Influence of multiple scattering on the resolution of an imaging system: a Cramér-Rao analysis

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Abstract: We revisit the notion of resolution of an imaging system in the light of a probabilistic concept, the Cramér-Rao bound (CRB). We show that the CRB provides a simple quantitative estimation of the accuracy one can expect in measuring an unknown parameter from a scattering experiment. We then investigate the influence of multiple scattering on the CRB for the estimation of the interdistance between two objects in a typical two-sphere scattering experiments. We show that, contrarily to a common belief, the occurence of strong multiple scattering does not automatically lead to a resolution enhancement.

References and links

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1. Introduction

Many quantitative imaging systems, in the optical, micro-wave or acoustical domains, resort to numerical inversion procedures to reconstruct the image of the sample from its diffracted far-

field. Several authors have pointed out that, accounting for multiple scattering in the inversion model, could ameliorate the image resolution beyond that expected with the Rayleigh criterion $\lambda/(2NA)$, where λ is the incident radiation and NA is the numerical aperture of the system [1, 2, 3]. Similarly, it has been observed, in the context of time reversal focusing that, the spot width of the time reversed wave is smaller when the wave propagates in a random medium, underlying multiple scattering, than when it propagates in a homogeneous medium [4, 5]. Hence, the question whether multiple scattering within the sample may lead to a better resolution than that expected with a single-scattering analysis stirs considerable interest. The usual argument in favor of this thesis is that multiple scattering permits the conversion of evanescent waves into propagative ones so that the diffracted far-field conveys information on the sample's high spatial frequencies which, hopefully, can be recovered with an adequate inversion procedure [1, 3]. Yet, to our knowledge, sole qualitative studies on this subject have been conducted and a quantitative analysis of the influence of multiple scattering on the resolution has never been done. To address this problem, the first step consists in clarifying the notion of resolution. Indeed, in quantitative imaging, the resolution depends strongly on the chosen physical model, the *a priori* information therein, and the signal to noise ratio (SNR). In this case, the Rayleigh criterion, which depends solely on the configuration of the set-up, is not sufficient and a statistical criterion accounting for the SNR and the fitting physical model appears more adequate [6, 7].

In this paper, we quantify the influence of multiple scattering on the resolution of an imaging system by means of the Cramér-Rao bound (CRB). This signal processing notion permits one to evaluate the lower bound of the variance of the parameters retrieved by an inversion procedure from noisy experimental data. The smaller the variance, the more accurate the estimation of the parameter. More precisely, our experiment consists in illuminating two small spheres by a plane wave and collecting their diffracted intensity in far-field with a certain number of detectors within a certain angular cone. To simplify the analysis to its core, we limit the imaging problem to the sole estimation of the interdistance α between the spheres centers. The variance of the estimate $\hat{\alpha}$ obtained from the data of the diffracted intensity is calculated for several values of α, radius and permittivity of the spheres, the latter being chosen to generate more or less interaction between the spheres.

2. The Cramér-Rao bound

The Cramér-Rao inequality provides a universal lower bound for the variance of an arbitrary unbiased estimator (e.g. Ref. [8]). This bound is optimal in the sense that the equality can be reached in extreme cases. We will recall the main result of the Cramér-Rao inequality in the context of an optical imaging system. We call $\mathbf{I}^m = [I_1^m, ..I_N^m]$ the vector of the N measured far-field intensities at different angles in one realization of the experiment. We denote α the unknown parameter of the scattering object to be estimated. In our experiment α is the center-to-center distance of two homogeneous identical dielectric spheres of given diameter d and optical index n. The Cramér-Rao analysis also applies to the multi-parameter case, but for sake of simplicity we will restrict ourselves to a single parameter. Now, the measurement of the scattered intensities $\mathbf{I}^m$ has limited accuracy since it is affected by various sources of noise. We denote $\mathbf{I}^t = [I_1^t, ..I_N^t]$ the "true" values of the intensity, that is the values that would be measured in a noiseless experimental system. An estimator $\hat{\alpha}$ for the unknown parameter can only be derived on the basis of the noisy data. Hence, the variance of this estimator depends drastically on the level of noise in the measurement. The Cramér-Rao inequality sets an absolute lower bound for this variance in the case of an unbiased estimator:

$$\mathbb{E}\left[(\hat{\alpha}-\alpha)^2\right] \geq \left(\mathbb{E}\left[\partial_\alpha L(\alpha)\right]^2\right)^{-1},\tag{1}$$

where $\mathbb{E}$ is the mathematical expectation over the realizations of noise. We have introduced the log-likelihood function $L(\alpha)$, which is the logarithm of the density of probability of measuring the values $\mathbf{I}^m$ for a given value of the parameter α, and its derivative $\partial_\alpha L$ with respect to the parameter α. We see from Eq. (1) that the CRB is a measure of the sensitivity of the experiment to a parameter. Indeed, a sharp variation of the detected intensity, as the parameter α varies, results in large derivative of the log-likelihood function and a small CRB. More explicit calculation requires a model of noise. We will adopt some reasonable assumptions that render the analysis tractable.

Gaussian additive noise   The source-independent background noise is often modelled as a Gaussian additive noise. We therefore write the measures I_j^m as:

$$ I_j^m = I_j^t + N_j, \ j = 1,..,N, \tag{2} $$

where N_j are Gaussian, centered, and perfectly uncorrelated identical variables:

$$ N_j \sim \mathcal{N}(0, I_0^2), \ \mathbb{E}[N_i N_j] = I_0^2 \delta_{ij}, \tag{3} $$

where I_0 is a constant intensity and δ_{ij} the Kronecker symbol. With these hypotheses the CRB can be easily derived:

$$ CRB = I_0^2 \left(\sum_{j=1}^N \left[\partial_\alpha I_j^t \right]^2 \right)^{-1} \tag{4} $$

Multiplicative noise   Another cause of noise in an optical or micro-wave experiment is the fluctuation of the source. This is in general modelled by a Gamma distributed multiplicative noise:

$$ I_j^m = I_j^t N_j, \ j = 1,..,N. \tag{5} $$

The variables N_j will again be assumed perfectly uncorrelated. They will be taken to follow a Gamma law of order L and mean μ:

$$ N_j \sim \Gamma(\mu, L), \ \mathbb{E}[N_i N_j] = \frac{\mu^2}{L} \delta_{ij} + \mu^2. \tag{6} $$

Direct calculations lead to a similar expression as in the additive case:

$$ CRB = \frac{\mu^2}{L} \left(\sum_{j=1}^N \left[\frac{\partial_\alpha I_j^t}{I_j^t} \right]^2 \right)^{-1} \tag{7} $$

Note that in the case of a multiplicative noise, the SNR is kept constant in all configurations.

The CRB is the basis of our definition of the resolution of an imaging system. Precisely, we define the resolution on parameter α as the square root of the relative minimal variance predicted by the Cramér-Rao bound:

$$ \text{Resolution}(\alpha) = \sqrt{\frac{CRB}{\alpha^2}} \tag{8} $$

This quantity gives an estimation of the relative accuracy one can expect from a parameter estimation based on a scattering experiment. With this quantitative tool we will now study the influence of multiple scattering on the resolution of an optical imaging experiment.

3. A scattering experiment

The experimental set-up is as follows. Two homogeneous dielectric spheres of diameter d and optical index n are placed in vacuum. They are separated by a distance α and aligned along the $\widehat{\mathbf{x}}$- or $\widehat{\mathbf{z}}$-axis. An incoming plane wave with wave vector $\mathbf{K}_0 = 2\pi/\lambda\widehat{\mathbf{z}}$ illuminates the spheres. The plane of observation is $(\widehat{\mathbf{x}},\widehat{\mathbf{z}})$ and the scattering angle θ_j defines the direction of observation $\widehat{\mathbf{K}}_j = \cos\theta_j\widehat{\mathbf{z}} - \sin\theta_j\widehat{\mathbf{x}}$. This incident wave is polarized along $\widehat{\mathbf{x}}$ or $\widehat{\mathbf{y}}$. The scattered intensity is recorded in a 30 degrees aperture cone around the forward or around the backward direction. The different configurations with their nomenclatures are depicted on Fig. 1. These configurations have been chosen to enhance or disminish the role of single scattering in the estimation of α.
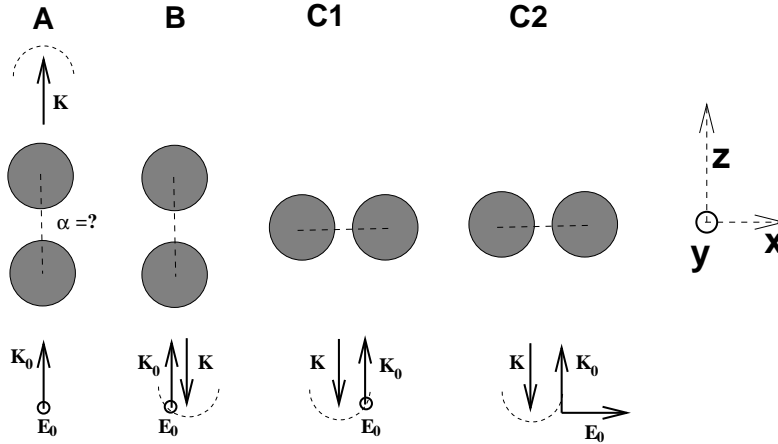

Fig. 1. The set-up of the scattering experiment. In the first configuration, the spheres are aligned in the direction of the incident wave vector $\mathbf{K}_0$. The scattered waves ($\mathbf{K}$) are recorded in a 30 degree aperture cone in the scattering plane $(\widehat{\mathbf{z}},\widehat{\mathbf{x}})$ around the forward (A) or backward (B) direction. The incident polarization $\mathbf{E}_0$ is perpendicular to the scattering plane. In the second configuration, the sphere alignment is perpendicular to $\mathbf{K}_0$ and only the backward direction is investigated. The polarization $\mathbf{E}_0$ is perpendicular (C1) or parallel (C2) to the scattering plane.

3.1. Single scattering analysis

When the interaction between the spheres is neglected, the scattered intensity in the $\widehat{\mathbf{K}}$ direction, $I(\widehat{\mathbf{K}})$, is given by the classical interference formula:

$$I(\widehat{\mathbf{K}}) = 2I_s(\widehat{\mathbf{K}})\left(1 + \cos[\alpha\Phi(\widehat{\mathbf{K}})]\right) \tag{9}$$

where $I_s(\widehat{\mathbf{K}})$ is the scattering intensity of one single sphere, $\Phi(\widehat{\mathbf{K}}) = \frac{2\pi}{\lambda}(\widehat{\mathbf{z}} - \widehat{\mathbf{K}})\cdot\widehat{\mathbf{u}}$ and $\widehat{\mathbf{u}}$ is the direction of alignment of the spheres. Assuming a large number of angular observations, so that the discrete summation can be replaced by a continous one, we obtain a simple analytical expression for the CRB in the case of a multiplicative noise:

$$CRB^{-1} \simeq \text{const} \times \int_{\Phi_{min}}^{\Phi_{max}} \frac{\Phi^2\sin^2(\alpha\Phi)}{|1+\cos(\alpha\Phi)|^2}\mathrm{d}\Phi \tag{10}$$
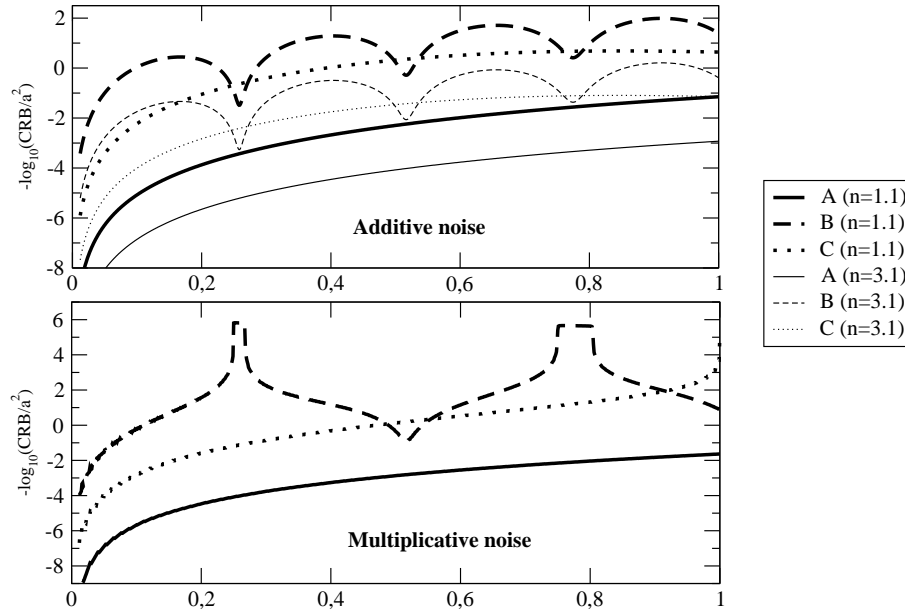
Fig. 2. Relative CRB predicted by a single scattering analysis in different configurations for small spheres ($d = 0.06\lambda$) and two different indices $n = 1.1$ and $n = 3.1$. The proportionality constants in eq. (10) and (11) have been taken to 1. Changing the index results in a vertical translation of the CRB in the case of additive noise (top), but has no influence in the case of multiplicative noise (bottom).

where the proportionality constant is independent of the configuration and the bounds of the integral are the extreme values assumed by the phase Φ in the inspected angular interval. The remarkable feature of this expression is that it does not depend on the shape and size of the scattering objects, since the one-particle intensity I_s is cancelled out in the calculation. A similar formula can be derived in the additive case, assuming in addition that I_s is quasi-constant in the angular region of observation (this is actually the case for small spheres):

$$CRB^{-1} \simeq \text{const} \times I_s \int_{\Phi_{min}}^{\Phi_{max}} \Phi^2 \sin^2(\alpha\Phi)\mathrm{d}\Phi \qquad (11)$$

At this stage we can draw several conclusions from the single scattering analysis of the CRB. In the case of an additive noise, the resolution is trivially ameliorated as the scattering power I_s of one single sphere increases. This not surprising since it corresponds to an increase of the SNR. For small spheres and small contrasts it is well known that the scattering power grows like $I_s \sim \left|(n^2 - 1)/(n^2 + 2)\right|^2$ as the optical index is increased. Hence, doubling the optical constrast $n^2 - 1$ will result in an almost four times smaller CRB. In the case of a multiplicative noise, the resolution is not related to the scattering power of the spheres.

This simple study shows that a fair comparison of the resolution of different scattering configurations requires a constant SNR, which amounts to consider multiplicative noise only. To mimick the classical experiment of point source imaging, we consider two spheres whose diameter is much smaller than the electromagnetic wavelength ($d = 0.06\lambda$). Figure 2 shows the

evolution of the inverse relative CRB as a function of the normalized interdistance α/λ for the three scattering configurations depicted on Fig. 1 (C_1 and C_2 give almost identical results) and the two types of noise. It appears from these plots that the configurations B,C and A listed in this order correspond to an increasing degree of resolution. This is in agreement with the widely accepted rule that finer details of the scatterer can be obtained as the Ewald vector $\mathbf{K} - \mathbf{K}_0$ is increased, since this corresponds to the spatial frequency probed by the interrogating wave in the Rayleigh or Born regime. Indeed we recall that:

$$I(\widehat{\mathbf{K}}) \sim |\widehat{\chi}(\mathbf{K} - \mathbf{K}_0)|^2, \tag{12}$$

where $\widehat{\chi}$ is the Fourier transform of the permittivity function of the object. Note that the peaks observed for the CRB curves in the B configuration are basically an artefact of the multiplicative noise. Indeed, they correspond to interdistances $\alpha = \lambda/4 + p\lambda/2$, where p is an integer, for which the scattered intensity in the backward z direction is null due to destructive interference. Thus, the noise is also null for this direction.

We will now consider the following question: at a given SNR, is a configuration with strong multiple scattering better resolved than a configuration with single scattering ? To answer this question, we consider the same diffraction experiments as previously but we now account for the interaction between the spheres. Since we now cannot calculate the CRB analytically, we use a rigorous method [9] to simulate the intensity scattered by the two spheres and we build numerically the CRB.

3.2. Multiple scattering analysis

In Fig. 3 the CRB is plotted for multiplicative noise in the C configuration for the three different values of the sphere optical index. The multiplicative constant μ^2/L has been set to 1. We consider two incident polarizations, orthogonal (polarization 1, (3a)) or parallel (polarization 2, (3b)) to the plane of observation. Not surprisingly, we observe that as α/λ is increased, the CRB obtained in the rigorous experiments rejoins that obtained when the interaction between the spheres is neglected. On the contrary, when the coupling between the spheres becomes important, the CRB gets lower than that given by the single scattering analysis. We recall that, for a given α, the electromagnetic interaction between the spheres is more important in polarization 1 than in polarization 2 [10] and that it increases with the optical index of the spheres. This analysis clearly shows that, in this configuration, the coupling between the spheres can ameliorate the resolution of the image. A similar conclusion is obtained with the analysis of the CRB in the A configuration as seen in Fig. (4b). On the other hand, the analysis of the CRB in the B configuration leads to a quite different result. Indeed, we observe in Fig. (4a) that, in this case, the interaction between the spheres does not have any effect on the CRB. Hence, we see from these different exemples that the potential amelioration of the resolution due to multiple scattering is only obtained when the experimental configuration is incomplete, i.e. when the Ewald vector scanned in the experiment does not reach the highest possible spatial frequencies.

To increase further the coupling strength between the objects, we have conducted the same experiments as in Fig. 4 with spheres of larger diameter ($d = 0.3\lambda$). The analysis of the CRB in this case (Fig. 5) stresses even more the influence of the configurations. Indeed, in configuration A (Fig. 5b) the amelioration of the resolution due to multiple scattering is patent. Moreover, we notice by comparing Fig. (4b) and (5b) that the estimation of α can be more accurate for larger spheres than for smaller ones. This result confirms the study of Belkebir et al [3] in which it was shown that, with a non-linear inversion algorithm and an experimental set-up close to the A configuration, two big cubes will be distinguished more easily than two small cubes with the same center to center distance. On the other hand, it is seen that, in the B configuration (Fig. (5a)), multiple scattering does not permit an amelioration of the resolution and can even
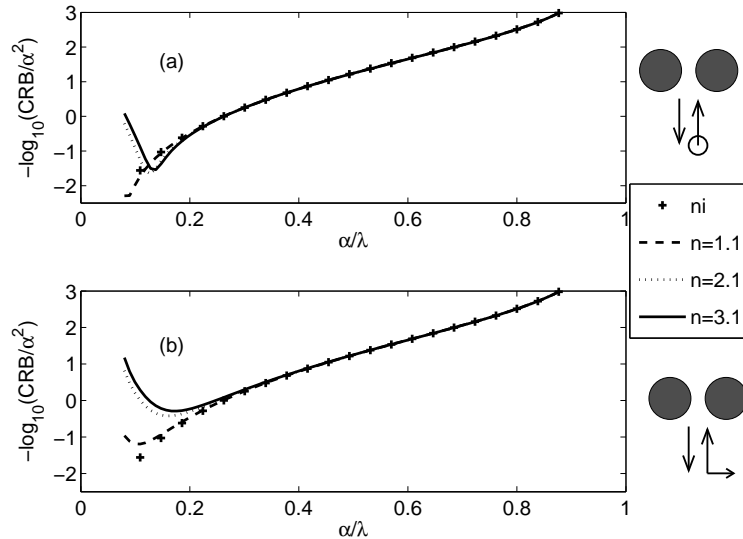
deteriorate it.

Fig. 3. Evolution of the relative CRB as a function of the normalized sphere interdistance α/λ, in configuration C1 (top) and C2 (bottom). The spheres have diameter 0.06λ and the noise is multiplicative. The curve labelled 'ni' (for 'no interaction') is the result of ignoring the coupling between the spheres, for the three optical indices.

4. Conclusion

We have proposed a statistical analysis of the resolution of a non-linear quantitative imaging system accounting for the noise and the fitting model. This approach has allowed us to point out the role of multiple scattering in the resolution of the system. More precisely, we have studied different angular configurations of an optical diffraction tomography experiment plagued with additive or multiplicative noise. We have studied the accuracy of the estimation, from the scattered intensities, of the center to center distance of two spheres. We have shown that when the Ewald vectors, (or momentum transfer) scanned in the experiment are small, multiple scattering permits a clear improvement of the interdistance estimate. On the other hand, if the Ewald vectors are large, multiple scattering does not modify the accuracy with which the interdistance is evaluated. This analysis is in agreement with previous studies in which an amelioration of the resolution, due to multiple scattering, was observed in incomplete experimental configurations.
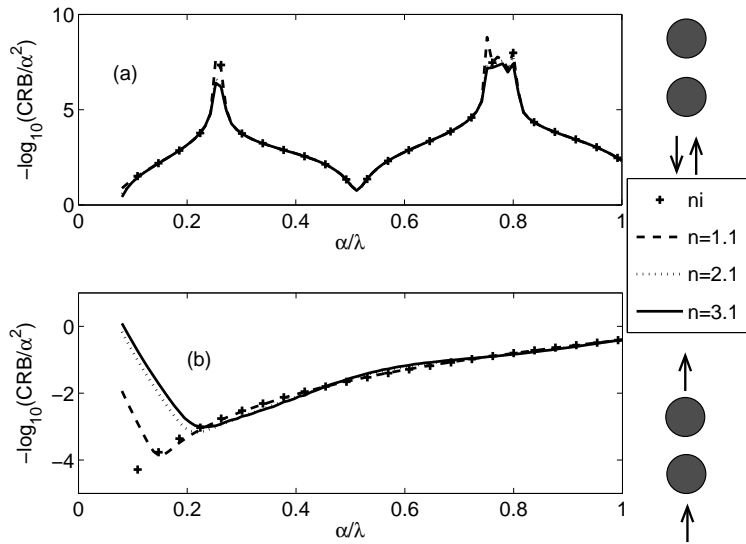
Fig. 4. Same as Fig. 3 in configuration A (bottom) and B (top). The peaks are due to the zeros of intensity, that make the inverse CRB explode (see eq. 7)
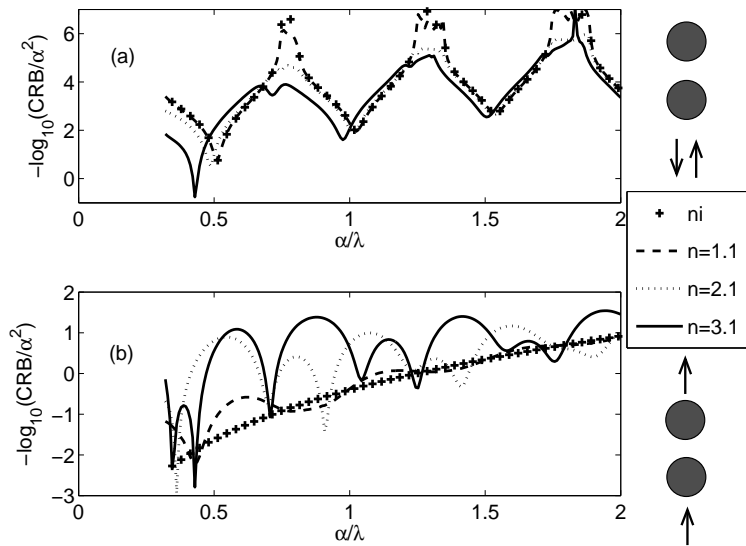

Fig. 5. Same as Fig. 4 for spheres of diameter 0.3λ. The coupling between the spheres is visible on a wider range as for small spheres. In (a), the peaks occur, again, at the distances that lead to destructive interference between the spheres.