of the tangential flux density in the real situation, and the double layer density $\mathbf{K}_D$ corresponds to a discontinuity of the vector potential. That can be written as:

$$\mu_0\mathbf{K}_S = \mathbf{B}_{t2} - \mathbf{B}_{t1} \tag{52}$$

$$\mu_0 e\mathbf{K}_D = \mathbf{A}_2 - \mathbf{A}_1 \tag{53}$$

where the indices 1 and 2 are for the opposite sides of the plate. Note that in this case the relevant quantity is not $\mathbf{K}_D$ but the dipole moment $e\mathbf{K}_D$, i.e. the product of the double layer density with the thickness of the plate. It is in fact here and only here that the actual thickness of the plate reappears as a parameter of the formulation.

$\mathbf{B}_{t\,out}$, defined by eqn (48), is the sum of all the external influences except the local influence of the currents $\mathbf{K}_1$ and $\mathbf{K}_2$. $\mathbf{B}_{t\,out}$ combines with the discontinuous contributions of $\mathbf{K}_1$ and $\mathbf{K}_2$ to the tangential flux density to give the values $\mathbf{B}_{t1}$ and $\mathbf{B}_{t2}$ for the opposite sides outside the plate and the value $\mathbf{B}_{t\,in}$ inside the plate. The following relations are obtained:

$$\mathbf{B}_{t\,in} = \mathbf{B}_{t\,out} + \mu_0\mathbf{K}_1/2 + \mu_0\mathbf{K}_2/2$$
$$= \mathbf{B}_{t\,out} + \mu_0\mathbf{K}_D \tag{54}$$

$$\mathbf{B}_{t1} = \mathbf{B}_{t\,out} - \mu_0\mathbf{K}_1/2 + \mu_0\mathbf{K}_2/2$$
$$= \mathbf{B}_{t\,out} - \mu_0\mathbf{K}_S/2 \tag{55}$$

$$\mathbf{B}_{t2} = \mathbf{B}_{t\,out} + \mu_0\mathbf{K}_1/2 - \mu_0\mathbf{K}_2 2$$
$$= \mathbf{B}_{t\,out} + \mu_0\mathbf{K}_S/2 \tag{56}$$

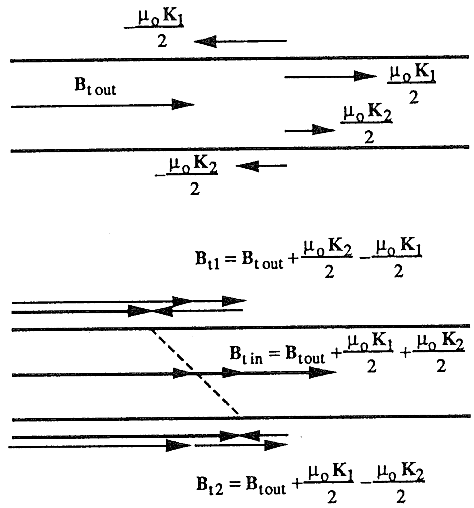

Fig. 5. Flux densities inside and outside the plate.

Those relations are graphically illustrated in Fig. 5. $\mathbf{B}_{t1}$, $\mathbf{B}_{t2}$ and $\mathbf{B}_{t\,in}$ are the physical values of the tangential flux densities inside and outside the plate while $\mathbf{B}_{t\,out}$ is only a fictitious quantity used in the computations.

The difference between eqns (56) and (55) gives eqn (52) while their arithmetic mean gives:

$$\mathbf{B}_{t\,out} = (\mathbf{B}_{t2} + \mathbf{B}_{t1})/2 \tag{57}$$

The combination of eqns (50)–(53) together with eqn (57) gives transmission conditions involving only the vector potential and the tangential flux density:

$$\frac{\frac{\mu_1 + \mu_2}{2} - \mu_3}{\frac{\mu_1 + \mu_2}{2}}\frac{(\mathbf{B}_{t1} + \mathbf{B}_{t2})e}{2} = \mathbf{A}_2 - \mathbf{A}_1 \tag{58}$$

$$\mathbf{B}_{t1}/\mu_1 = \mathbf{B}_{t2}/\mu_2 \tag{59}$$

This last equation is the classical relation expressing the continuity of the tangential magnetic field across a boundary between two media with different permeabilities. The tangential magnetic field is not perturbed by the presence of the plate. The modelling of a thin magnetic plate by the direct boundary element method consists of considering the plate as a boundary between two media and imposing the discontinuity of the vector potential according to eqn (58) instead of its continuity. Note that only the tangential part of the vector potential is concerned with the discontinuity.

The modelling of thin magnetic plates in the framework of the scalar potential model is also possible. In this case, the computations are exactly the same as in the two-dimensional case[2,7] but with ϕ, H_n and ν instead of $\mathbf{A}$, $\mathbf{B}_t$ and μ, and they give:

$$\frac{\frac{\nu_1 + \nu_2}{2} - \nu_3}{\frac{\nu_1 + \nu_2}{2}}\frac{(H_{n1} + H_{n2})e}{2} = \phi_2 - \phi_1 \tag{60}$$

$$H_{n1}/\nu_1 = H_{n2}/\nu_2 \tag{61}$$

CONCLUSION

The vector formulation of the three-dimensional magnetostatics requires a careful definition of the integrals of the vector singular kernels involved. Beyond its importance from a theoretical point of view, this is necessary for the numerical computation of the integrals. On the basis of these theoretical developments, a workable method for three-dimensional thin magnetic plates which avoids the numerical computation of nearly singular integrals has been developed.

REFERENCES

1. Brebbia, C. A. *The Boundary Element Method for Engineers*, Pentech Press, London, 1984.
2. Nicolet, A. Modélisation du champ magnétique dans les systèmes comprenant des milieux non linéaires, PhD thesis, University of Liège, Liège, May 1991.
3. Nicolet, A., Dular, P., Genon, A. & Legros, W. Boundary element singularities in 3D magnetostatic problems based on the vector potential, Proc. Betech Conf., 3–5 June 1992, Albuquerque, *Boundary Element Technology VII*, ed. C. A. Brebbia & M. S. Ingber, Computational Mechanics Publications, Southampton, Elsevier Applied Science, London, pp. 319–29.
4. Schwartz, L. *Méthodes Mathématique pour les Sciences Physiques*, Hermann, Paris, 1983.
5. Dautray, R. & Lions, J. L. *Analyse Mathématique et Calcul Numérique pour les Sciences et les Techniques, Vol.* 6, Masson, Paris, 1987.
6. Adriaens, J. P., Delincé, F., Dular, P., Genon, A., Legros, W. & Nicolet, A., Vector potential boundary element method for three dimensional magnetostatic. *IEEE Trans. Magnetics*, 1991, **27**(5), 3808–10.
7. Nicolet, A., Delincé, F., Dular, P., Genon, A. & Legros, W. Indirect and direct BEM for thin magnetic plates, Proc. BEM 14 Conf. 3–6 Nov. 1992, Seville *Boundary Elements XIV, Vol. 1: Field Problems and Applications*, ed. C. A. Brebbia, J. Dominguez & F. Paris, Computational Mechanics Publications, Southampton, Elsevier Applied Science, London, pp. 437–48.
8. Palamara Orsi, A. Nested quadrature rules for Cauchy principal value integrals, *J. Computational & Appl. Maths.*, 1989, **25**, 251–66.
9. Hayami, K. High precision numerical integration methods for 3-D boundary element analysis, *IEEE Trans. Magnetics*, 1990, **26**(2), 603–6.
10. Bamps, N., Delincé, F., Genon, A., Legros, W. & Nicolet, A. Comparison of various methods for modeling of thin magnetic plates, *J. Appl. Phys.*, 1991, **69**(8), 5047–9.